Nash Equilibria for Voronoi Games on Transitive Graphs*

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Abstract

In a *Voronoi game,* each of $\kappa \geq 2$ *players* chooses a vertex in a graph $\mathsf{G} = \langle \mathsf{V}(\mathsf{G}), \mathsf{E}(\mathsf{G}) \rangle$. The *utility* of a player measures her *Voronoi cell*: the set of vertices that are closest to her chosen vertex than to that of another player; each vertex contributes uniformly to the utilities of players whose Voronoi cells the vertex belongs to. In a *Nash equilibrium,* unilateral deviation of a player to another vertex is not profitable;. so, the existence of a Nash equilibrium is determined from the cardinalities of Voronoi cells. We focus on various, symmetry-possessing classes of *transitive graphs*: the *vertex-transitive* and *generously vertex-transitive graphs,* and the more restricted class of *friendly graphs* we introduce; the latter encompasses as special cases the popular *d-dimensional bipartite torus* $\mathsf{T}_d = \mathsf{T}_d(2p_1, \ldots, 2p_d)$ with even *sides* $2p_1, \ldots, 2p_d$ and *dimension* $d \geq 2$ (the *d-dimensional hypercube* H_d being a special case), and a subclass of the *Johnson graphs.*

How easily would transitivity enable bypassing the explicit enumeration of Voronoi cells? To argue in favor, we resort to a technique using *automorphisms,* which suffices alone for generously vertex-transitive graphs with $\kappa = 2$.

To go beyond the case $\kappa = 2$, we show the (perhaps surprising) *Two-Guards Theorem for Friendly Graphs*: whenever two of the three players are located at an *antipodal* pair of vertices in a friendly graph G, the third player receives a utility of $\frac{|\mathsf{V}(\mathsf{G})|}{4} + \frac{|\Omega|}{12}$, where Ω is the intersection of the three Voronoi cells. If the friendly graph G is *bipartite* and has odd *diameter,* the utility of the third player is fixed to $\frac{|\mathsf{V}(\mathsf{G})|}{4}$; this allows discarding the third player when establishing that such a triple of locations is a Nash equilibrium. Combined with appropriate automorphisms *and without explicit enumeration,* the *Two-Guards Theorem* implies the existence of a Nash equilibrium for *any* friendly graph G with $\kappa = 4$, with colocation of players allowed; if colocation is forbidden, existence still holds under the additional assumption that G is bipartite and has odd diameter.

For the case where $\kappa = 3$, *we have been unable to bypass the explicit enumeration of Voronoi cells.* Combined with appropriate automorphisms and *explicit enumeration,* the *Two-Guards Theorem* implies the existence of a Nash equilibrium for *(i)* the 2-dimensional torus T_2 with odd diameter $\sum_{j \in [2]} p_j$ and $\kappa = 3$, and *(ii)* the hypercube H_d with odd d and $\kappa = 3$. In conclusion, *transitivity does not seem sufficient for bypassing explicit enumeration:* far-reaching challenges in combinatorial enumeration are in sight, even for values of κ as small as 3.

1 Introduction

$\boxed{\textbf{The Voronoi Game.}}$ Recently, there has been a considerable amount of research dealing with *non-cooperative games* on *networks,* inspired from diverse application domains from computer and communication networks, such as resource allocation, routing, scheduling, caching, multicasting and facility location. In this work, we shall extend the study of the *Nash equilibria* [17, 18] associated with

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a particular game inspired from facility location and called the *Voronoi game*; it was introduced by Dürr and Thang in [4] and further studied in [16]. We shall ony consider *pure* Nash equilibria.

The Voronoi game [4, 16] is reminiscent of the classical *Hotelling games* [12], where there is a number of *vendors* in some continuous *metric space*. Each vendor comes with goods for sale; simultaneously with other vendors, she must choose a *location* for her *facility* (in the metric space). The objective for each vendor is to maximize the region of points that are closest to her than to any other vendor, called her *Voronoi cell*. For example, consider a number of *ice-cream vendors* and *tourists* on a beach, modeled as a straight-line segment. Assuming that each tourist buys ice-cream from the closest vendor, each vendor seeks a location on the beach attracting the maximum number of tourists. In a *Nash equilibrium* [17, 18], no vendor can increase her profit by switching to a different point. (In the example with *two* ice-cream vendors, there is a *unique* Nash equilibrium where both vendors are located at the middle of the segment; more generally, there is a Nash equilibrium if and only if the number of vendors is even, and it is then unique.) Hotelling games (and extensions of them incorporating *prices*) have been studied extensively in Economics Theory; see, e.g., the surveys in [6, 13].

The Voronoi game is a discrete analog of Hotelling games, where an undirected graph $\mathsf{G} = \langle \mathsf{V}(\mathsf{G}), \mathsf{E}(\mathsf{G})\rangle$ is used instead of a metric space. There are κ *players,* each choosing a vertex; they may be thought of as *Internet providers* located at the nodes of some network with *customers*. A player's *utility* measures her *Voronoi cell*: the set of vertices closest to her than to another player; so, it reflects the number of customers attaching themselves to their *closest* Internet provider. A *boundary vertex* is closest to more than one player; it contributes uniformly to the utilities of its closest players. A significant difference between Voronoi games and Hotelling games [12] is that the boundary vertices in a Voronoi game need to be taken into account, while the boundary points in a Hotelling game have measure zero and can be discarded. The Voronoi game is represented as the pair $\langle \mathsf{G}, [\kappa]\rangle$.

In a *Nash equilibrium* [17, 18], no player can unilaterally increase her utility by switching to another vertex; loosely speaking, her Voronoi cell does not increase when she deviates. The Voronoi game distinguishes itself among the non-cooperative games on networks considered so far within *Algorithmic Game Theory* in that *the existence of Nash equilibria is contingent upon enumeration properties of sets of vertices* — the (*ex*) Voronoi cells associated with a given collection of locations for the players, and the (*post*) Voronoi cell resulting from a player's deviation. Consequently, *explicit enumeration of Voronoi cells manifests itself as a combinatorial bottleneck to identifying Nash equilibria, even if the locations of the players are given. Is this bottleneck always inherent?*

Previous Work. Dürr and Thang [4, Section 4] proved that it is $\mathcal{NP}$-complete to decide the existence of a Nash equilibrium for an *arbitrary* Voronoi game $\langle \mathsf{G}, [\kappa]\rangle$. (For a *constant* κ, the decision problem is in $\mathcal{P}$ through exhaustively checking all collections of locations for the κ players on a given graph G.) Further, Dürr and Thang [4, Section 4] presented a simple counterexample of a (*not* vertex-transitive) graph with no Nash equilibrium for $\kappa = 2$. Subsequent work by Mavronicolas *et al.* [16] provided combinatorial characterizations of Nash equilibria for *rings*, determining the ring size allowing for a Nash equilibrium. Still for rings, Dürr and Thang [4] and Mavronicolas *et al.* [16] presented bounds on the induced *Social Cost Discrepancy* and *Price of Anarchy* [14], respectively.

