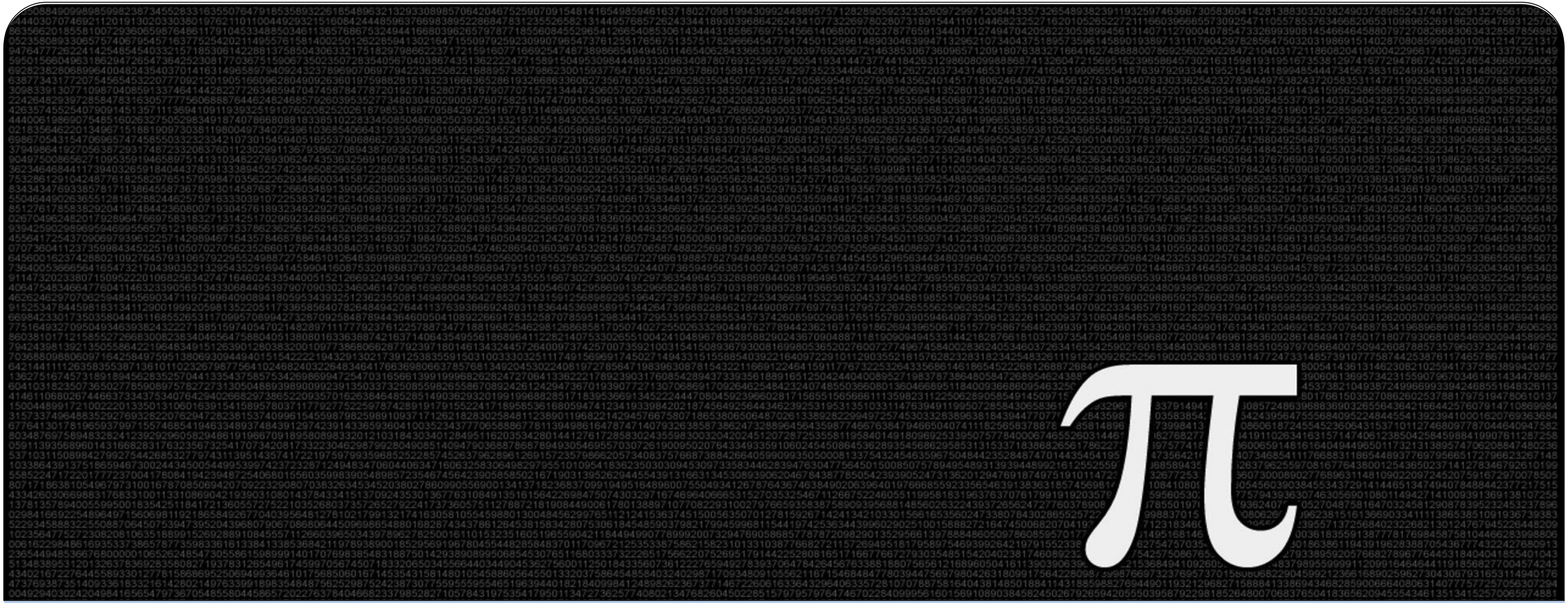


Γραφικά Υπολογιστών

Basic mathematics: Linear Algebra

Andreas Aristidou
andarist@ucy.ac.cy
http://www.andreasaristidou.com

Description

A. Basic Geometry

- Initial Coordinates
- Points
- Vectors
- Lines
- Planes
- Spheres

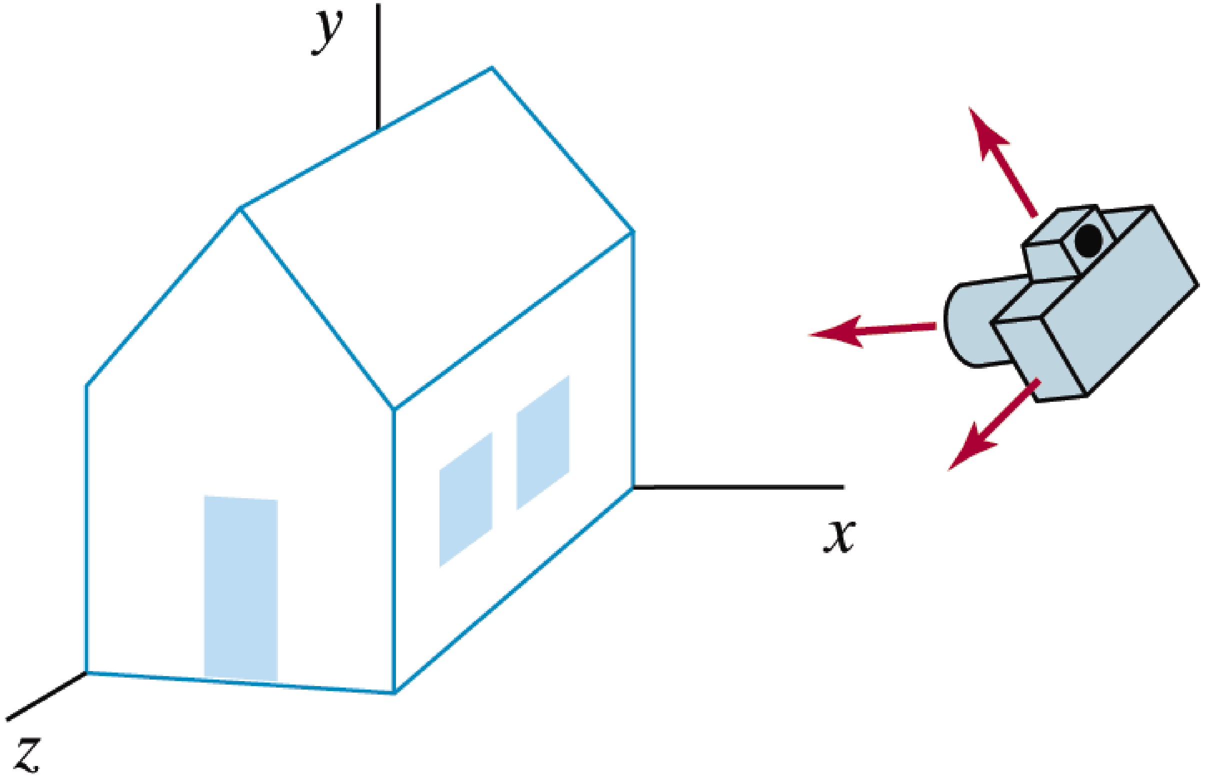
B. Matrices

- Transformations with matrices

- ***Computer Graphics are mathematics!***
- *Although maths used in computer graphics are not **difficult,** we need to have a good understanding of them before defining some techniques.*

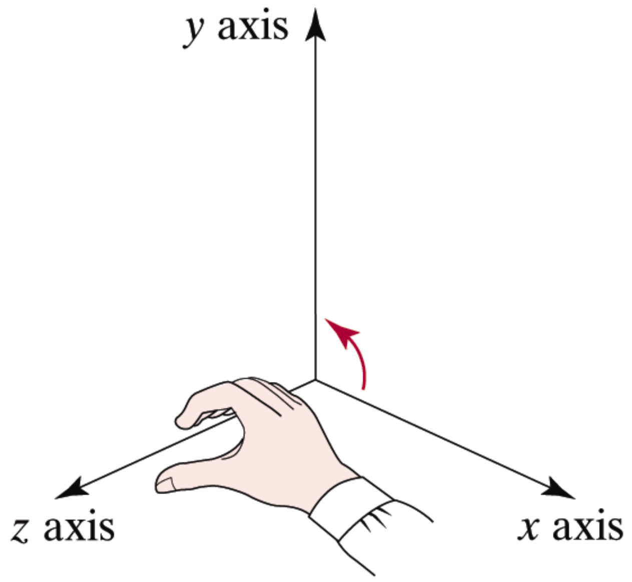
Basic idea

Images taken from Hearn & Baker, "Computer Graphics with OpenGL" (2004)



Right-handed or Left-handed Reference System?

- There are two different ways in which we can set 3D coordinates – right-handed or left-handed.



Right-Handed Reference System

Mainly, we will use the right-handed reference system.

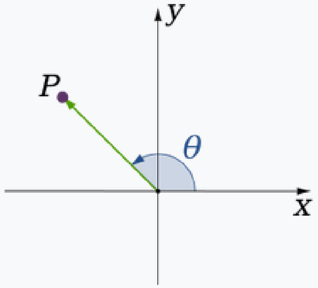
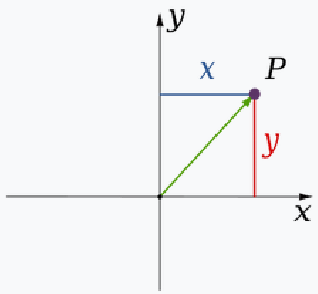
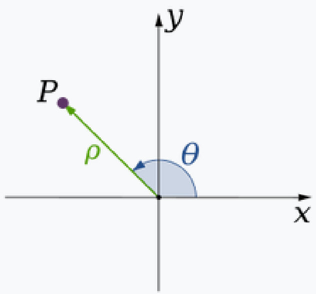
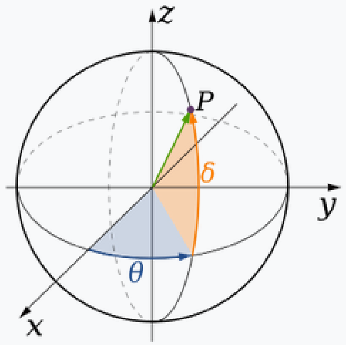
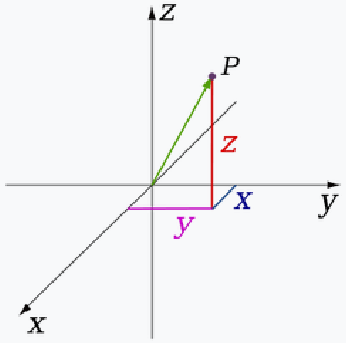
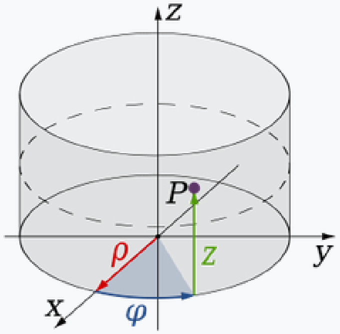
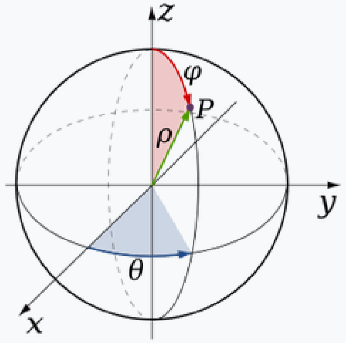
Left-Handed Reference System

Images taken from Hearn & Baker, “Computer Graphics with OpenGL” (2004)

PART A- Basic Geometry

Dimension

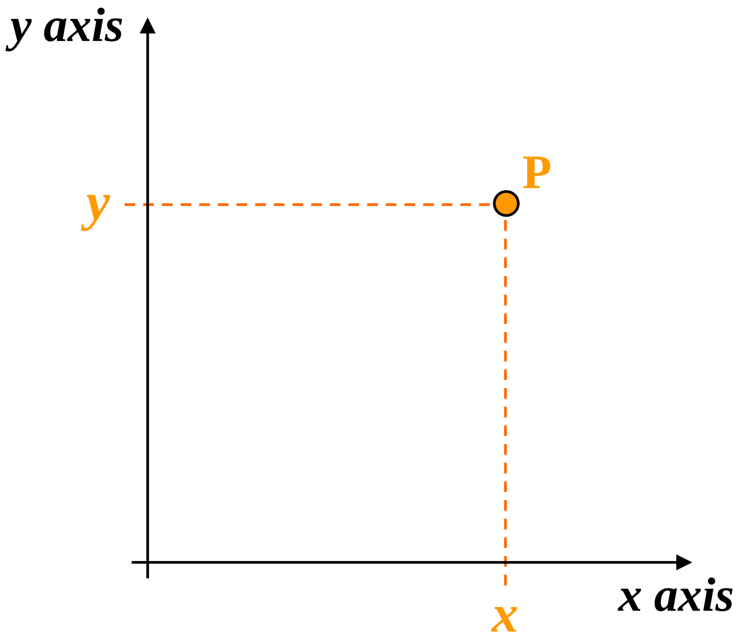
- How much freedom has an "object" to *move around* in space?
- How many variables do we need to define an exact position in space?