Motivated by (competitive) facility location as well, several works [1, 3, 5] have considered *repeated* Voronoi games in continuous (*geometric*) domains, where *two* players alternate in choosing a number of points from the domain; at the last round, the player with the largest Voronoi cell wins. Those works did *not* consider the associated Nash equilibria. Neither did an earlier work by Teramoro *et al.* [19] on corresponding (repeated) Voronoi games on graphs. Zhao *et al.* [20] proposed recently the *isolation game* on an arbitrary *metric space* as a generalization of the Voronoi game where each player has now objectives other than maximizing the size of her Voronoi cell; for example, players may seek to be away from each other as much as possible. Several results on the Nash equilibria associated with isolation games were shown in [20].

Motivation, Framework and Techniques. Here is our motivation in two sentences: *How easily would transitivity enable bypassing the explicit enumeration of Voronoi cells? What are the broadest classes of (transitive) graphs for which transitivity would so succeed for a given number of players?*

To materialize our motivation, we embark on the broad class of *vertex-transitive* graphs, which enjoy a rich structure; roughly speaking, a vertex-transitive graph "looks" the same from each vertex. (The ring considered in [4, 16] is vertex-transitive.) However, we shall focus on restricted classes of vertex-transitive graphs. To start with, in a *generously vertex-transitive graph,* an arbitrary pair of vertices can be swapped (cf. [8, Section 12.1] or [11, Section 4.3]). Although there are good examples of vertex-transitive graphs that are *not* generously vertex-transitive (e.g., the *cube-connected-cycles* [15, Section 3.2.1]), the class of generously vertex-transitive graphs includes sufficiently rich subclasses; one such is that of *friendly graphs.* We define a *friendly graph* as a generously vertex-transitive graph where, in addition, every vertex is on some shortest path between an *antipodal* pair of vertices (Definition 2.1). Our prime example of a friendly graph is the *d-dimensional, bipartite torus* T_d, which encompasses the d-dimensional hypercube H_d as a special case (Lemma 2.1). Yet, we identify a special subclass of the *Johnson graphs* [9, Section 1.6] as another example of a friendly graph (Lemma 2.2).*

In this endeavor, we seek to exploit the algebraic and combinatorial structure of friendly graphs in devising techniques to *compare the cardinalities of the (ex and post) Voronoi cells without explicitly enumerating them*; such techniques will allow *establishing the existence of Nash equilibria by bypassing explicit enumeration.* This idea is naturally inspired from the technique of *bijective proofs* in Combinatorics (see, for example, [7, Section 2] and references therein), which shows that two (finite) sets have the same cardinality by providing a bijection between them. In particular, we shall resort to *automorphisms* of friendly graphs.

Contribution and Significance. *Resorting to automorphisms suffices to settle the case of generously vertex-transitive graphs with* $\kappa = 2$. Specifically, we prove that *every* location for the two players yields a Nash equilibrium for a generously vertex-transitive graph (Proposition 4.1). Unfortunately, this simple idea may not extend beyond generously vertex-transitive graphs in a general way: we prove that some particular vertex-transitive but not generously vertex-transitive graph, namely the cube-connected-cycles, has no Nash equilibrium for $\kappa = 2$ (Corollary 4.3). This fact follows immediately from a general *necessary condition* we establish for *any* vertex-transitive graph to admit a Nash equilibrium: There is a pair of vertices to locate the two players so that they receive different utilities (Proposition 4.2). This counterexample extends the earlier one of Dürr and Thang [4, Section 4].

We have been unable to go beyond the case $\kappa = 2$ *without assuming some additional structure on the graph* G. Towards this end, we establish the (perhaps surprising) *Two-Guards Theorem for Friendly Graphs* concerning the case $\kappa = 3$ (Theorem 5.2): If two of the players are located at an antipodal pair of vertices in a friendly graph G, the third player receives a utility of $\frac{|\mathsf{V}(\mathsf{G})|}{4} + \frac{|\Omega|}{12}$, where Ω denotes the intersection of the three Voronoi cells. For a *bipartite* friendly graph with *odd* diameter, the *Two-Guards Theorem for Friendly Graphs* has an interesting extension: independently of her location, the third player receives a *fixed* utility of $\frac{|\mathsf{V}(\mathsf{G})|}{4}$ (Corollary 5.3). So, a corresponding paradigm emerges for establishing the existence of a Nash equilibrium: *locate two of players at an antipodal pair and prove that none of them can unilaterally improve.* For this paradigm to succeed, it remains to devise techniques to argue the impossibility of unilateral improvement for either of the antipodal players.

Through this paradigm, we have been able to *bypass explicit enumeration for the case* $\kappa = 4$. Assuming that *colocation of players is allowed,* we establish, through a simple proof, the existence of a Nash equilibrium for *(i)* an *arbitrary* friendly graph with $\kappa = 4$ (Theorem 6.1). However, *forbidding colocation* has required (still for $\kappa = 4$) the additional assumption that *(ii)* the friendly graph is bipartite and has odd diameter (Theorem 6.2); the proof is more challenging and uses suitable automorphisms. The key idea for the proofs of both results has been that when one of the four players deviates, *there still remain two players located at an antipodal pair of vertices*; in turn, this allows applying the *Two-Guards Theorem for Friendly Graphs* and its extension.

For the case $\kappa = 3$, *we have developed techniques for the explicit enumeration of Voronoi cells, which*

*For the record, we were initially interested in considering just tori and hypercubes for their associated Nash equilibria, as they provide some of the most versatile architectures for parallel computation (cf. [15]). However, along the way, we introduced friendly graphs in an effort to isolate out into a separate abstraction a *minimal* set of properties of the torus T_d that sufficed for the specialized proofs of some initial results about tori we had derived; the convenience provided by the resulting abstraction is reflected into the name of *friendly graphs* we chose to adopt. In turn, the abstraction allowed the results for such tori to apply immediately to other examples of friendly graphs we were able to identify (e.g., some Johnson graphs). The resulting proofs for the more general results about friendly graphs have been, in turn, less complex, since they manage to decrystallize the essential proof ingredients in a clean way.

make the most technically challenging part of this work. Employed on top of the paradigm of the *Two-Guards Theorem for Friendly Graphs* (and its extension) and enriched with appropriate automorphisms, these enumeration techniques have enabled settling the existence of a Nash equilirium in the following special cases: *(iii)* The 2-dimensional torus T_2 with odd diameter $\sum_{j\in[2]} p_j$ (Theorem 7.1), and *(iv)* the hypercube H_d with odd d (Theorem 7.9); for the proof of *(iv),* we have derived explicit combinatorial formulas (as nested sums of binomial coefficients) for the utilities of three players located arbitrarily in the hypercube H_d. Although the hypercube H_d is a special case of the torus T_d, these two existence results are *incomparable*: *(iv)* applies to the hypercube H_d with odd d, while *(iii)* applies to the torus T_d with $d = 2$.

To complement the existence results for $\kappa = 3$ in *(iii)* and *(iv)*, we have carried out an extensive set of experiments. The experimental results provide strong evidence that there is *no* Nash equilibrium for the cases of *(v)* the 2-dimensional torus T_2 with *even* diameter $\sum_{j\in[2]} p_j$, and *(vi)* the hypercube H_d with *even* d; so, they suggest that the assumptions made for *(iii)* and *(iv)* are *essential.*

Our results provide some evidence that *transitivity on* G *might be insufficient for bypassing explicit enumeration of Voronoi cells for the Voronoi game* $\langle \mathsf{G}, [\kappa]\rangle$ *with arbitrary* κ; far-reaching challenges in combinatorial enumeration have been demanded even for values of κ as small as 3.

2 Vertex-Transitive Graphs

Preliminaries. We shall consider a simple, connected and undirected ***graph*** $\mathsf{G} = \langle \mathsf{V}(\mathsf{G}), \mathsf{E}(\mathsf{G})\rangle$. A ***path*** in G is a sequence $v_0, v_1, \ldots, v_\ell$ of vertices such that for each index $i \in [\ell]$, $\{v_{i-1}, v_i\} \in \mathsf{E}(\mathsf{G})$; the ***length*** of the path is the number ℓ of its edges. A ***cycle*** is a path $v_0, v_1, \ldots, v_\ell$ with $v_\ell = v_0$. For a pair of vertices $u, v \in \mathsf{V}(\mathsf{G})$, the ***distance*** between u and v, denoted as $\mathsf{dist}_\mathsf{G}(u, v)$ (or just $\mathsf{dist}(u, v)$ when G is clear from context), is the length of the shortest path between u and v. The ***diameter*** of G is given by $\mathsf{diam}(\mathsf{G}) = \max_{u,v\in\mathsf{V}(\mathsf{G})} \mathsf{dist}(u, v)$. Say that the pair of vertices $u, v \in \mathsf{V}(\mathsf{G})$ is ***antipodal*** if $\mathsf{dist}(u, v) = \mathsf{diam}(\mathsf{G})$; so, u (resp., v) is an ***antipode*** to v (resp., u). For a set of vertices $\mathsf{V}' \subseteq \mathsf{V}(\mathsf{G})$, denote $\Omega(\mathsf{V}') = \{u \in \mathsf{V}(\mathsf{G}) \mid$ the distance $\mathsf{dist}(u, v)$ is the same for all vertices $v \in \mathsf{V}'(\mathsf{G})$ $\}$; so, $\Omega(\mathsf{V}')$ is the set of all vertices that have the same distance from each vertex in V'. Note that for a *bipartite* graph G, if the set V' contains two vertices at odd distance from each other, then $\Omega(\mathsf{V}') = \emptyset$. (This is a very useful property of bipartite graphs, which will explain later their prominent role in this work.)

Automorphisms. Two graphs $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ and $\mathsf{G}' = (\mathsf{V}', \mathsf{E}')$ are ***isomorphic*** if there is a bijection $\varphi : \mathsf{V} \to \mathsf{V}'$ such that for each pair of vertices $u, v \in \mathsf{V}$, $\{u, v\} \in \mathsf{E}$ if and only if $\{\varphi(u), \varphi(v)\} \in \mathsf{E}'$; so, φ preserves both edges and non-edges. The bijection φ is called an ***isomorphism*** from G to G'. An ***automorphism*** of G is an isomorphism from G to itself. Note that for an automorphism φ, for each pair of vertices $u, v \in \mathsf{V}(\mathsf{G})$, $\mathsf{dist}_\mathsf{G}(u, v) = \mathsf{dist}_{\mathsf{G}'}(\varphi(u), \varphi(v))$. Denote as ι the *identity automorphism.*

(Generously) Vertex-Transitive Graphs. We continue with some notions of *transitivity*; for a more detailed treatment, we refer the reader to [2, Chapters 15 & 16], [8, Section 12.1], [9, Chapter 3], [10, Section 6.1], or [11, Section 4.3]. Say that the graph G is ***vertex-transitive*** if for each pair of vertices $u, v \in \mathsf{V}$, there is an automorphism ϕ of G such that $\phi(u) = v$; roughly speaking, a vertex-transitive graph "looks" the same no matter from which vertex it is viewed. Say that the graph G is ***generously vertex-transitive*** if for each pair of vertices $u, v \in \mathsf{V}$, there is an automorphism ϕ of G such that $\phi(u) = v$ and $\phi(v) = u$; so, each pair of vertices can be swapped.

Friendly Graphs. To the best of our knowledge, the following definition is new.

Definition 2.1 (Friendly Graph) *A graph* G *is* ***friendly*** *if the following two conditions hold:*

(F.1) G *is generously vertex-transitive.*

(F.2) *For any pair of antipodal vertices* $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathsf{V}(\mathsf{G})$, *and for any arbitrary vertex* $\boldsymbol{\gamma} \in \mathsf{V}(\mathsf{G})$, $\boldsymbol{\gamma}$ *is on a shortest path between* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$.

We remark that each vertex in a friendly graph has a *unique* antipode. (In fact, vertex-transitivity (rather than Condition (F.1)) and Condition (F.2) suffice for this property to hold.) We continue to consider two subclasses of friendly graphs.

Tori. Fix an arbitrary integer $d \geq 2$, called the ***dimension***, and a sequence of integers $p_1, \ldots, p_d \geq 1$, called the ***sides***. The ***d-dimensional bipartite torus*** $\mathsf{T}_d = \mathsf{T}_d[2p_1, \ldots, 2p_d]$ is the graph T_d with $\mathsf{V}(\mathsf{T}_d) = \{0, 1, \ldots, 2p_1 - 1\} \times \ldots \times \{0, 1, \ldots, 2p_d - 1\}$ and

$$\mathsf{E}(\mathsf{T}_d) \quad = \quad \{\{\boldsymbol{\alpha}, \boldsymbol{\beta}\} \mid \boldsymbol{\alpha} \text{ and } \boldsymbol{\beta} \text{ differ in exactly one component } j \in [d] \text{ and } |\alpha_j - \beta_j| \equiv 1 \pmod{2p_j}\}\,;$$