Number of dimensions	Example co-ordinate systems		
1	 Number line	 Angle	
2	 **Cartesian** (two-dimensional)	 Polar	 Latitude and longitude
3	 **Cartesian** (three-dimensional)	 Cylindrical	 Spherical

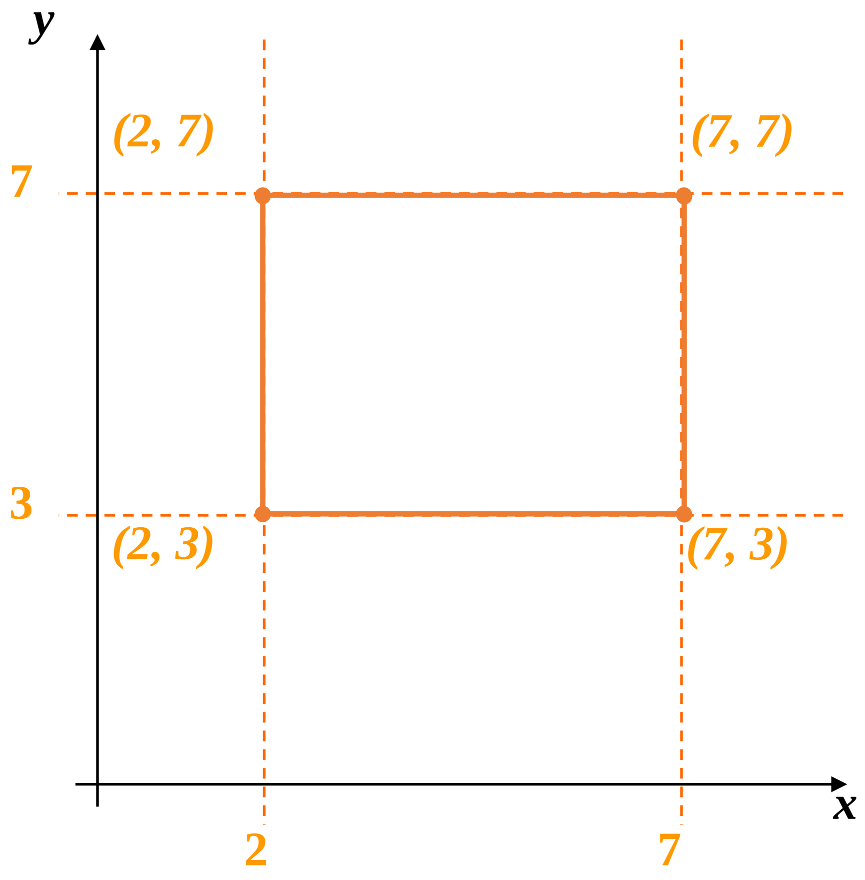
https://en.wikipedia.org/wiki/Dimension

Reference Point – 2D

- When we create a scene in computer graphics, we are essentially defining the scene with simple geometry.
- For 2D scenes we use simple two-dimensional Cartesian coordinates.
- All objects are defined by simple pairs of coordinates

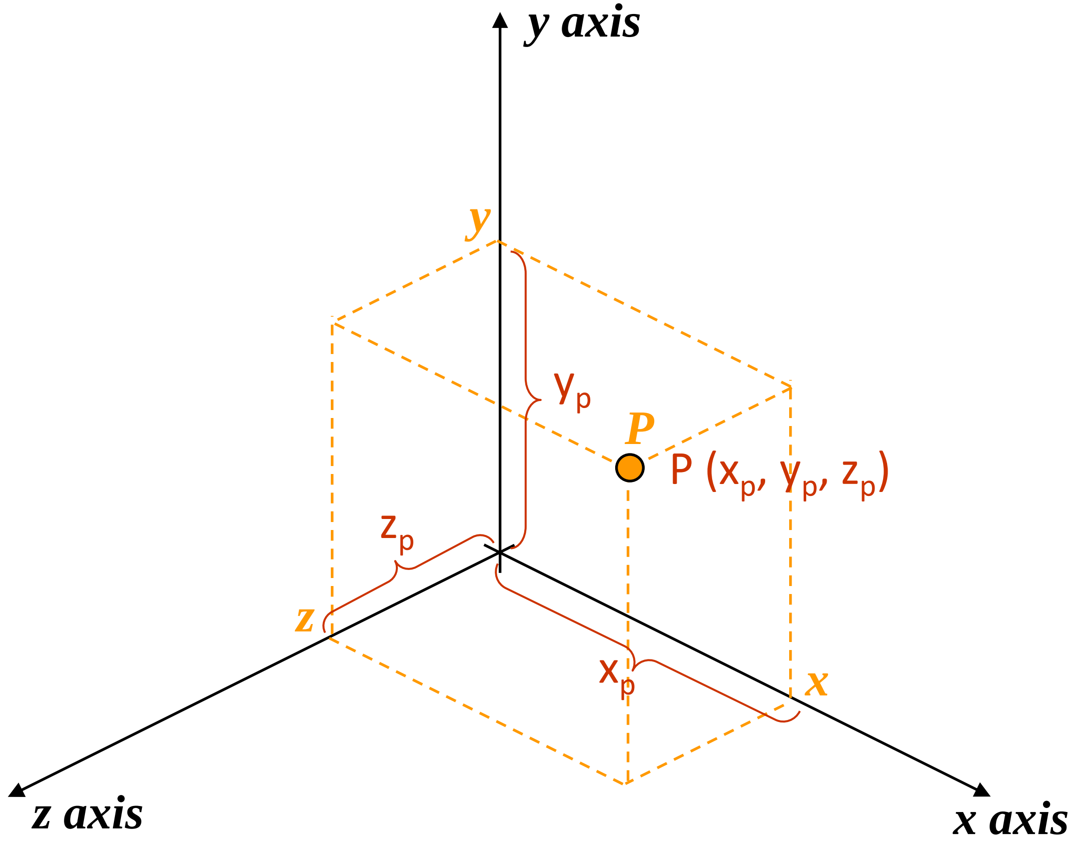


Reference Point – 2D



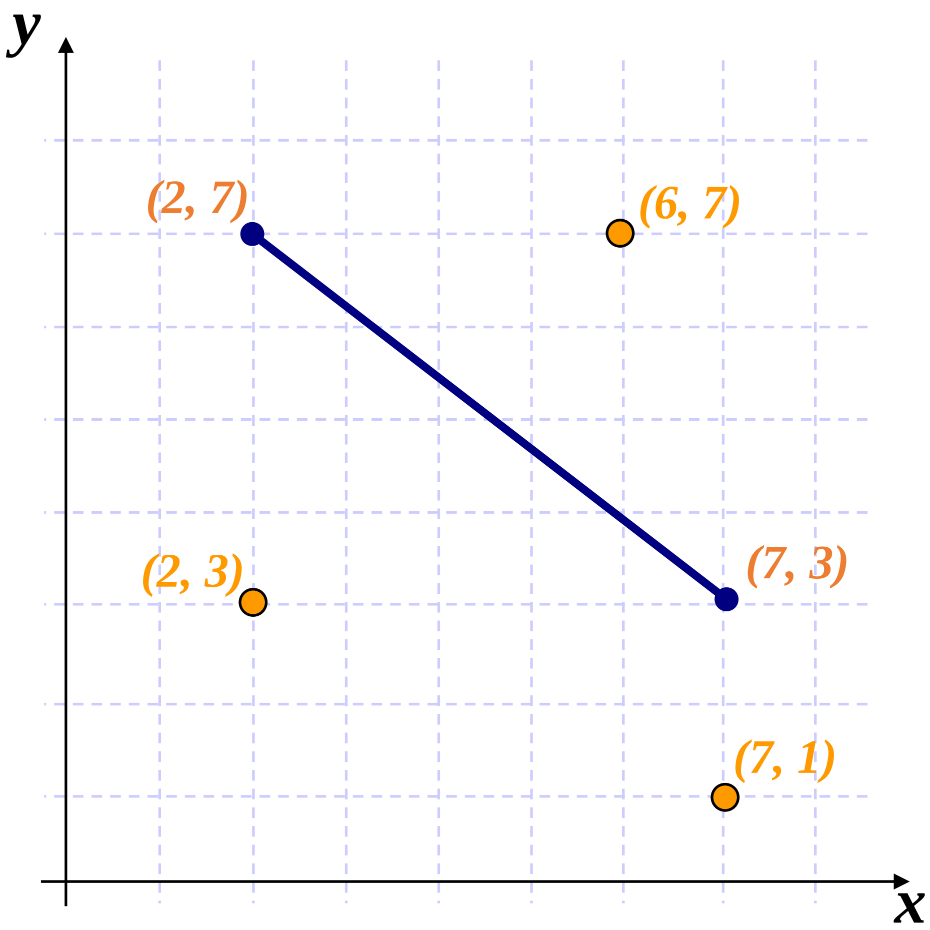
Reference Point – 3D

- For three-dimensional scenes we just add an extra coordinate.
- P (x, y, z)



Points & Lines

Line from
(2, 7) to (7, 3)



The equation of a straight line

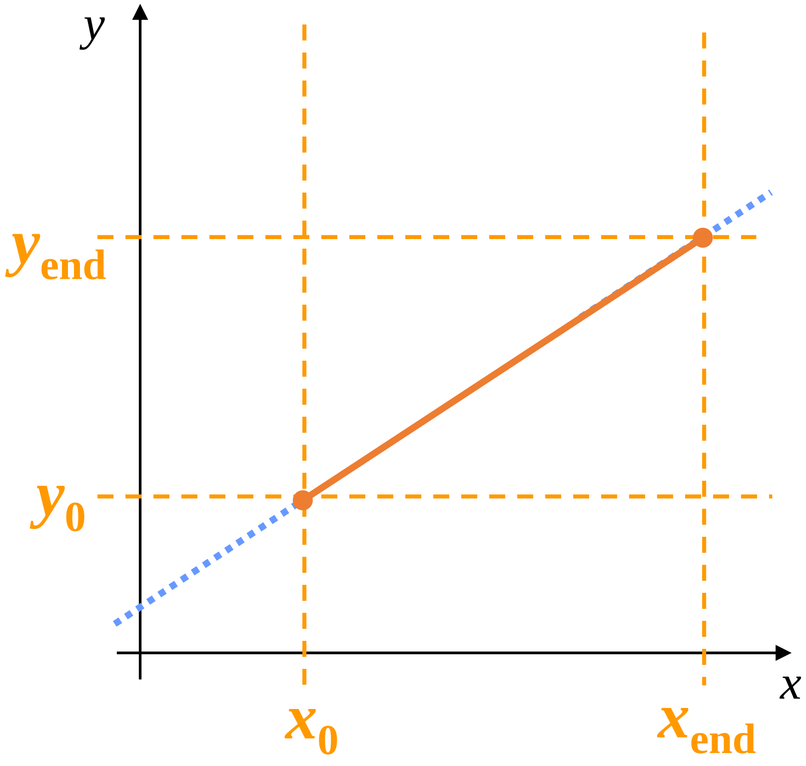
- The slope equation for a straight line is:

$$y = m \cdot x + b$$

- when:

$$m = \frac{y_{end} - y_0}{x_{end} - x_0}$$

$$b = y_0 - m \cdot x_0$$



- This straight line equation gives us the corresponding point y for each point x.

A simple example

- Let's see a part of the line given by the equation:

$$y = \frac{3}{5}x + \frac{4}{5}$$

- What is the y coordinate for every x point?

A simple example

- For each value x we calculate the value of y:

$$y(2) = \frac{3}{5} \cdot 2 + \frac{4}{5} = 2$$

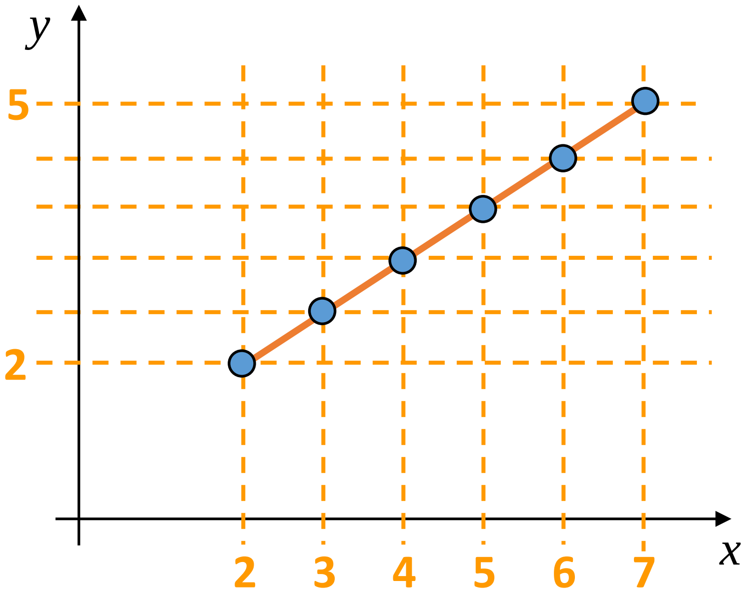
$$y(3) = \frac{3}{5} \cdot 3 + \frac{4}{5} = 2\frac{3}{5}$$

$$y(4) = \frac{3}{5} \cdot 4 + \frac{4}{5} = 3\frac{1}{5}$$

$$y(5) = \frac{3}{5} \cdot 5 + \frac{4}{5} = 3\frac{4}{5}$$

$$y(6) = \frac{3}{5} \cdot 6 + \frac{4}{5} = 4\frac{2}{5}$$

$$y(7) = \frac{3}{5} \cdot 7 + \frac{4}{5} = 5$$

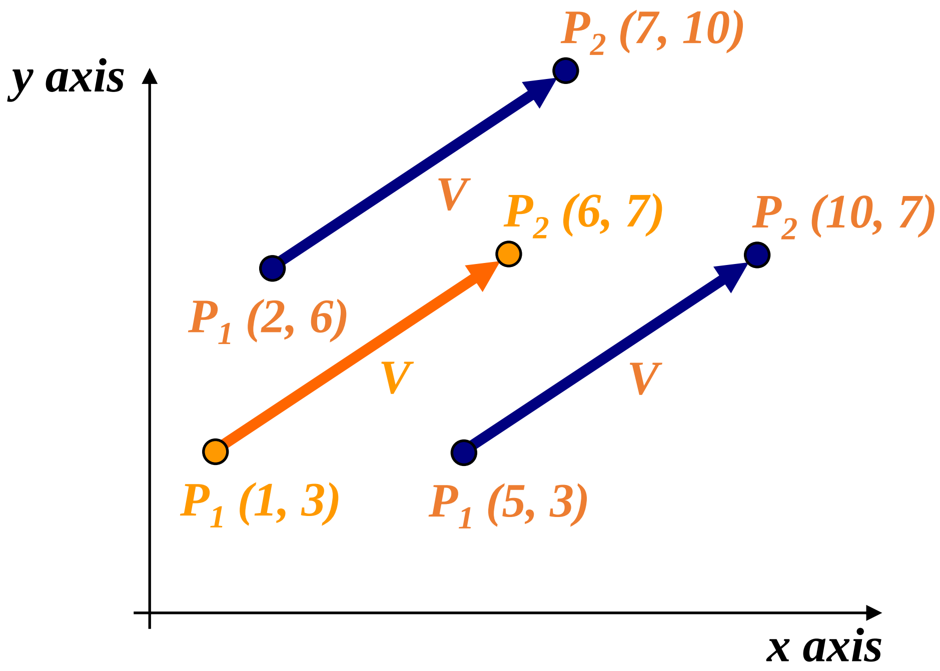


Vectors

- Vector:
  - A vector is defined as the difference between two points.
  - The important thing, however, is that each vector has a direction and a length.
- Where are the vectors used?
  - A vector shows us how and how much an object will move from one point to another.
  - Vectors are very important in graphics, especially in the transformations that we will see later (translation)