the ***dimension*** of the edge $\{\boldsymbol{\alpha}, \boldsymbol{\beta}\} \in \mathsf{E}(\mathsf{T}_d)$ is the dimension $j \in [d]$ in which $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ differ. Note that the bipartite graph T_d is the cartesian product of d even cycles, where the cycle in dimension $j \in [d]$ has length $2p_j$. We shall often abuse notation to call each integer $j \in [d]$ a ***dimension*** of the graph T_d; so, a vertex in the graph T_d is a d-dimensional vector $\boldsymbol{\alpha} = \langle \alpha_1, \ldots, \alpha_d \rangle$.
Fix a pair of vertices $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathsf{V}(\mathsf{T}_d)$. Then, $\mathsf{dist}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{j \in [d]} \mathsf{dist}_j(\alpha_j, \beta_j)$, where for each dimension $j \in [d]$, $\mathsf{dist}_j(\alpha_j, \beta_j)$ is the distance between the components α_j and β_j on the cycle of length $2p_j$ in dimension j. Note that a pair of vertices $\boldsymbol{\alpha} = \langle \alpha_1, \ldots, \alpha_d \rangle$ and $\overline{\boldsymbol{\alpha}} = \langle (\alpha_1 + p_1) \bmod 2p_1, \ldots, (\alpha_d + p_d) \bmod 2p_d \rangle$ is antipodal in the torus T_d. Clearly, $\mathsf{diam}(\mathsf{T}_d) = \sum_{j \in [d]} p_j$. Since an even cycle fulfils Condition (F.2) and T_d is the cartesian product of even cycles, so does T_d. Induced by an arbitrary pair of vertices $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathsf{V}(\mathsf{T}_d)$ is the automorphism $\Psi : \mathsf{V}(\mathsf{T}_d) \to \mathsf{V}(\mathsf{T}_d)$ where: for each vertex $\boldsymbol{\chi}$, $\Psi(\boldsymbol{\chi}) = \langle \psi_1(\chi_1), \ldots, \psi_d(\chi_d) \rangle$, where for each dimension $j \in [d]$, $\psi_j(\chi_j) = (\alpha_j + \beta_j - \chi_j) \bmod (2p_j)$; clearly, $\Psi(\boldsymbol{\alpha}) = \boldsymbol{\beta}$ and $\Psi(\boldsymbol{\beta}) = \boldsymbol{\alpha}$, and (F.1) follows. Hence, we obtain:

Lemma 2.1 *The d-dimensional bipartite torus $\mathsf{T}_d = \mathsf{T}_d[2p_1, \ldots, 2p_d]$ is friendly.*

We remark that the *non-bipartite* torus is *not* friendly. As a special case, the ***d-dimensional hypercube*** H_d is the d-dimensional torus $\mathsf{T}_d[2, \ldots, 2]$; so, each vertex in $\mathsf{V}(\mathsf{H}_d)$ is a binary vector $\boldsymbol{\alpha} \in \{0, 1\}^d$, and the distance between two vertices is the usual *Hamming distance* between the two binary vectors. So, the diameter of H_d equals the dimension d. The ***d-dimensional cube-connected-cycles*** CCC_d is constructed from the d-dimensional hypercube H_d as follows (cf. [15, Section 3.2.1]). Each vertex of H_d is replaced with a cycle of d vertices $1, \ldots, d$ in CCC_d; each dimension j edge incident to a vertex of H_d is connected to vertex j of the corresponding cycle in CCC_d.
It is simple to verify that the cube-connected-cycles fails Condition (F.2) in Definition 2.1; so, it is not friendly. We will later conclude that the cube-connected-cycles fails also Condition (F.1): it is *not* generously vertex-transitive; however, it is vertex-transitive. It follows that the class of generously vertex-transitive graphs is a *strict* restriction of the class of vertex-transitive graphs; hence, so is the subclass of friendly graphs.
Johnson Graphs. Let ν, k and ℓ be fixed positive integers with $\nu \geq k \geq \ell$; let $\mathcal{U}$ be a fixed *ground set* of size ν. Define the graph $\mathsf{J}(\nu, k, \ell)$ as follows (cf. [9, Section 1.6]). The vertices of $\mathsf{J}(\nu, k, \ell)$ are the subsets of $\mathcal{U}$ with size k; two subsets are adjacent if their intersection has size ℓ. If φ is a permutation of $\mathcal{U}$ and $S \subseteq \mathcal{U}$, then define $\varphi(S) = \{\varphi(s) \mid s \in S\}$. Clearly, each permutation of $\mathcal{U}$ determines a permutation of the subsets of $\mathcal{U}$, and in particular a permutation of the subsets with size k. If $S, T \subseteq \mathcal{U}$, then $|S \cap T| = |\varphi(S) \cap \varphi(T)|$. So, φ is an automorphism of $\mathsf{J}(\nu, k, \ell)$. For $\nu \geq 2k$, the graph $\mathsf{J}(\nu, k, k-1)$ is known as a *Johnson graph*. We prove:

Lemma 2.2 *The graph $\mathsf{J}(\nu, k, \ell)$ is generously vertex-transitive for all $\nu \geq k \geq \ell$. It is friendly if $\nu = 2k$ and $\ell = k - 1$.*

3 Voronoi Games

The Voronoi Game $\langle \mathsf{G}, [\kappa] \rangle$. Fix any integer $\kappa \geq 2$; denote $[\kappa] = \{1, \ldots, \kappa\}$. The ***Voronoi game*** $\langle \mathsf{G}, [\kappa] \rangle$ is the strategic game $\langle [\kappa], \{S_i\}_{i \in [\kappa]}, \{\mathsf{U}_i\}_{i \in [\kappa]} \rangle$, where for each ***player*** $i \in [\kappa]$, *(i)* $S_i = \mathsf{V}(\mathsf{G})$ and *(ii)* for each ***profile*** $\mathbf{s} \in S_1 \times \ldots \times S_\kappa$, the ***utility*** of player i in the profile $\mathbf{s}$ is given by $\mathsf{U}_i(\mathbf{s}) = \sum_{v \in \mathsf{Vor}_i(\mathbf{s})} \dfrac{1}{\mu_v(\mathbf{s})}$, where the ***Voronoi cell*** of player $i \in [\kappa]$ in the profile $\mathbf{s}$ is the set

$$\mathsf{Vor}_i(\mathbf{s}) \quad = \quad \{v \in \mathsf{V}(\mathsf{G}) \mid \mathsf{dist}(s_i, v) \leq \mathsf{dist}(s_{i'}, v) \text{ for each player } i' \in [\kappa]\}\,,$$

and the ***multiplicity*** of vertex $v \in \mathsf{V}(\mathsf{G})$ in the profile $\mathbf{s}$ is the integer $\mu_v(\mathbf{s}) = |\{i' \in [\kappa] \mid v \in \mathsf{Vor}_{i'}(\mathbf{s})\}|$. Clearly, for a profile $\mathbf{s}$, $\sum_{i \in [\kappa]} \mathsf{U}_i(\mathbf{s}) = |\mathsf{V}(\mathsf{G})|$; so, the Voronoi game $\langle \mathsf{G}, [\kappa] \rangle$ is *constant-sum.*

For a profile $\mathbf{s}$ and a player $i \in [\kappa]$, $\mathbf{s}_{-i} \oplus v$ denotes the profile obtained by replacing vertex s_i in $\mathbf{s}$ with vertex v. Say that $\mathbf{s}$ is a ***Nash equilibrium*** [17, 18] (for the Voronoi game $\langle \mathsf{G}, [\kappa] \rangle$) if for each player $i \in [\kappa]$, for each vertex $v \in \mathsf{V}(\mathsf{G})$, $\mathsf{U}_i(\mathbf{s}) \geq \mathsf{U}_i(\mathbf{s}_{-i} \oplus v)$; so, no player has an incentive to unilaterally switch from her chosen vertex in a Nash equilibrium.

$\boxed{\textbf{Profiles.}}$ The ***support*** of the profile $\mathbf{s}$ is the set $\mathsf{support}(\mathbf{s}) = \{s_i \mid i \in [\kappa]\}$, the set of vertices chosen by the players. Given a profile $\mathbf{s}$, an automorphism ϕ of G maps each strategy s_i with $i \in [\kappa]$ to the strategy $\phi(s_i)$; so, ϕ induces an ***image profile*** $\phi(\mathbf{s}) = \langle \phi(s_1), \ldots, \phi(s_\kappa) \rangle$. Say that profiles $\mathbf{s}$ and $\mathbf{t}$ are ***equivalent*** if there is an automorphism ϕ of G such that $\mathbf{t} = \phi(\mathbf{s})$. We observe:

Observation 3.1 *For a pair of equivalent profiles* $\mathbf{s}$ *and* $\mathbf{t}$*, and for each player* $i \in [\kappa]$*,* $\mathsf{U}_i(\mathbf{s}) = \mathsf{U}_i(\mathbf{t})$.

Given a profile $\mathbf{s}$, an automorphism ϕ of G induces an ***image support*** $\phi(\mathsf{support}(\mathbf{s})) = \mathsf{support}(\phi(\mathbf{s}))$. A pair of players $i, i' \in [\kappa]$ is ***symmetric*** for the profile $\mathbf{s}$ if there is an automorphism ϕ of G such that *(i)* $\phi(\mathsf{support}(\mathbf{s})) = \mathsf{support}(\mathbf{s})$, and *(ii)* $\phi(s_i) = s_{i'}$. Say that $\mathbf{s}$ is ***colocational*** if there is a pair of distinct players $i, i' \in [3]$ such that $s_i = s_{i'}$; say that $\mathbf{s}$ is ***balanced*** if for each pair of vertices $u, v \in \mathsf{V}(\mathsf{G})$, $|\{i \in [\kappa] \mid s_i = u\}| = |\{i \in [\kappa] \mid s_i = v\}|$. (Note that a non-colocational profile is balanced.) We observe:

Observation 3.2 *For a symmetric pair of players* $i, i' \in [\kappa]$ *for the balanced profile* $\mathbf{s}$*,* $\mathsf{U}_i(\mathbf{s}) = \mathsf{U}_{i'}(\mathbf{s})$.

Say that a profile $\mathbf{s}$ is ***symmetric*** if each pair of players $i, i' \in [\kappa]$ is symmetric for $\mathbf{s}$. By Observation 3.2, it immediately follows:

Observation 3.3 *For any symmetric profile* $\mathbf{s}$*, and for any pair of players* $i, i' \in [\kappa]$*,* $\mathsf{U}_i(\mathbf{s}) = \mathsf{U}_{i'}(\mathbf{s})$.

The profile $\mathbf{s}$ is ***antipodal*** if its support includes an antipodal pair of vertices.

4 Two Players

For the case $\kappa = 2$, we show:

Proposition 4.1 *Assume that* G *is generously vertex-transitive and* $\kappa = 2$*, and fix an arbitrary profile* $\mathbf{s}$*. Then,* $\mathbf{s}$ *is a Nash equilibrium with* $\mathsf{U}_1(\mathbf{s}) = \mathsf{U}_2(\mathbf{s}) = \frac{|\mathsf{V}|}{2}$.

Proof: Since G is generously vertex-transitive, it follows that $\mathbf{s}$ is symmetric. Hence, by Observation 3.3, $\mathsf{U}_1(\mathbf{s}) = \mathsf{U}_2(\mathbf{s}) = \frac{|\mathsf{V}|}{2}$. To prove that $\mathbf{s}$ is a Nash equilibrium, fix any player $i \in [2]$ and a vertex $u \in V$. Since G is generously vertex-transitive, it follows that $\mathbf{s}_{-i} \oplus u$ is symmetric. Hence, by Observation 3.3, $\mathsf{U}_i(\mathbf{s}_{-i} \oplus u) = \mathsf{U}_{[2]\setminus\{i\}}(\mathbf{s}_{-i} \oplus u) = \frac{|\mathsf{V}|}{2}$. So, $\mathsf{U}_i(\mathbf{s}_{-i} \oplus u) = \mathsf{U}_1(\mathbf{s})$. Since i was chosen arbitrarily, it follows that $\mathbf{s}$ is a Nash equilibrium. ■

Compare Proposition 4.1 to a corresponding result for Hotelling games on a (finite) line segment with two players: there is only one Nash equilibrium where both players are located in the middle of the line segment and receive the same utility [12]. This result confirms to the more general *Principle of Minimum Differentiation* [12] (for Hotelling games), stating that in a Nash equilibrium, players must be indifferent. Since all vertices are indifferent in a vertex-transitive graph, Lemma 4.1 confirms to the analog of the *Principle of Minimum Differentiation* for Voronoi games. We next show:

Proposition 4.2 *Assume that* G *is vertex-transitive and* $\kappa = 2$*. Assume that there are vertices* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *such that* $\mathsf{U}_1(\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle) \neq \mathsf{U}_2(\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle)$*. Then, the Voronoi game* $\langle \mathsf{G}, [2] \rangle$ *has no Nash equiibrium.*

Proof: Assume, without loss of generality, that $\mathsf{U}_1(\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle) < \mathsf{U}_2(\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle)$. Consider an arbitrary profile $\langle \boldsymbol{\gamma}, \boldsymbol{\delta} \rangle$; we shall prove that $\langle \boldsymbol{\gamma}, \boldsymbol{\delta} \rangle$ is *not* a Nash equilibrium. We proceed by case analysis.

1. Assume first that $\mathsf{U}_1(\langle\boldsymbol{\gamma},\boldsymbol{\delta}\rangle) \neq \mathsf{U}_2(\langle\boldsymbol{\gamma},\boldsymbol{\delta}\rangle)$. Without loss of generality, take that $\mathsf{U}_1(\langle\boldsymbol{\gamma},\boldsymbol{\delta}\rangle) > \mathsf{U}_2(\langle\boldsymbol{\gamma},\boldsymbol{\delta}\rangle)$. So, $\mathsf{U}_2(\langle\boldsymbol{\gamma},\boldsymbol{\delta}\rangle) < \frac{\mathsf{V}(\mathsf{G})}{2}$. But, $\mathsf{U}_2(\langle\boldsymbol{\gamma},\boldsymbol{\gamma}\rangle) = \frac{\mathsf{V}(\mathsf{G})}{2}$, and player 2 improves by switching to vertex $\boldsymbol{\gamma}$.