Vectors (2D)

- To identify the vector between two points, we simply subtract them

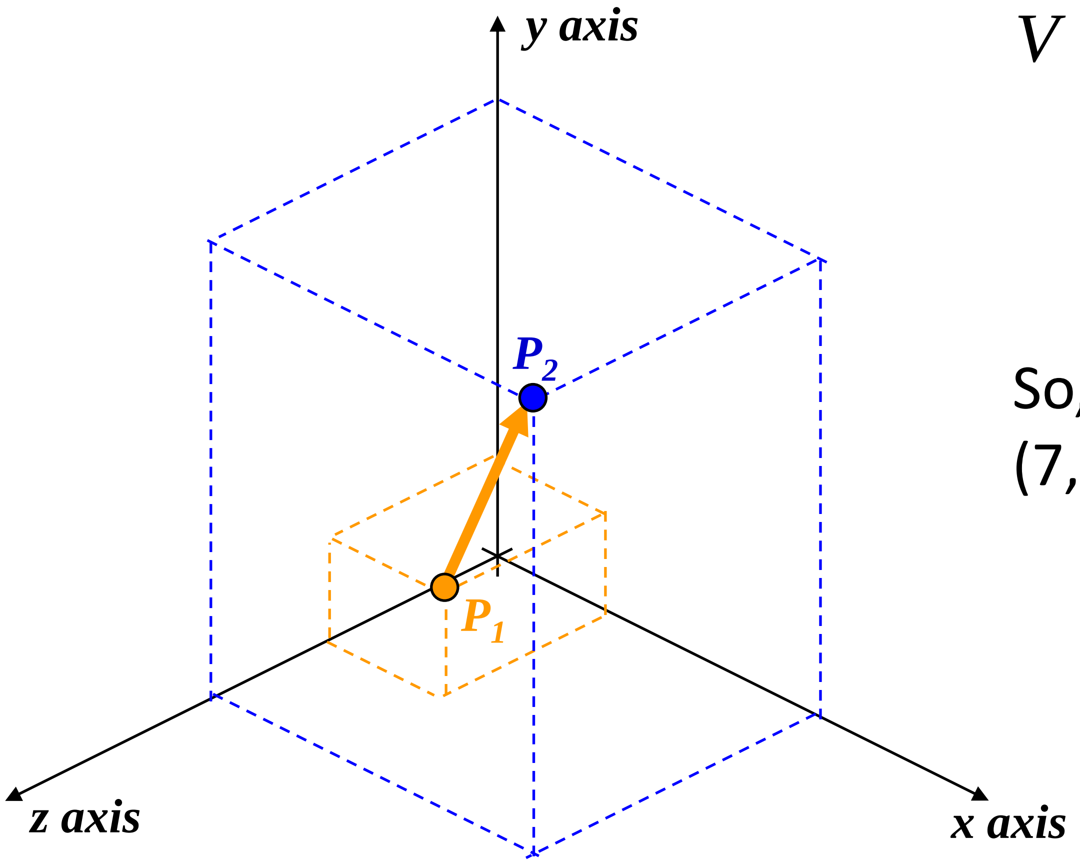


$$
\begin{aligned}
V &= P_2 - P_1 \\
&= (x_2 - x_1, y_2 - y_1) \\
&= (6 - 1, 7 - 3) \\
&= (5, 4)
\end{aligned}
$$

Attention: Many pairs of points have the same vector.

Vectors (3Δ)

- In the three dimensions, the vectors are calculated in the same way



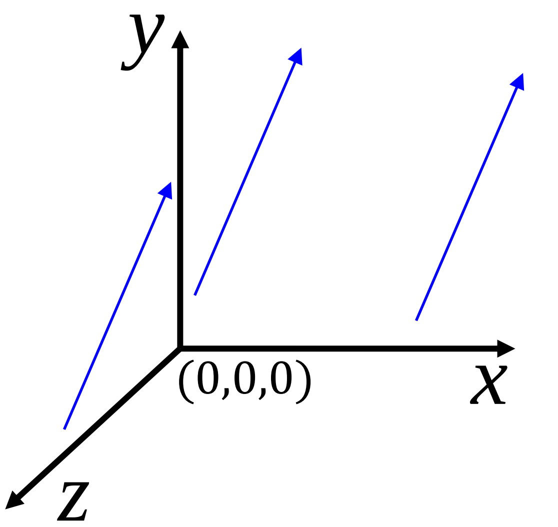
$$V = P_2 - P_1$$
$$= (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$
$$= (V_x, V_y, V_z)$$

So, from (2, 1, 3)to (7, 10, 5) we will have

$$= (7 - 2, 10 - 1, 5 - 3)$$
$$= (5,9,2)$$

Vectors

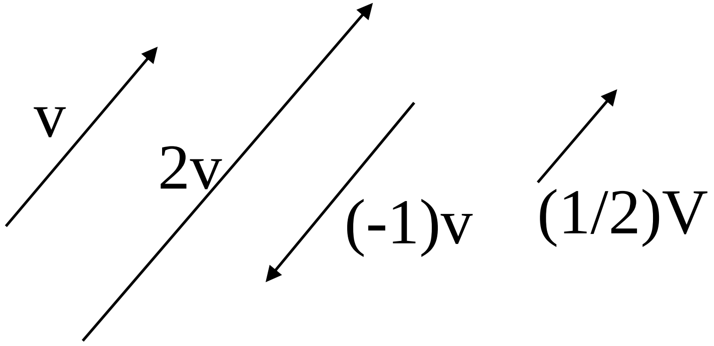
- There are many important functions we need to know in order to properly manage and process vectors:
  - Calculate the length of the vector
  - Adding vectors
  - Scalar multiplication of vectors
  - Inner product (Scalar or dot product)
  - Outer product (Vector or cross product)



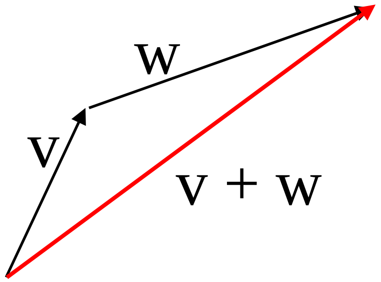
- **Points! = Vectors**
  - vector + vector = vector
  - point + vector = point
  - *point + point =;*
  - point - point = vector (why?)

Not defined! But it does make sense in calculating the center of mass of some points

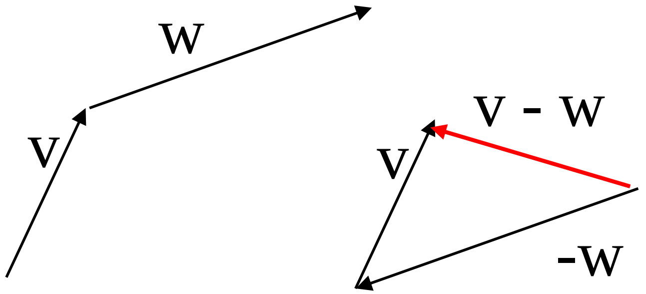
Vectors



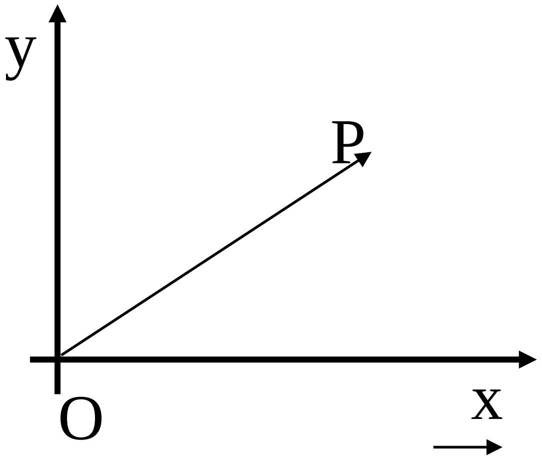
Scalar multiplication of vectors (keep the direction)



Add vectors: **v** + **w**



Subtract vectors : **v** - **w** = **v** + (-**w**)



The Vector $\overrightarrow{OP}$

Vectors: *The length of a vector*

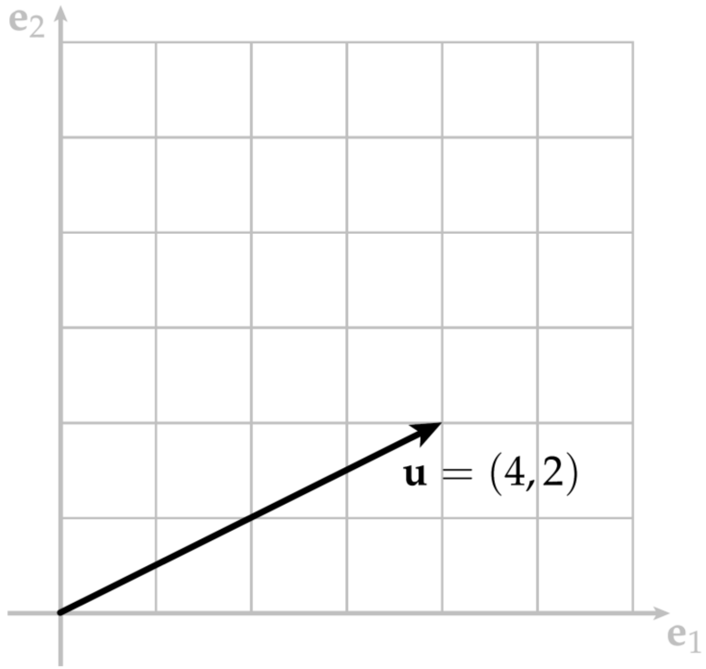
- The length of a vector (modulus) is easily calculated in two dimensions (Euclidean norm):

$$|V| = \sqrt{\sum_{i=1}^{n} V_i^2} = \sqrt{V_x^2 + V_y^2}$$

- And in 3 dimensions:

$$|V| = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

$$|V(x,y,z)| = \sqrt{x^2 + y^2 + z^2}$$



$$\mathbf{u} = (4,2)$$

$$|\mathbf{u}| = \sqrt{4^2 + 2^2} = 2\sqrt{5}$$

WARNING: this quantity does not represent the geometric length unless the vectors are coded on an orthonormal basis. **(Common source of bugs!)**