2. Assume now that $\mathsf{U}_1(\langle\boldsymbol{\gamma},\boldsymbol{\delta}\rangle) = \mathsf{U}_2(\langle\boldsymbol{\gamma},\boldsymbol{\delta}\rangle)$; so, $\mathsf{U}_2(\langle\boldsymbol{\gamma},\boldsymbol{\delta}\rangle) = \frac{\mathsf{V}(\mathsf{G})}{2}$. Since G is vertex-transitive, there is an automorphism ψ of G with $\psi(\boldsymbol{\alpha}) = \boldsymbol{\gamma}$. Then,

$$\begin{array}{rll} & \mathsf{U}_2(\langle\boldsymbol{\gamma},\psi(\boldsymbol{\beta})\rangle) & \\ = & \mathsf{U}_2(\langle\psi(\boldsymbol{\alpha}),\psi(\boldsymbol{\beta})\rangle) & \\ = & \mathsf{U}_2(\langle\boldsymbol{\alpha},\boldsymbol{\beta}\rangle) & \text{(by Observation 3.1)} \\ > & \mathsf{U}_1(\langle\boldsymbol{\alpha},\boldsymbol{\beta}\rangle) & \text{(by assumption)} \\ = & \mathsf{U}_1(\langle\psi(\boldsymbol{\alpha}),\psi(\boldsymbol{\beta})\rangle) & \text{(by Observation 3.1)} \\ = & \mathsf{U}_1(\langle\boldsymbol{\gamma},\psi(\boldsymbol{\beta})\rangle)\,. & \end{array}$$

So, $\mathsf{U}_2(\langle\boldsymbol{\gamma},\psi(\boldsymbol{\beta})\rangle) > \frac{\mathsf{V}(\mathsf{G})}{2}$, and player 2 improves by switching to vertex $\psi(\boldsymbol{\beta})$.

Hence, the profile $\langle\boldsymbol{\gamma},\boldsymbol{\delta}\rangle$ is *not* a Nash equilibrium, as needed. ■

We use Proposition 4.2 to show:

Corollary 4.3 *The Voronoi game* $\langle\mathsf{CCC}_3, [2]\rangle$ *has no Nash equilibrium.*

Proposition 4.1 and Corollary 4.3 imply that, in general, the cube-connected cycles CCC_d is *not* generously vertex-transitive. It is nice to observe that an impossibility result in Algebraic Graph Theory is concluded from an impossibility result about Nash equilibria.

5 Two-Guards Theorems

$\boxed{\textbf{Preliminaries.}}$ For a profile $\langle\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\rangle$. For each index $\ell \in \{0,1,2\}$, define the sets

$$\mathcal{A}_\ell(\langle\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\rangle) \;=\; \{\boldsymbol{\delta} \in \mathsf{V}(\mathsf{G}) \mid \mathsf{dist}_\mathsf{G}(\boldsymbol{\delta},\boldsymbol{\gamma}) \sim_\ell \mathsf{dist}_\mathsf{G}(\boldsymbol{\delta},\boldsymbol{\alpha})\}$$

and

$$\mathcal{B}_\ell(\langle\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\rangle) \;=\; \{\boldsymbol{\delta} \in \mathsf{V}(\mathsf{G}) \mid \mathsf{dist}_\mathsf{G}(\boldsymbol{\delta},\boldsymbol{\gamma}) \sim_\ell \mathsf{dist}_\mathsf{G}(\boldsymbol{\delta},\boldsymbol{\beta})\}\,,$$

where $\sim_0$ is $<$, $\sim_1$ is $=$, and $\sim_2$ is $>$. Clearly, $\mathcal{A}_0$, $\mathcal{A}_1$ and $\mathcal{A}_2$ (resp. $\mathcal{B}_0$, $\mathcal{B}_1$ and $\mathcal{B}_2$) partition $\mathsf{V}(\mathsf{G})$. So, $\mathcal{A}_0$ (resp., $\mathcal{B}_0$) contains all vertices that are closer to $\boldsymbol{\gamma}$ than to $\boldsymbol{\alpha}$ (resp., than to $\boldsymbol{\beta}$); $\mathcal{A}_1$ (resp., $\mathcal{B}_1$) contains all vertices that are equally close to each of $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ (resp., to each of $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$); $\mathcal{A}_2$ (resp., $\mathcal{B}_2$) contains all vertices that are closer to $\boldsymbol{\alpha}$ (resp., to $\boldsymbol{\beta}$) than to $\boldsymbol{\gamma}$. For each index $\ell \in \{0,1,2\}$, we shall use the shorter notations $\mathcal{A}_\ell$ and $\mathcal{B}_\ell$ for $\mathcal{A}_\ell(\langle\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\rangle)$ and $\mathcal{B}_\ell(\langle\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\rangle)$, respectively, when the profile $\langle\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\rangle)$ is clear from context. The sets $\mathcal{A}_\ell$ and $\mathcal{B}_\ell$, with $\ell \in \{0,1,2\}$ determine the utility of player 3 in the profile $\langle\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\rangle$ as

$$\mathsf{U}_3(\langle\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\rangle) \;=\; |\mathcal{A}_0 \cap \mathcal{B}_0| + \frac{1}{2}\,|\mathcal{A}_0 \cap \mathcal{B}_1| + \frac{1}{2}\,|\mathcal{A}_1 \cap \mathcal{B}_0| + \frac{1}{3}\,|\mathcal{A}_1 \cap \mathcal{B}_1|\,.$$

$\boxed{\textbf{The Meat.}}$ We first prove:

Lemma 5.1 *For any antipodal pair of vertices* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$, *and for any arbitrary vertex* $\boldsymbol{\gamma}$ *in a friendly graph* G, *consider an automorphism* Φ *of* G *such that* $\Phi(\boldsymbol{\beta}) = \boldsymbol{\gamma}$ *and* $\Phi(\boldsymbol{\gamma}) = \boldsymbol{\beta}$. *Then, for each vertex* $\boldsymbol{\chi} \in \mathsf{V}(\mathsf{G})$, *the following conditions hold:*

(C.1) *For each index* $\ell \in \{0,1,2\}$, $\boldsymbol{\chi} \in \mathcal{A}_\ell$ *if and only if* $\Phi(\boldsymbol{\chi}) \in \mathcal{A}_\ell$.

(C.2) $\chi \in \mathcal{B}_0$ *if and only if* $\Phi(\chi) \in \mathcal{B}_2$ *(and* $\chi \in \mathcal{B}_2$ *if and only if* $\Phi(\chi) \in \mathcal{B}_0$*).*

(C.3) $\chi \in \mathcal{B}_1$ *if and only if* $\Phi(\chi) \in \mathcal{B}_1$

The fact that Φ is an automorphism suffices for Conditions (C.2) and (C.3); the assumptions that *(i)* the pair $\boldsymbol{\alpha}, \boldsymbol{\beta}$ is antipodal, and *(ii)* G is friendly are only needed for Condition (C.1).

Proof: Since Φ is an automorphism, $\mathsf{dist}_{\mathsf{G}}(\chi, \boldsymbol{\beta}) = \mathsf{dist}_{\mathsf{G}}(\Phi(\chi), \boldsymbol{\gamma})$ and $\mathsf{dist}_{\mathsf{G}}(\chi, \boldsymbol{\gamma}) = \mathsf{dist}_{\mathsf{G}}(\Phi(\chi), \boldsymbol{\beta})$, so that χ is closer to $\boldsymbol{\gamma}$ than to $\boldsymbol{\beta}$ if and only if $\Phi(\chi)$ is closer to $\boldsymbol{\beta}$ than to $\boldsymbol{\gamma}$, so that Conditions (C.2) and (C.3) follow. We continue to prove Condition (C.1). Since G is friendly and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathsf{V}(\mathsf{G})$ is a pair of antipodal vertices in G, it follows from Condition (F.2) that $\mathsf{dist}_{\mathsf{G}}(\boldsymbol{\alpha}, \chi) + \mathsf{dist}_{\mathsf{G}}(\boldsymbol{\beta}, \chi) = \mathsf{dist}_{\mathsf{G}}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and $\mathsf{dist}_{\mathsf{G}}(\boldsymbol{\alpha}, \Phi(\chi)) + \mathsf{dist}_{\mathsf{G}}(\boldsymbol{\beta}, \Phi(\chi)) = \mathsf{dist}_{\mathsf{G}}(\boldsymbol{\alpha}, \boldsymbol{\beta})$. It follows that

$$\begin{array}{ll} & \mathsf{dist}_{\mathsf{G}}(\boldsymbol{\alpha}, \chi) - \mathsf{dist}_{\mathsf{G}}(\boldsymbol{\gamma}, \chi) \\ = & \mathsf{dist}_{\mathsf{G}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) - \mathsf{dist}_{\mathsf{G}}(\boldsymbol{\beta}, \chi) - \mathsf{dist}_{\mathsf{G}}(\boldsymbol{\gamma}, \chi) \\ = & \mathsf{dist}_{\mathsf{G}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) - \mathsf{dist}_{\mathsf{G}}(\Phi(\boldsymbol{\beta}), \Phi(\chi)) - \mathsf{dist}_{\mathsf{G}}(\Phi(\boldsymbol{\gamma}), \Phi(\chi)) \quad \text{(since } \Phi \text{ is an automorphism)} \\ = & \mathsf{dist}_{\mathsf{G}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) - \mathsf{dist}_{\mathsf{G}}(\boldsymbol{\gamma}, \Phi(\chi)) - \mathsf{dist}_{\mathsf{G}}(\boldsymbol{\beta}, \Phi(\chi)) \quad \text{(by definition of } \Phi\text{)} \\ = & \mathsf{dist}_{\mathsf{G}}(\boldsymbol{\alpha}, \Phi(\chi) - \mathsf{dist}_{\mathsf{G}}(\boldsymbol{\gamma}, \Phi(\chi))\,; \end{array}$$

so, χ is closer to $\boldsymbol{\alpha}$ than to $\boldsymbol{\gamma}$ if and only $\Phi(\chi)$ is closer to $\boldsymbol{\alpha}$ than to $\boldsymbol{\gamma}$, and Condition (C.1) follows. ■

We now show:

Theorem 5.2 (The Two-Guards Theorem for Friendly Graphs) *Fix an antipodal pair of vertices* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$*, and an arbitrary vertex* $\boldsymbol{\gamma}$ *in a friendly graph* G*. Then,* $\mathsf{U}_3(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle) = \frac{1}{4}\, |\mathsf{V}(\mathsf{G})| + \frac{1}{12}\, |\Omega(\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\})|$.

Proof: By Lemma 5.1 (Conditions (C.1) and (C.2)), it follows that for each index $\ell \in \{0, 1\}$, for each vertex $\chi \in \mathsf{V}(\mathsf{G})$, $\chi \in \mathcal{A}_\ell \cap \mathcal{B}_0$ if and only if $\Phi(\chi) \in \mathcal{A}_\ell \cap \mathcal{B}_2$. Since the function Φ is a bijection, the restriction $\Phi : \mathcal{A}_\ell \cap \mathcal{B}_0 \rightarrow \mathcal{A}_\ell \cap \mathcal{B}_2$ is a bijection. Hence, $|\mathcal{A}_\ell \cap \mathcal{B}_0| = |\mathcal{A}_\ell \cap \mathcal{B}_2|$. It follows that for each index $\ell \in \{0, 1\}$, $|\mathcal{A}_\ell| = |\mathcal{A}_\ell \cap \mathcal{B}_0| + |\mathcal{A}_\ell \cap \mathcal{B}_1| + |\mathcal{A}_\ell \cap \mathcal{B}_2| = 2\, |\mathcal{A}_\ell \cap \mathcal{B}_0| + |\mathcal{A}_\ell \cap \mathcal{B}_1|$. Hence,

$$\begin{array}{rcl} \mathsf{U}_3(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle) & = & |\mathcal{A}_0 \cap \mathcal{B}_0| + \dfrac{1}{2}\, |\mathcal{A}_0 \cap \mathcal{B}_1| + \dfrac{1}{2}\, |\mathcal{A}_1 \cap \mathcal{B}_0| + \dfrac{1}{3}\, |\mathcal{A}_1 \cap \mathcal{B}_1| \\ & = & \dfrac{1}{2}\, |\mathcal{A}_0| + \dfrac{1}{4}\, |\mathcal{A}_1| - \dfrac{1}{4}\, |\mathcal{A}_1 \cap \mathcal{B}_1| + \dfrac{1}{3}\, |\mathcal{A}_1 \cap \mathcal{B}_1| \\ & = & \dfrac{1}{2} \left(|\mathcal{A}_0| + \dfrac{1}{2}\, |\mathcal{A}_1| \right) + \dfrac{1}{12}\, |\mathcal{A}_1 \cap \mathcal{B}_1|\,. \end{array}$$

Consider the Voronoi game $\langle \mathsf{G}, [2] \rangle$, with players 1 and 3. Then, for the profile $\langle \boldsymbol{\alpha}, \boldsymbol{\gamma} \rangle$, $\mathsf{U}_3(\langle \boldsymbol{\alpha}, \boldsymbol{\gamma} \rangle) = |\mathcal{A}_0| + \frac{1}{2} |\mathcal{A}_1|$. By Lemma 4.1, $\mathsf{U}_3(\langle \boldsymbol{\alpha}, \boldsymbol{\gamma} \rangle) = \frac{1}{2}\, |\mathsf{V}(\mathsf{G})|$. Hence, $|\mathcal{A}_0| + \frac{1}{2} |\mathcal{A}_1| = \frac{1}{2}\, |\mathsf{V}(\mathsf{G})|$, so that

$$\begin{array}{rcl} \mathsf{U}_3(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle) & = & \dfrac{1}{4} |\mathsf{V}(\mathsf{G})| + \dfrac{1}{12}\, |\mathcal{A}_1 \cap \mathcal{B}_1| \\ & = & \dfrac{1}{4} |\mathsf{V}(\mathsf{G})| + \dfrac{1}{12}\, |\Omega(\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\})|\,, \end{array}$$

as needed. ■

(Note that $\dfrac{|\mathsf{V}(\mathsf{G})|}{4} + \dfrac{|\Omega|}{12}$ is strictly less than $\dfrac{|\mathsf{V}(\mathsf{G})|}{3}$ unless $\Omega = \mathsf{V}(\mathsf{G})$.) An immediate implication of Theorem 5.2 for a *bipartite* friendly graph G with odd diameter follows. Since $\mathsf{dist}_{\mathsf{G}}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is odd for an arbitrary antipodal pair $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, $\Omega(\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}) = \emptyset$ for an arbitrary vertex $\boldsymbol{\gamma}$; hence, we obtain:

Corollary 5.3 (Two-Guards Theorem for Bipartite, Odd-Diameter Friendly Graphs) *Fix an antipodal pair of vertices* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$*, and an arbitrary vertex* $\boldsymbol{\gamma}$ *in a bipartite friendly graph* G *with odd diameter. Then,* $\mathsf{U}_3(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle) = \frac{1}{4}\, |\mathsf{V}(\mathsf{G})|$.