Vectors: *Unit vector*

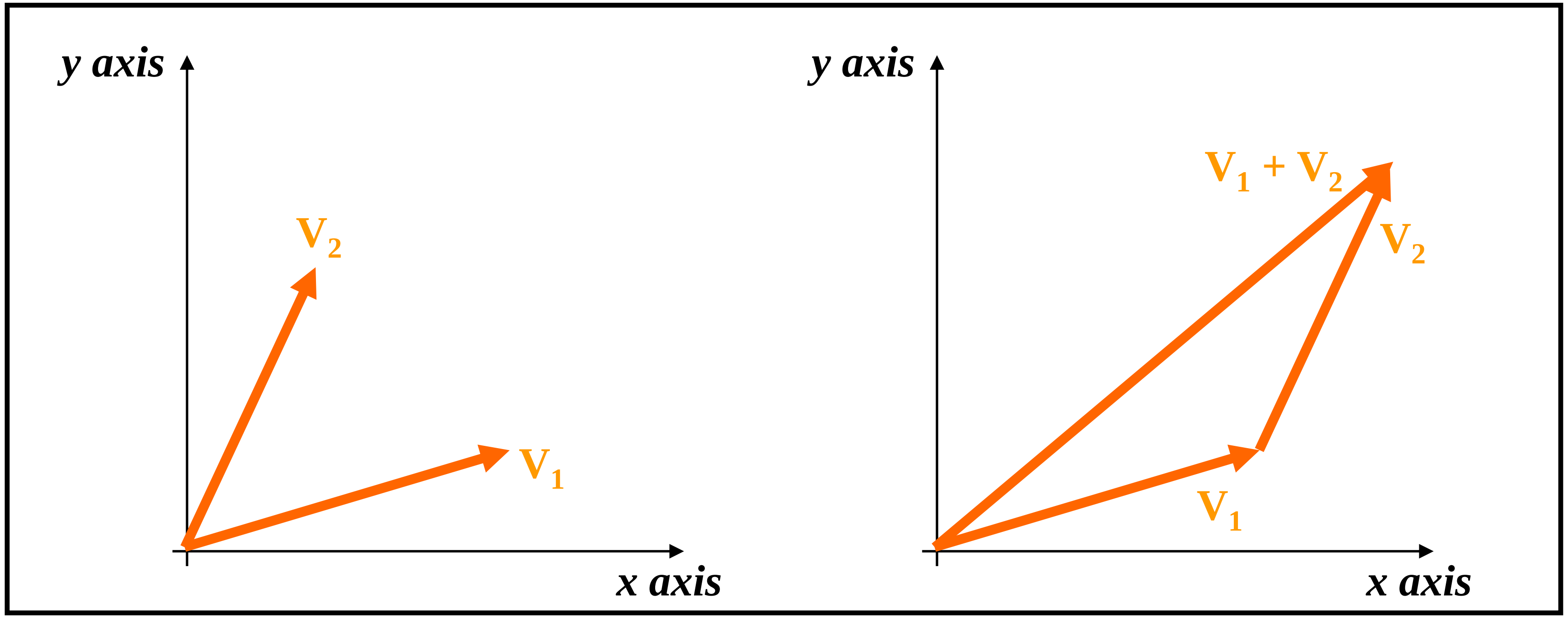
- Unit vector
  - Normalization

$$\hat{V} = \frac{\text{vector } V}{\text{modulus } V} = \frac{V}{|V|}$$

Vectors: *Adding vectors*

- The sum of two vectors is calculated by simply adding its individual elements

$$V_1 + V_2 = (V_{1x} + V_{2x}, V_{1y} + V_{2y})$$

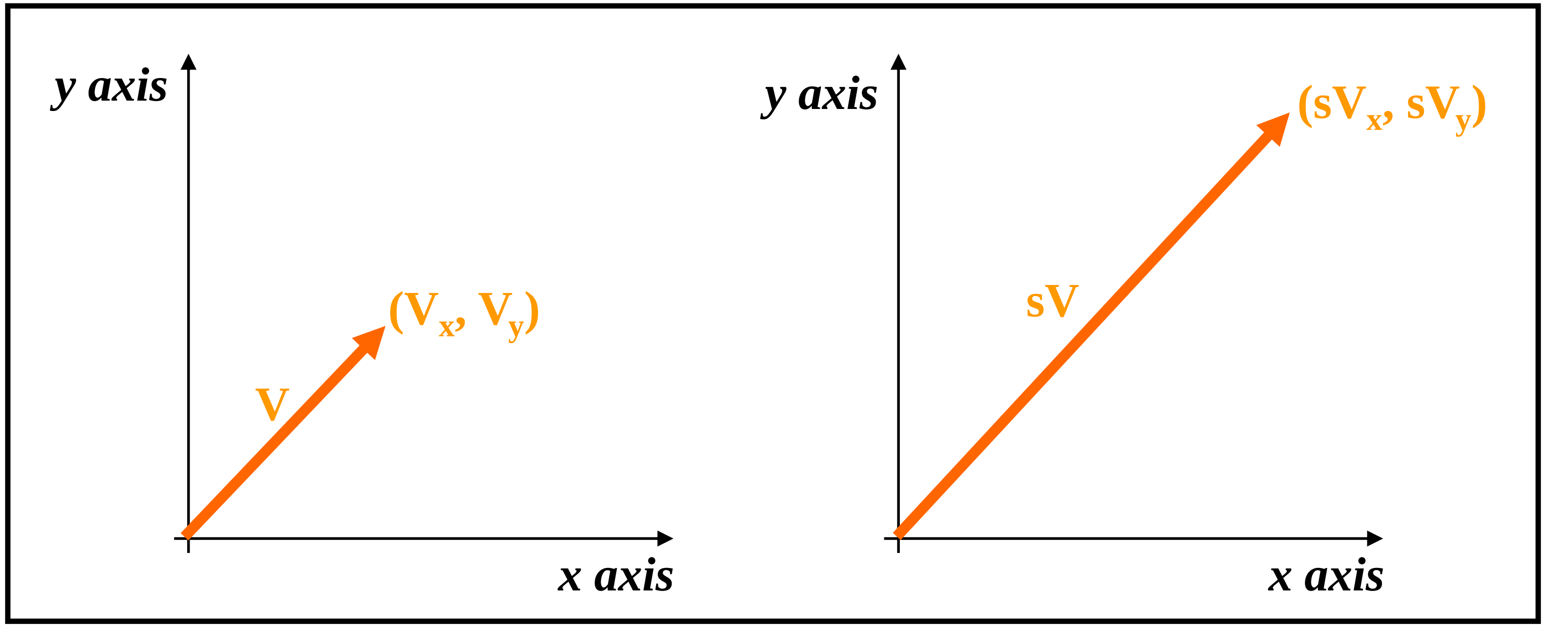


- Same in 3 dimensions.

Vectors: *Scalar multiplication of vectors*

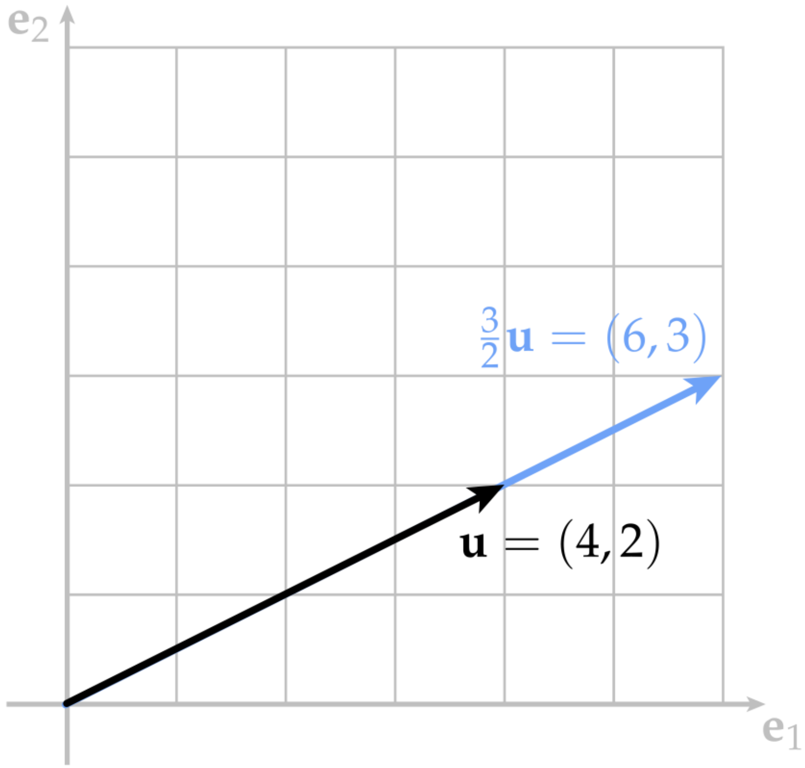
- The multiplication of a vector by a constant is calculated by simply multiplying its individual elements

$$sV = (sV_x, sV_y)$$



Vectors: *Scalar multiplication of vectors*

- Example



$$\frac{3}{2}\mathbf{u}$$

$$= \frac{3}{2}(4, 2)$$

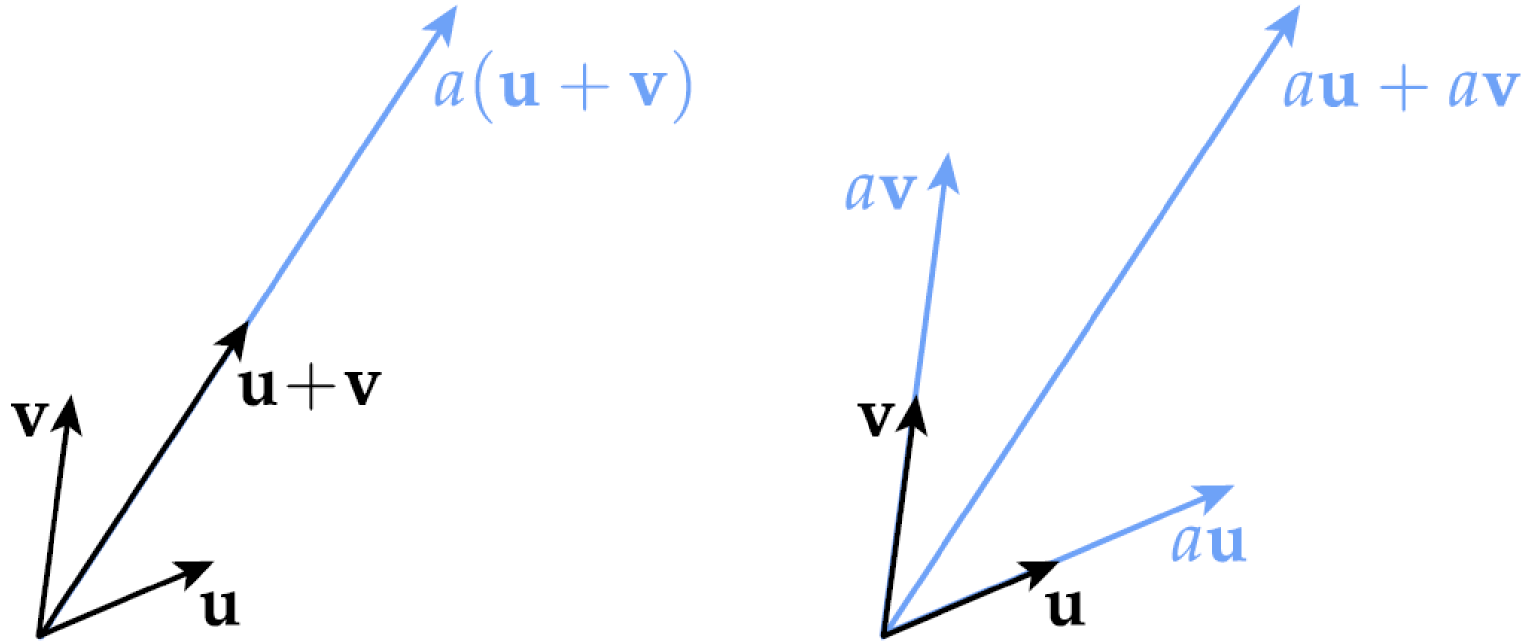
$$= (4 \cdot 3/2, 2 \cdot 3/2)$$

$$= (12/2, 6/2)$$

$$= (6, 3)$$

Vectors: *Scalar multiplication of vectors & Addition*

- What if we try to add two scalable vectors? Or change the scale to two vectors added together?



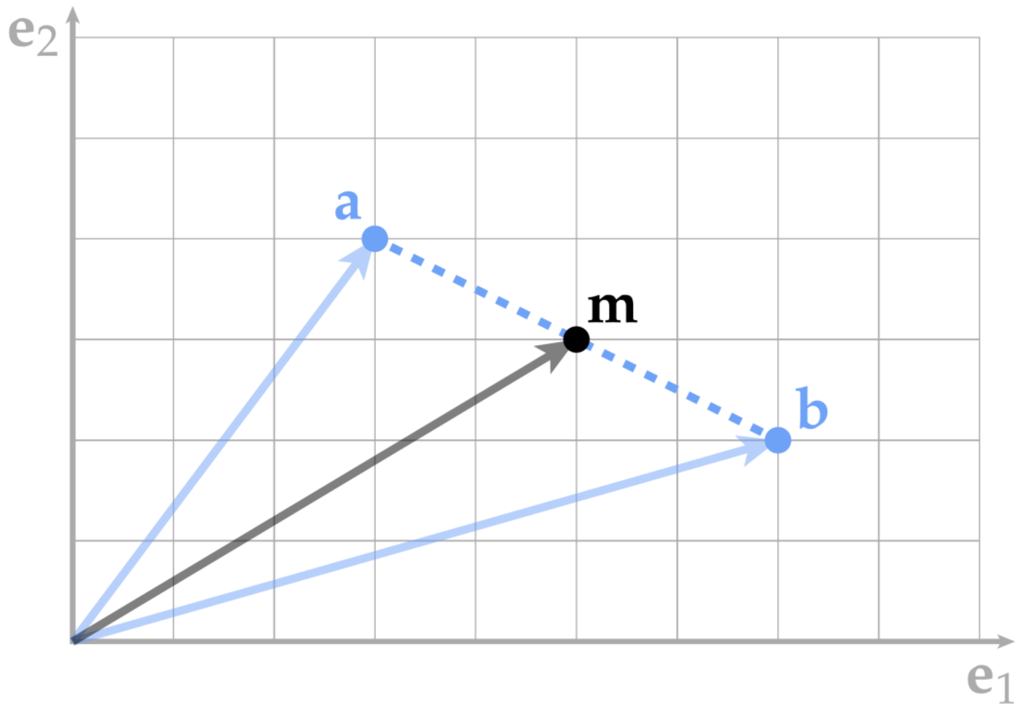
Ενδιαφέρον - φαίνεται ότι έχουμε το ίδιο αποτέλεσμα όπως και να έχει: $a(\mathbf{u}+\mathbf{v}) = a\mathbf{u} + a\mathbf{v}$

Vectors: *Properties*

- For each vector **u, v, w,** and constants a, b
  - $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  - $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{v} + \mathbf{u}) + \mathbf{w}$
  - $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$
  - $\mathbf{v} + (-\mathbf{v}) = (-\mathbf{v}) + \mathbf{v} = \mathbf{0}$
  - $a(b\mathbf{v}) = (ab)\mathbf{v}$
  - $a(\mathbf{v} + \mathbf{u}) = a\mathbf{v} + a\mathbf{u}$
  - $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

Vectors: *Midpoint*

- What is the midpoint $\mathbf{m}$ between $\mathbf{a} = (3,4)$ and $\mathbf{b} = (7,2)$;



$$\mathbf{m} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$$

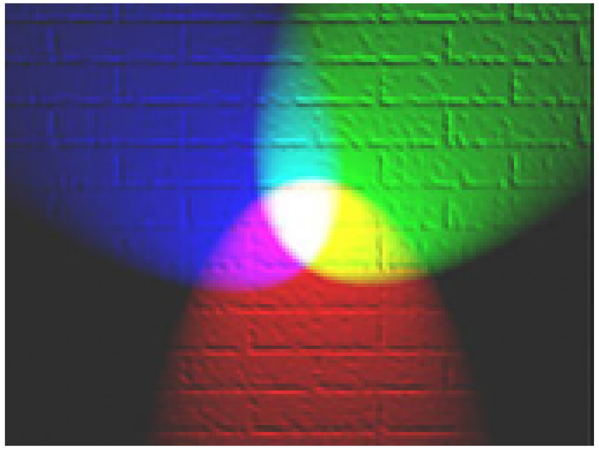
$$= \frac{1}{2}((3,4) + (7,2))$$

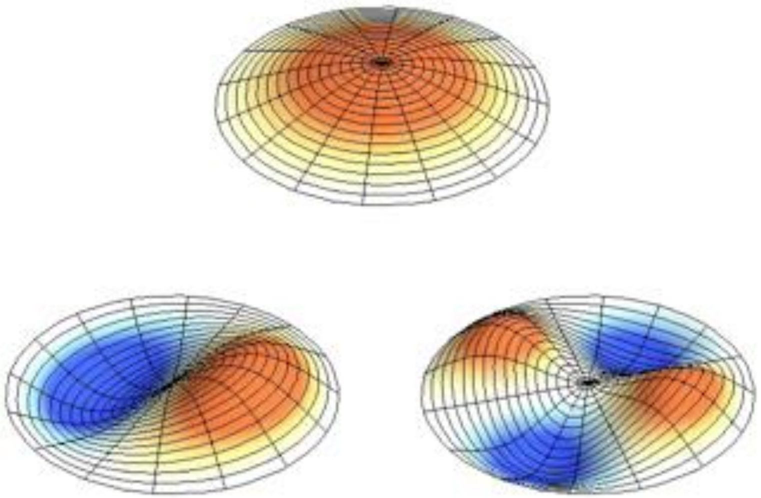
$$= \frac{1}{2}(10,6)$$

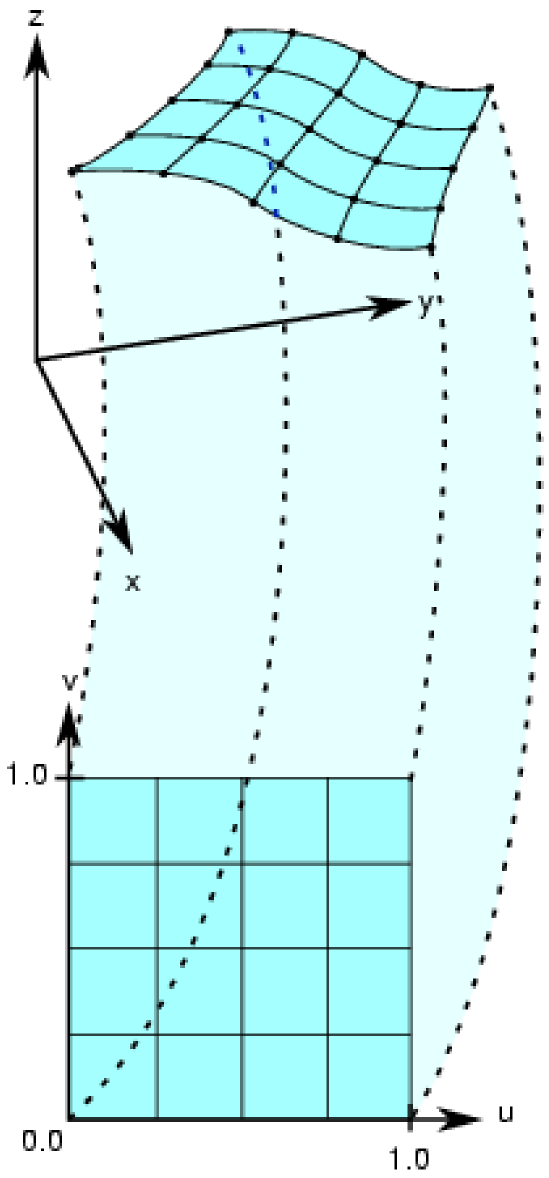
$$= (5,3)$$

Vectors: *Functions*

- Another very important example of the use of vectors in computer graphics is in functions.
- Why? Because many of the objects we want to process in graphics are the result of functions! (e.g., images, glow from a light source, surfaces, vibrations, ...)

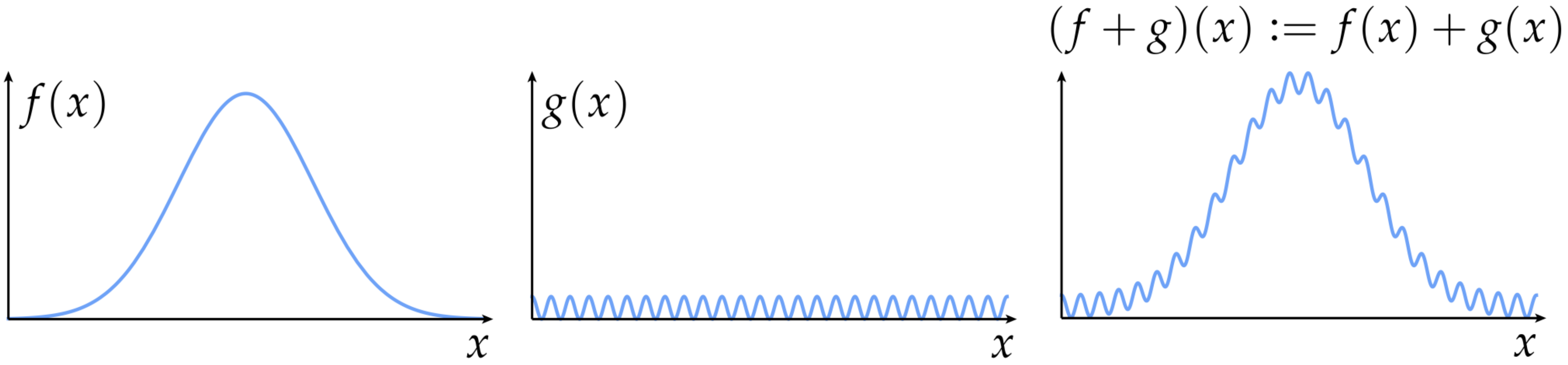




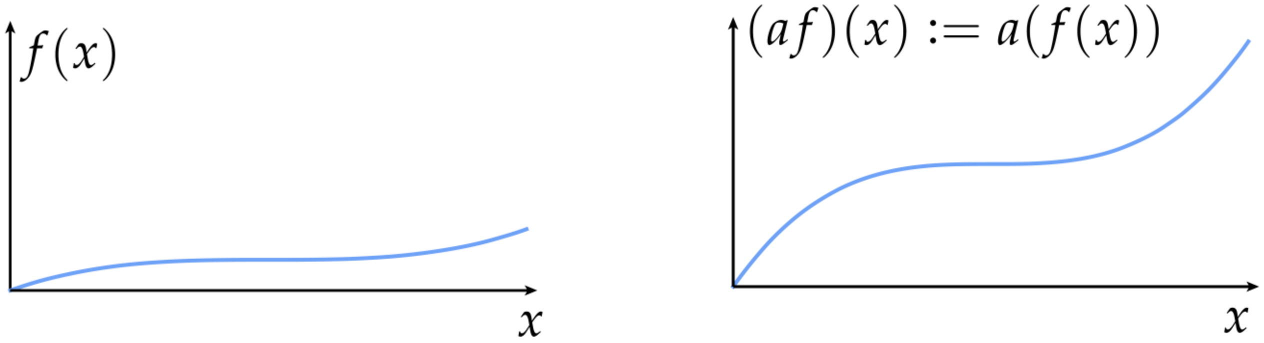


Vectors: *Functions*

- E.g. Adding two functions



- Scalar multiplication in one function

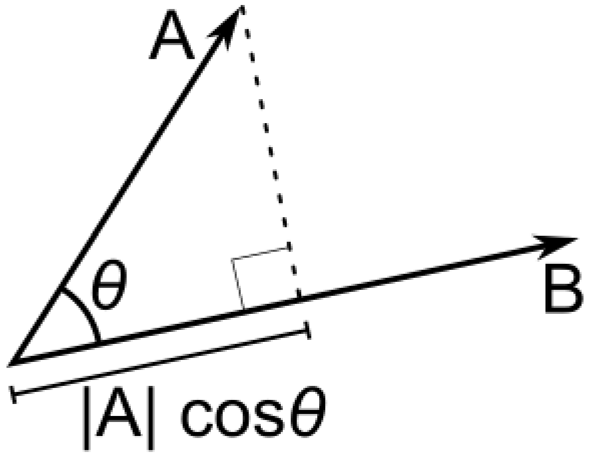


Vectors: *Inner product (dot product)*

- $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \sum_{i=1}^{n} a_i b_i$

- $\boldsymbol{a}.\boldsymbol{b} = a_x b_x + a_y b_y + a_z b_z$

- $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| cos\theta$
  - $\boldsymbol{cos\theta} = \frac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$

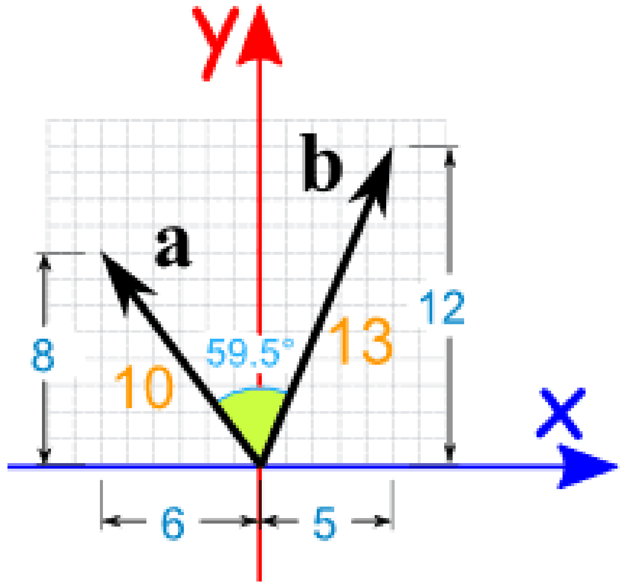


- What if the vectors were normalized?
- What if the inner product was == 0 or == 1;

- **The result is a simple value, not a vector!**

Vectors: *Inner product (dot product)*

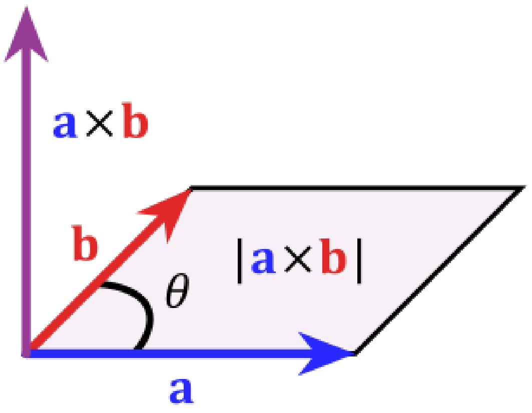
- **Example (A method)**
  - $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \times \cos(\theta)$
  - $\mathbf{a} \cdot \mathbf{b} = 10 \times 13 \times \cos(59.5°)$
  - $\mathbf{a} \cdot \mathbf{b} = 10 \times 13 \times 0.5075...$
  - $\mathbf{a} \cdot \mathbf{b} = 66$



- **Example (B method)**
  - $\mathbf{a} \cdot \mathbf{b} = a_x \times b_x + a_y \times b_y$
  - $\mathbf{a} \cdot \mathbf{b} = \mathbf{-6} \times 5 + 8 \times 12$
  - $\mathbf{a} \cdot \mathbf{b} = -30 + 96$
  - $\mathbf{a} \cdot \mathbf{b} = 66$

Vectors: *Outer product (cross product)*

- The result is not a scalar number, but **a vector perpendicular to the plane of the others 2.**
- We find the vector using the determinant.

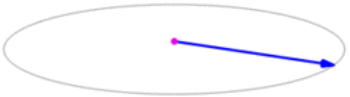


$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin(\theta) \, \mathbf{n}$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ u_1 & u_2 & u_3 \\ \upsilon_1 & \upsilon_2 & \upsilon_3 \end{vmatrix} \quad \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ u_1 & u_2 & u_3 \\ \upsilon_1 & \upsilon_2 & \upsilon_3 \end{vmatrix} \quad \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ u_1 & u_2 & u_3 \\ \upsilon_1 & \upsilon_2 & \upsilon_3 \end{vmatrix}$$

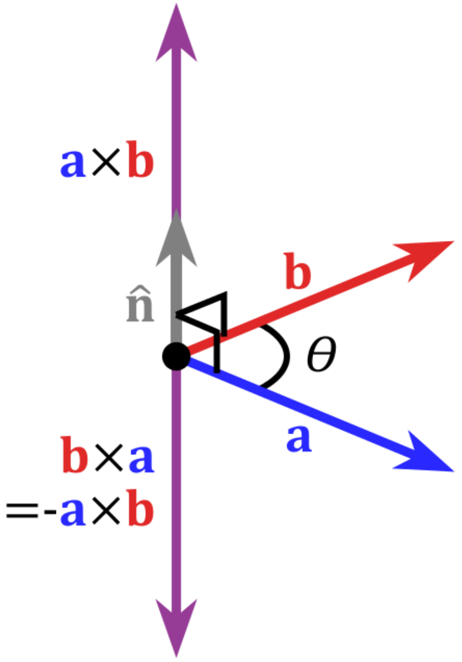
$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

$$= (u_2 v_3 - u_3 v_2)\mathbf{i} - (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}$$



Vectors: *Outer product (cross product)*

- The size of the vector is equal to the area of the rectangle
  - |**a** x **b**| = |**a**||**b**|sinθ
- And its direction is perpendicular to the 2 vectors
  - **… But in which direction?**
  - Use the Right-Handed Reference System!
- The outer product of a vector with itself (or in parallel vectors e.g. angle ϑ between them is 0° or 180°) is the zero vector.

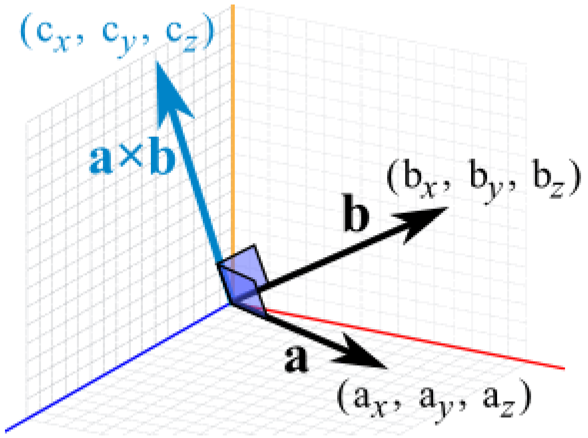


Vectors: *Outer product (cross product)*

Example 1

The outer product of **a** = (2,3,4) and **b** = (5,6,7)

- $c_x = a_y b_z - a_z b_y = 3\times7 - 4\times6 = -3$
- $c_y = a_z b_x - a_x b_z = 4\times5 - 2\times7 = 6$
- $c_z = a_x b_y - a_y b_x = 2\times6 - 3\times5 = -3$



- Consequently: **a × b** = (−3,6,−3)

Vectors: *Outer product (cross product)*

Example 2

Calculate the outer product of vectors $\mathbf{a} = (3, -3, 1)$ and $\mathbf{b} = (4, 9, 2)$.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -3 & 1 \\ 4 & 9 & 2 \end{vmatrix}$$

$$= \mathbf{i}(-3 \cdot 2 - 1 \cdot 9) - \mathbf{j}(3 \cdot 2 - 1 \cdot 4) + \mathbf{k}(3 \cdot 9 + 3 \cdot 4)$$

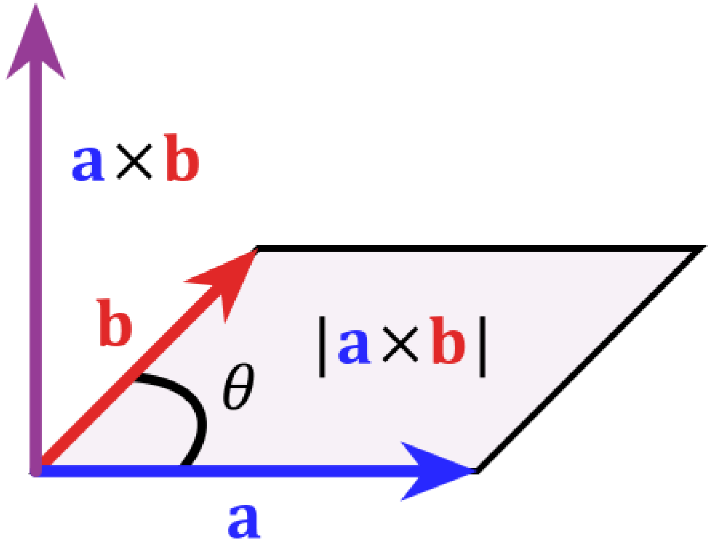
$$= -15\mathbf{i} - 2\mathbf{j} + 39\mathbf{k}$$

Vectors: *Outer product (cross product)*

Example 3

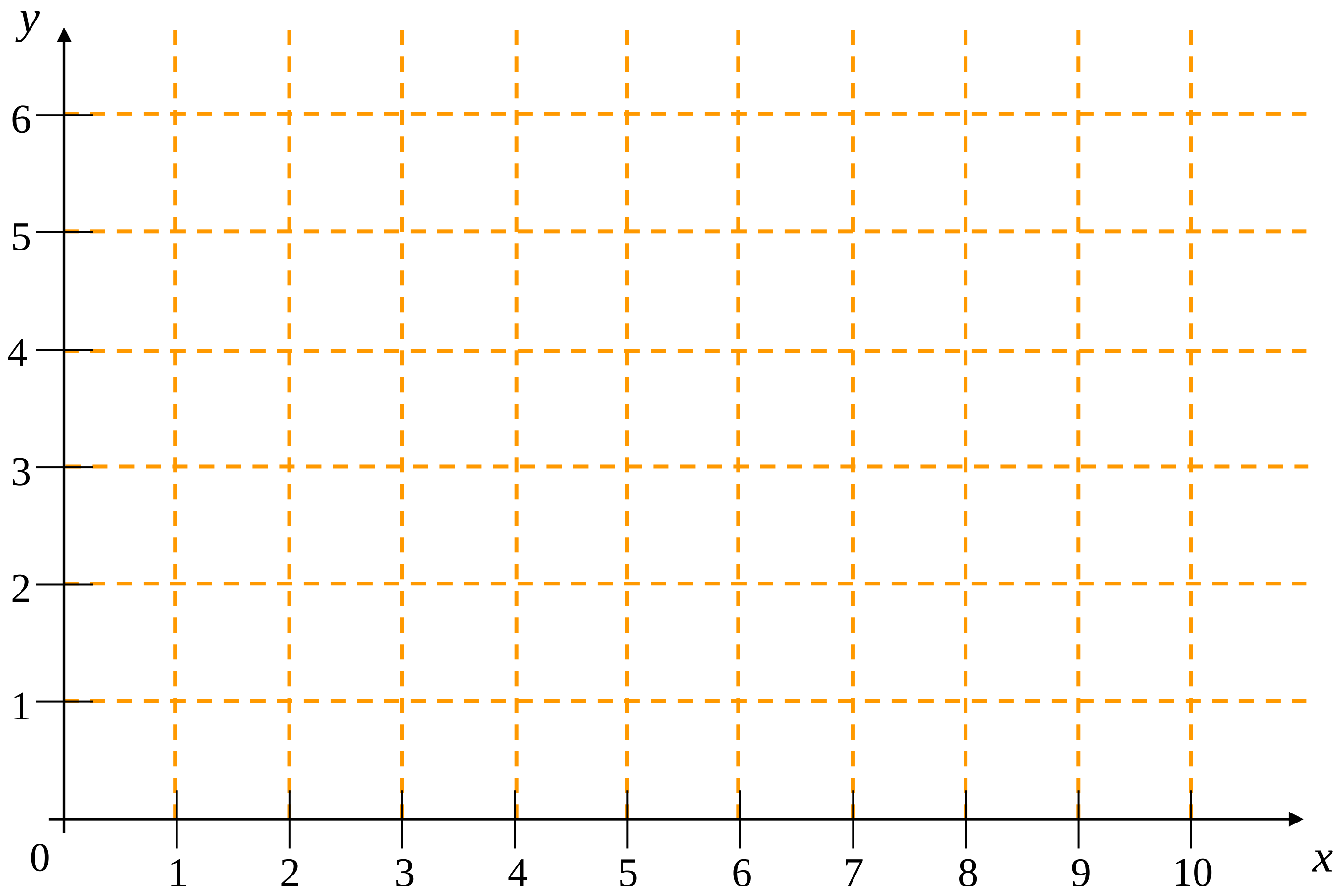
- Calculate the area of the rectangle you create from the vectors **a**=(3,−3,1) and **b**=(4,9,2).

- The area equals ‖**a**×**b**‖ = $\sqrt{15^2+2^2+39^2}$ = $5\sqrt{70}$.



Exercise 1

- Draw the straight line $y = \frac{1}{2}x + 2$ from the point $x = 1$ to $x = 9$

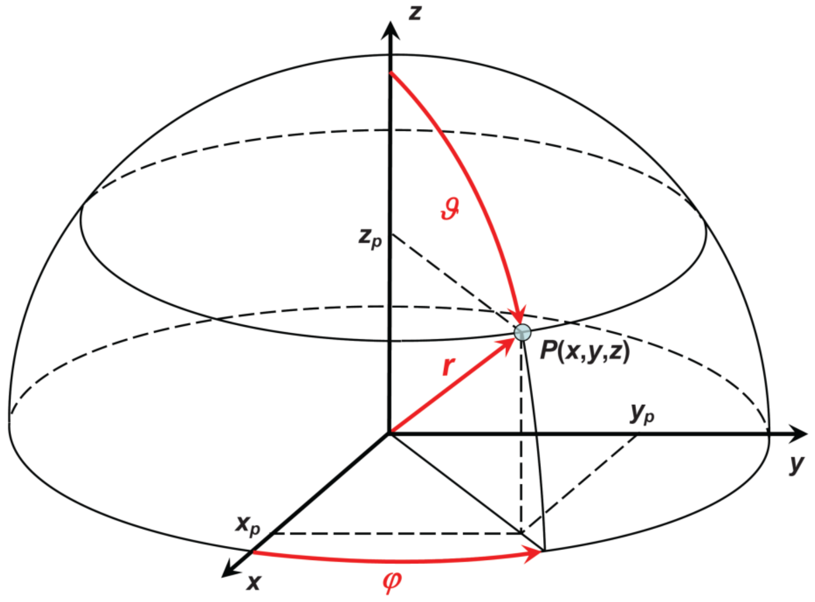


Polar Co-ordinates

- A vector V (x, y, z) can be expressed at spherical coordinates with 3 values: length and two angles ϑ, φ.

- From Cartesian to spherical:
- $r = \sqrt{(x^2+y^2+z^2)}$ ($\sqrt{}$ is the square root)
  - $\vartheta = \arccos(z/r)$
  - $\varphi = \text{atan}(y/x)$

- From spherical to Cartesian :
  - $x = r \sin\vartheta \cos\varphi$
  - $y = r \sin\vartheta \sin\varphi$
  - $z = r \cos\vartheta$

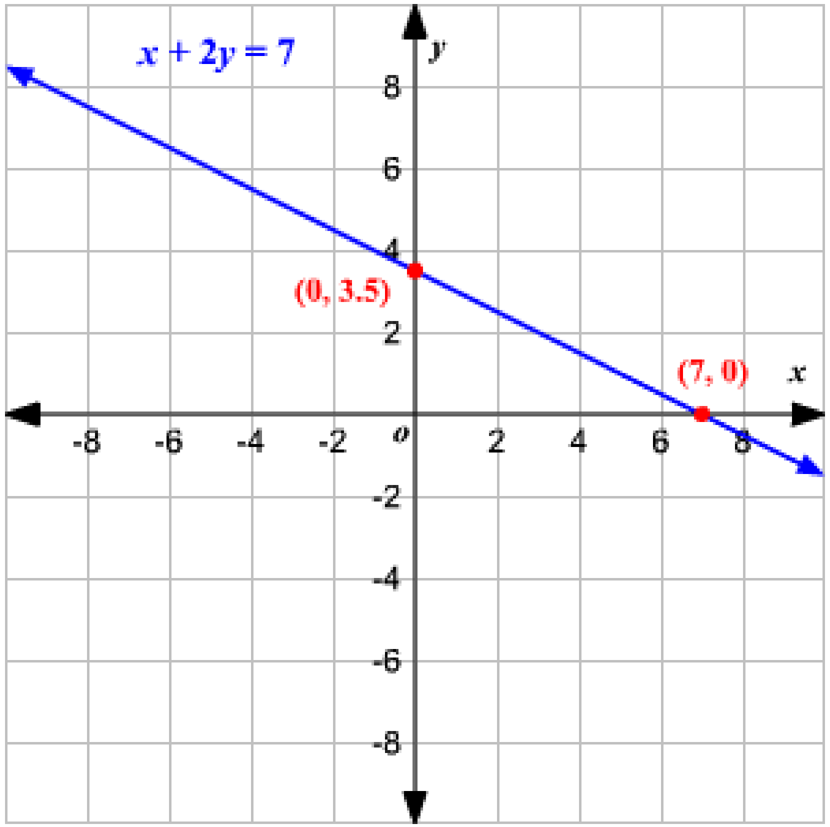


Parametric equation of straight line – radius

- Suppose we have two points $P_0 = (x_0, y_0, z_0)$ and $P_1 = (x_1, y_1, z_1)$, we can express the line that joins them as follows:

$$P(t) = P_0 + t(P_1 - P_0) = \begin{cases} x(t) = x_0 + t(x_1 - x_0) \\ y(t) = y_0 + t(y_1 - y_0) \\ z(t) = z_0 + t(z_1 - z_0) \end{cases}$$

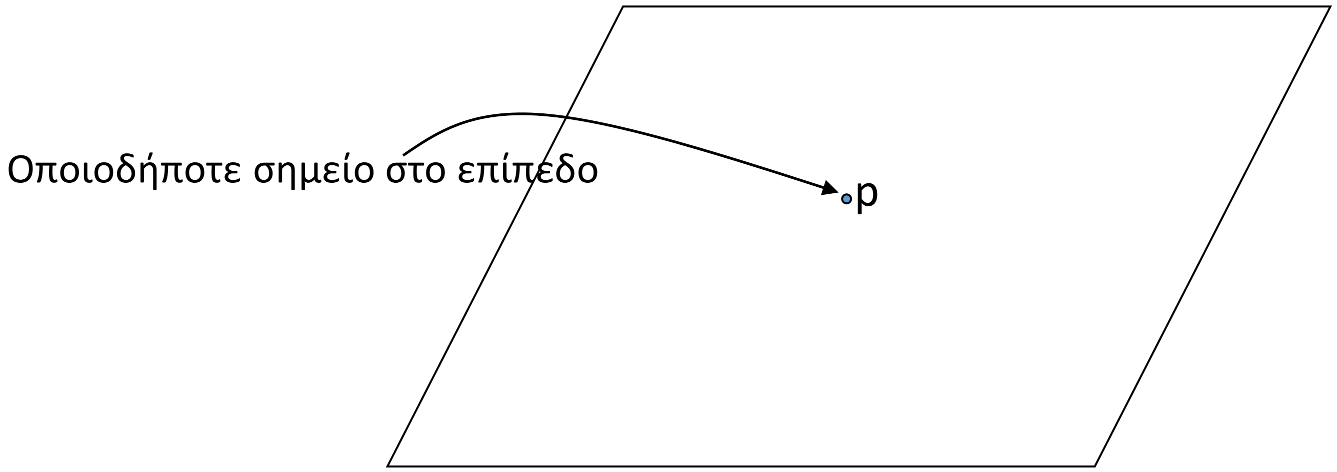
where $-\infty < t < \infty$



Equation of a plane

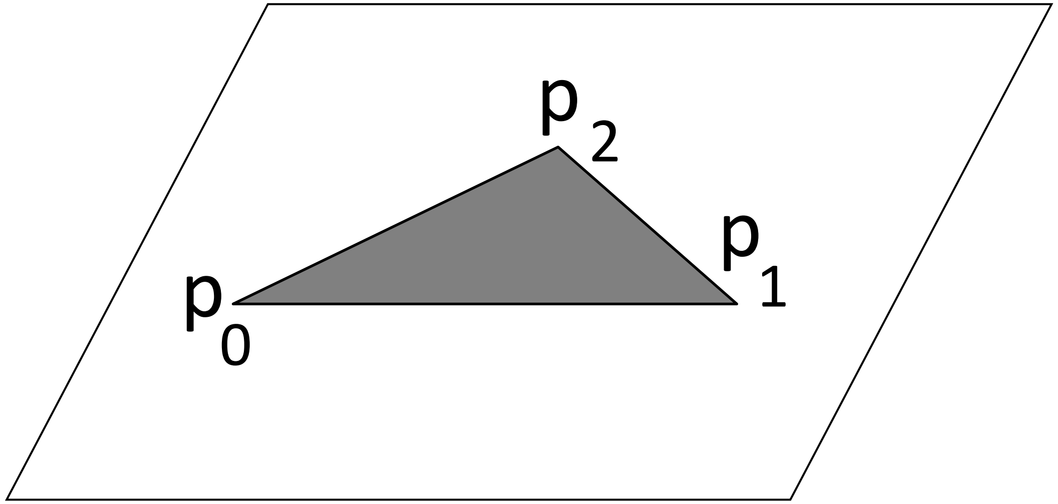
- Even a, b, c and d are constants that define a unique plane in space.
- A, b and c give us the slash in the plane.
- Some point p (x, y, z) you find in the layer if and only if it satisfies the equation.
-

$$ax + by + cz + d = 0$$



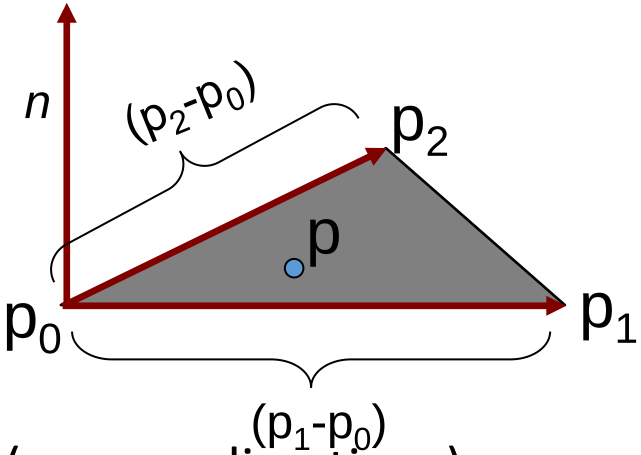
Equation of a plane

- If we have 3 points we can calculate the equation of the plane:
- We create 2 vectors and find the outer product, this gives us the "(a, b, c)"
  - We replace any of the 3 points in the equation $ax + by + cz + d = 0$ and gives us the d.



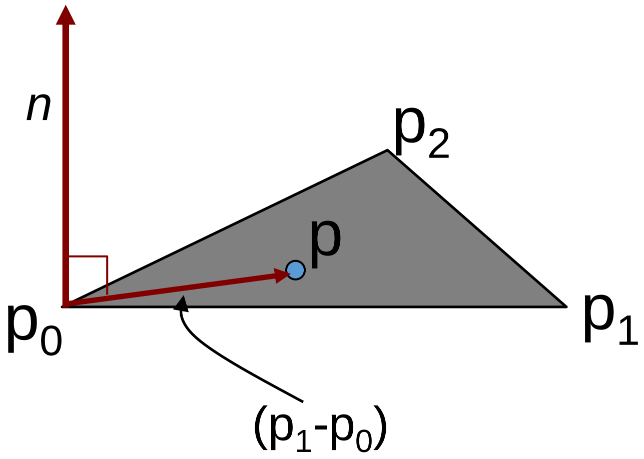
How to find the a,b,c & d

- The outer product defines the perpendicular to the plane with n = (a, b, c)
-



$$n = (p_1 - p_0) \times (p_2 - p_0)$$

- We have 2 perpendiculars (reverse directions)
- The vectors in the plane are perpendicular to the perpendicular
- From ax + by + cz + d = 0
  - ⇨ d = - (ax + by + cz) if we replace the p_0
  - ⇨ d = n.p_0 = $-(n_1 * x_0 + n_2 * y_0 + n_3 * z_0)$



How to find the a,b,c & d

Example:

- Consider the points P = (1, 1, 1), Q = (1, 2, 0), and R = (-1, 2, 1). We look for the constants of the equation ax + by + cz = d, where P, Q and R satisfy the equation:
- $$a + b + c = d$$
  $$a + 2b + 0c = d$$
  $$-a + 2b + c = d$$

  By subtracting the first equation from the second, and then adding the first equation to the third, we eliminate a.

  $$b - c = 0$$
  $$4b + c = 2d$$
- By adding the equations we have 5b = 2d, or b = (2/5)d, so solving as to c = b = (2/5)d, and a = d - b - c = (1/5)d.
- So the equation is $x + 2y + 2z = 5$

Half-Space

- One plane divides the space into 2 half-spaces
- Let's define:

$$l(x, y, z) = ax + by + cz + d$$

- if l(p) =0
  - the point p is in the plane
  - if l(p) > 0
  - The p-point is in the positive half-space
- if l(p) <0
  - the p-point is in the negative half-space

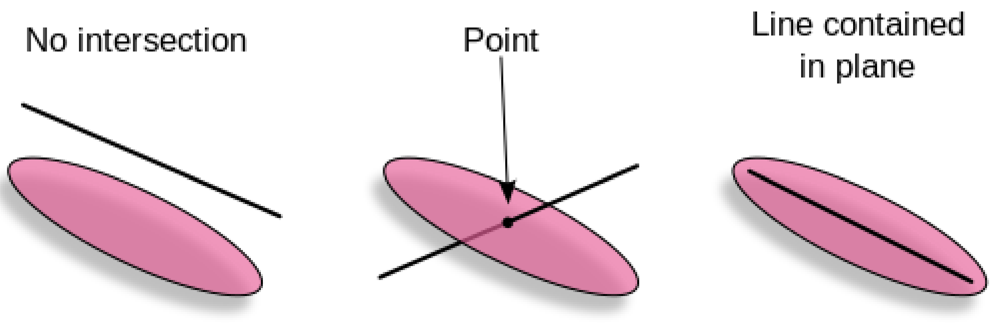
Intersection of line-plane

- Straight line:

$$P(t) = P_0 + t(P_1 - P_0) = \begin{cases} x(t) = x_0 + t(x_1 - x_0) \\ y(t) = y_0 + t(y_1 - y_0) \\ z(t) = z_0 + t(z_1 - z_0) \end{cases}$$

- Plane: ax + by + cz + d = 0

- We replace x, y, z in the equation of the layer and solve against t



Intersection of line-plane

Example:

- Let the plane be: $2x + y - 4z = 4$
- Let the straight line be

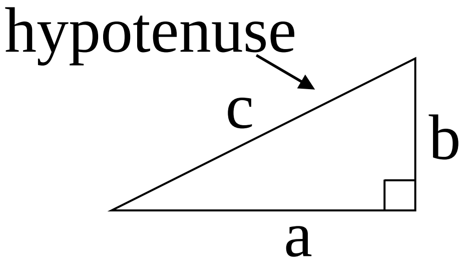
$$P(t) = P_0 + t(P_1 - P_0) = \begin{cases} x(t) = t \\ y(t) = 2 + 3t \\ z(t) = t \end{cases}$$

- We replace x(t), y(t), z(t) in the equation of the plane and solve against t
- $2t + (2+3t) - 4t = 4 \quad \rightarrow \quad t=2$
  - So for t=2, the intersection point is the (2,8,2)

Circle (2D)

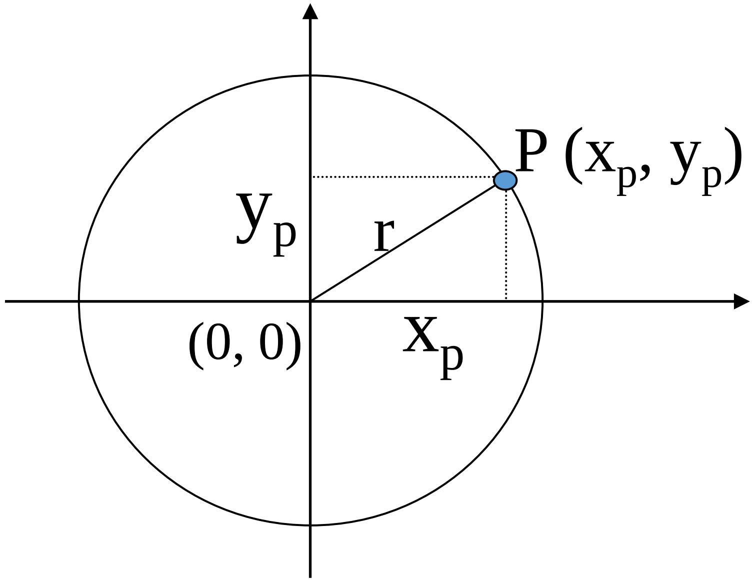
- The Pythagorean theorem:

$$a^2 + b^2 = c^2$$



- If we have a circle at the origin of the axes, with a radius r, then for each point P in it, we have:

$$x_p^2 + y_p^2 = r^2$$



Circle (2D)

- If the circle is not set at the origin of the system (0, 0), then again we have:

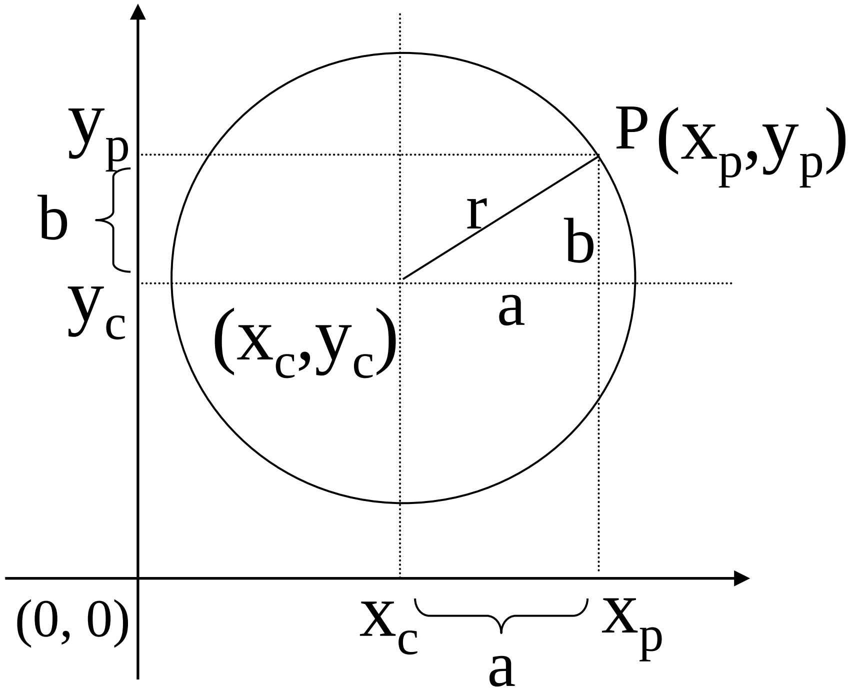
but

$$a^2 + b^2 = r^2$$

So the general form becomes:

$$a = x_p - x_c$$
$$b = y_p - y_c$$

$$(x - x_c)^2 + (y - y_c)^2 = r^2$$



Sphere (3D)

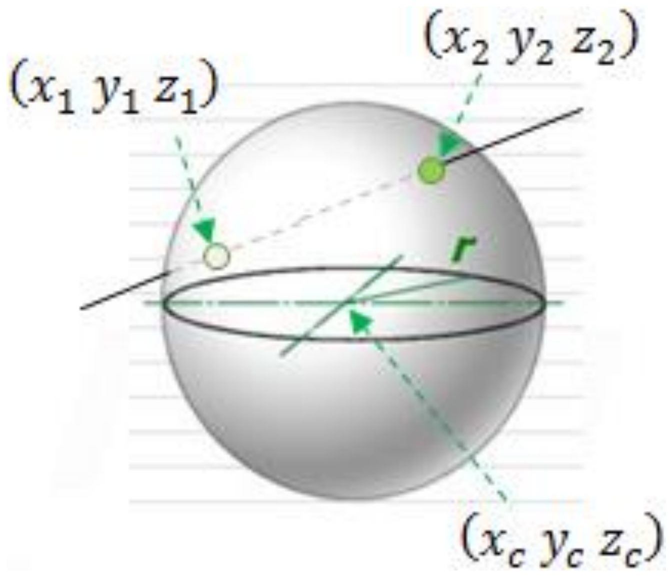
- The Pythagorean theorem is generalized in the 3D by giving $a^2 + b^2 + c^2 = d^2$. Based on this we can easily prove that the equation of the sphere is:
-

$$(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 = r^2$$

- And in the (0,0,0):

$$x^2 + y^2 + z^2 = r^2$$

Intersection sphere with line



$$(x-x_c)^2 + (y-y_c)^2 + (z-z_c)^2 = r^2$$ sphere

Straight line
$$\begin{cases} x(t) = x_1 + t(x_2 - x_1) \\ y(t) = y_1 + t(y_2 - y_1) \\ z(t) = z_1 + t(z_2 - z_1) \end{cases}$$

By replacing the values x, y and z in the equation of the sphere, we will have an equation of the form: $at^2 + bt + c = 0$

$$t = -b \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Case for intersection: $b^2 - 4ac > 0$

Case when tangential: $b^2 - 4ac = 0$

There is no intersection when: $b^2 - 4ac < 0$

PART B - Matrices

Vectors and Matrices

- Matrix is a set of values arranged in M rows with N columns

$$\begin{bmatrix} 1 & 11 & 13 \\ 10 & 4 & -3 \\ 2 & 0 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 4.3 \\ 6.7 \\ 1.2 \end{bmatrix} \quad \begin{bmatrix} 4 & 8 & 15 \\ 16 & 23 & 42 \end{bmatrix}$$

- Example
  - Matrix 3×6
  - The element *2,3* is the *7*
  - We will see only 2-dimensional matrices

$$\begin{pmatrix} 3 & 0 & 0 & -2 & 1 & -2 \\ 1 & 1 & 7 & 4 & 1 & -1 \\ -5 & 2 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- A vector can be thought of as a 1×M matrix

$$v = \begin{pmatrix} x & y & z \end{pmatrix}$$

Types of matrices

- Identity

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Diagonal

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

- Symmetric

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

  - The diagonal matrices are (of course) symmetrical
  - The identity matrices are (of course) diagonal

Operations in matrices: *Scalar multiplications*

- To multiply the elements of an array by a constant simply multiply each of its elements

$$s*\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} s*a & s*b & s*c \\ s*d & s*e & s*f \\ s*g & s*h & s*i \end{bmatrix}$$

- Example:

$$3*\begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix} = \begin{bmatrix} 6 & 12 & 18 \\ 24 & 30 & 36 \\ 42 & 48 & 54 \end{bmatrix}$$

Operations in matrices: *Addition*

- To add two tables simply add their individual elements

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix} = \begin{bmatrix} a+r & b+s & c+t \\ d+u & e+v & f+w \\ g+x & h+y & i+z \end{bmatrix}$$

- Example:

$$\begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix} + \begin{bmatrix} 3 & 5 & 7 \\ 9 & 11 & 13 \\ 15 & 17 & 19 \end{bmatrix} = \begin{bmatrix} 5 & 9 & 13 \\ 17 & 21 & 25 \\ 29 & 33 & 37 \end{bmatrix}$$

Both matrices must be the same size

Operations in matrices: *Πολλαπλασιασμός πινάκων*

- If we have two matrices with dimensions $N_1 x M_1$ and $N_2 x M_2$ then the multiplication can be done if and only if $M_1 = N_2$

- the result matrix will be N_1 x M_2

- e.g. Matrix A is 2 x 3 and table B 3 x 4
- the result matrix will be 2 x 4

- **Because A x B is possible does not mean that B x A is possible!**
- Attention! The multiplication of tables is not transitional, so:

$$AB \neq BA$$

Operations in matrices: *Multiplication of matrices*

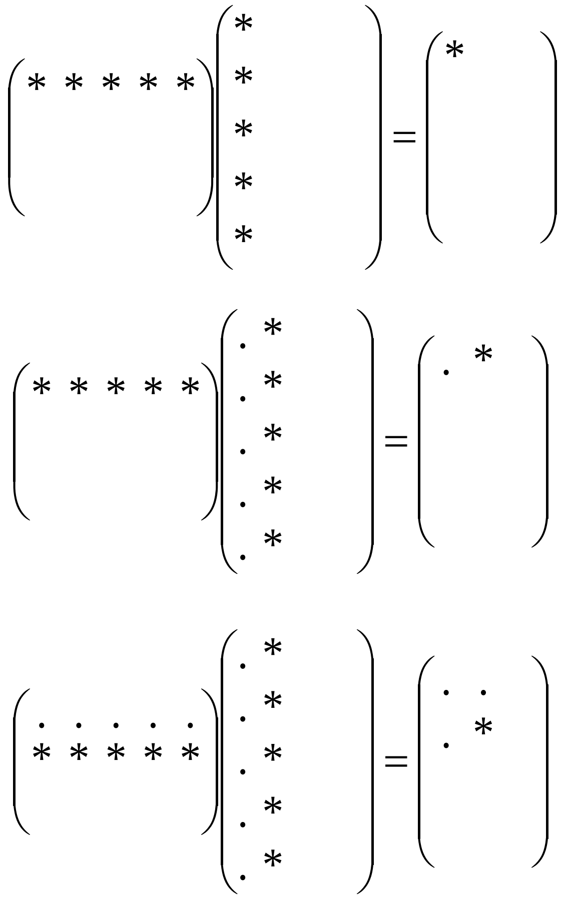
Suppose that

- A is n x k
- B is k x m

Then

- C = A x B is defined by

$$c_{ij} = \sum_{l=1}^{k} a_{il} b_{lj}$$



Attention

- B x A not necessarily equal to A x B

Operations in matrices: *Multiplication of matrices*

- Example:

$$\begin{bmatrix} 0 & -1 \\ 5 & 7 \\ -2 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0*1+(-1)*3 & 0*2+(-1)*4 \\ 5*1+7*3 & 5*2+7*4 \\ -2*1+8*3 & -2*2+8*4 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 26 & 38 \\ 22 & 28 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1*4+2*5+3*6 \end{bmatrix} = \begin{bmatrix} 32 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4*1 & 4*2 & 4*3 \\ 5*1 & 5*2 & 5*3 \\ 6*1 & 6*2 & 6*3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{bmatrix}$$

Operations in matrices: *Transpose Matrix*

- The transpose matrix M, is expressed by M^{T} and is achieved simply by changing the rows and columns in the table
- For instance:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 9 \\ 5 & 2 & 8 \\ 6 & 7 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 5 & 6 \\ 4 & 2 & 7 \\ 9 & 8 & 3 \end{bmatrix}$$

Operations in matrices: *Inverse Matrix*

- If A x B = I and B x A = I then
  $A = B^{-1}$ και $B = A^{-1}$

- $A A^{-1} = A^{-1} A = I$

- $A^{-1} = \frac{1}{\det(A)}(\text{cofactor matrix of A})^T$

Operations in matrices: *Inverse Matrix*

- Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$$

$$A_{11} = \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 24 \qquad A_{12} = -\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = 5 \qquad A_{13} = \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} = -4$$

$$A_{21} = -\begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} = -12 \qquad A_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3 \qquad A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$$

$$A_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2 \qquad A_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = -5 \qquad A_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4$$

cofactor of A $= \begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}$ (cofactor of A)^T $= \begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}$

Operations in matrices: *Inverse Matrix*

- Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$$

- $\det(A) = 1 \times 4 \times 6 + 2 \times 5 \times 1 + 3 \times 0 \times 0 - 3 \times 4 \times 1 - 5 \times 0 \times 1 - 6 \times 2 \times 0 = 24 + 10 - 12 = 22$

$$A = \frac{1}{22}\begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 24/22 & -12/22 & -2/22 \\ 5/22 & 3/22 & -5/22 \\ -4/22 & 2/22 & 4/22 \end{bmatrix}$$

Operations in matrices: *Exercises*

- Run the following additions to a matrix:
-

$$\begin{bmatrix} -11 & -19 & -15 & 5 \end{bmatrix} + \begin{bmatrix} -1 & -14 & -5 & 1 \end{bmatrix} = \begin{bmatrix} \_\_ & \_\_ & \_\_ & \_\_ \end{bmatrix}$$

$$\begin{bmatrix} 3 & 20 & -11 & 0 \\ -14 & 5 & -3 & 6 \\ 15 & 2 & 9 & -18 \\ -19 & 4 & -15 & 10 \end{bmatrix} + \begin{bmatrix} 16 & 10 & -12 & -11 \\ 10 & -15 & 15 & 5 \\ -1 & 14 & -9 & 0 \\ 3 & -3 & -16 & -5 \end{bmatrix} = \begin{bmatrix} \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ \end{bmatrix}$$

$$\begin{bmatrix} 13 & -3 & -4 \\ 19 & 9 & 8 \end{bmatrix} + \begin{bmatrix} -1 & -15 & -7 \\ -14 & 5 & 17 \end{bmatrix} = \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix}$$

Operations in matrices: *Exercises*

- Perform the following multiplications in a matrix:

$$\begin{bmatrix} 8 & 15 & 19 & 4 \\ 7 & -4 & 12 & 3 \end{bmatrix} * \begin{bmatrix} -15 & 19 \\ -12 & -19 \\ 0 & -13 \\ 10 & 7 \end{bmatrix} = \begin{bmatrix} \_\_\_\_\_\_\_\_ & \_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_ & \_\_\_\_\_\_\_\_ \end{bmatrix} = \begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix}$$

$$\begin{bmatrix} 16 & 10 & -12 & -11 \\ 10 & -15 & 15 & 5 \\ -1 & 14 & -9 & 0 \\ 3 & -3 & -16 & -5 \end{bmatrix} * \begin{bmatrix} 4 \\ 11 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} \_\_\_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_\_\_\_\_ \end{bmatrix} = \begin{bmatrix} \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \end{bmatrix}$$

Operations in matrices: *Exercises*

- Perform the following scalar multiplications in a matrix:
- $$6*\begin{bmatrix} 15 & 19 \\ 2 & 5 \\ 0 & -1 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \\ \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix}$$

- Compute the transpose matrix

$$\begin{bmatrix} 3 & 11 \\ 6 & 4 \\ 23 & 7 \end{bmatrix}^T = \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix}$$