6 Four Players

With Colocation. We show:

Theorem 6.1 *Consider a friendly graph* G*. Then, the Voronoi game* $\langle \mathsf{G}, [4] \rangle$ *has a Nash equilibrium. Specifically, for any arbitrary antipodal pair* $\boldsymbol{\alpha}$*,* $\boldsymbol{\beta}$*, the profile* $\mathbf{s} = \langle \boldsymbol{\alpha}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta} \rangle$ *is a Nash equilibrium.*

Proof: Consider a pair of players $i \in \{1, 2\}$ and $i' \in \{3, 4\}$; so, $s_i = \boldsymbol{\alpha}$ and $s_{i'} = \boldsymbol{\beta}$. Since G is doubly vertex-transitive, there is an automorphism ϕ of G such that $\phi(s_i) = s_{i'}$ and $\phi(s_{i'}) = s_i$. Note that by the construction of $\mathbf{s}$, $\phi(\mathsf{support}(\mathbf{s})) = \mathsf{support}(\mathbf{s})$. Hence, the pair of players i, i' is symmetric for $\mathbf{s}$. Since $\mathbf{s}$ is balanced, Observation 3.2 implies that $\mathsf{U}_i(\mathbf{s}) = \mathsf{U}_{i'}(\mathbf{s})$. It follows that $\mathsf{U}_i(\langle \boldsymbol{\alpha}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta} \rangle) = \frac{|\mathsf{V}(\mathsf{G})|}{4}$ for each player $i \in [4]$.
We now prove that no player $i \in [4]$ improves by switching to vertex $\boldsymbol{\alpha}'$. Without loss of generaity, fix $i = 1$. We shall prove that $\mathsf{U}_i(\langle \boldsymbol{\alpha}', \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta} \rangle) \leq \mathsf{U}_i(\langle \boldsymbol{\alpha}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta} \rangle)$. To do so, consider now the Voronoi game $\langle \mathsf{G}, [3] \rangle$ with players 1, 2 and 3. By the *Two-Guards Theorem for Friendly Graphs* (Theorem 5.3), $\mathsf{U}_1(\langle \boldsymbol{\alpha}', \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle) = \frac{|\mathsf{V}(\mathsf{G})|}{4} + \frac{|\Omega(\{\boldsymbol{\alpha}', \boldsymbol{\alpha}, \boldsymbol{\beta}\})|}{12}$.
The utility of player 1 decreases from $\langle \boldsymbol{\alpha}', \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle$ to $\langle \boldsymbol{\alpha}', \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta} \rangle$ at least due to the fact that the vertices in $\Omega(\{\boldsymbol{\alpha}', \boldsymbol{\alpha}, \boldsymbol{\beta}\})$ will be shared with player 4 (additionally to the players 1, 2 and 3); this partial decrease is $\frac{|\Omega(\{\boldsymbol{\alpha}', \boldsymbol{\alpha}, \boldsymbol{\beta}\})|}{3} - \frac{|\Omega(\{\boldsymbol{\alpha}', \boldsymbol{\alpha}, \boldsymbol{\beta}\})|}{4} = \frac{|\Omega(\{\boldsymbol{\alpha}', \boldsymbol{\alpha}, \boldsymbol{\beta}\})|}{12}$. So,

$$\begin{aligned} \mathsf{U}_1(\langle \boldsymbol{\alpha}', \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta} \rangle) &\leq \mathsf{U}_1(\langle \boldsymbol{\alpha}', \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle) - \frac{|\Omega(\{\boldsymbol{\alpha}', \boldsymbol{\alpha}, \boldsymbol{\beta}\})|}{12} \\ &= \frac{|\mathsf{V}(\mathsf{G})|}{4}, \end{aligned}$$

and the claim follows. ■

Without Colocation. We show:

Theorem 6.2 *Consider a bipartite friendly graph* G *with odd diameter. Then, the Voronoi game* $\langle \mathsf{G}, [4] \rangle$ *has a Nash equilibrium without colocation. Specifically, for any aribtrary pair of two distinct antipodal pairs* $\boldsymbol{\alpha}, \boldsymbol{\beta}$ *and* $\boldsymbol{\gamma}$*,* $\boldsymbol{\delta}$*, respectively, the profile* $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta} \rangle$ *is a Nash equilibrium.*

Proof: Consider first the bijection $\psi : \mathsf{V}(\mathsf{G}) \rightarrow \mathsf{V}(\mathsf{G})$ which maps each vertex to its unique antipode. (Since G is friendly, such a bijection exists.) So, $\psi(\boldsymbol{\alpha}) = \boldsymbol{\beta}$ and $\psi(\boldsymbol{\gamma}) = \boldsymbol{\delta}$; also, $\psi^2 = \iota$. We prove that ψ is an automorphism of G:

> Since ψ is a bijection, we only have to prove that ψ preserves edges. So, consider an edge $\{u, v\} \in \mathsf{E}(\mathsf{G})$; we shall prove that $\{\psi(u), \psi(v)\} \in \mathsf{E}(\mathsf{G})$.
>
> - Recall that ψ maps u to its *unique* antipode $\psi(u)$; so, $\mathsf{dist}_{\mathsf{G}}(u, \psi(u)) = \mathsf{diam}(\mathsf{G})$. Since ψ is a bijection $\psi(u) \neq \psi(v)$. It follows that $\mathsf{dist}_{\mathsf{G}}(u, \psi(v)) \neq \mathsf{diam}(\mathsf{G})$.
> - Now assume, by way of contradiction, that $\mathsf{dist}_{\mathsf{G}}(u, \psi(v)) < \mathsf{diam}(\mathsf{G}) - 1$. Then, the path consisting of the edge $\{v, u\}$ and the shortest path from u to $\psi(v)$. establishes that $\mathsf{dist}_{\mathsf{G}}(v, \psi(v)) < \mathsf{diam}(\mathsf{G})$, a contradiction. It follows that $\mathsf{dist}_{\mathsf{G}}(u, \psi(v)) \geq \mathsf{diam}(\mathsf{G}) - 1$.
>
> Hence, $\mathsf{dist}_{\mathsf{G}}(u, \psi(v)) = \mathsf{diam}(\mathsf{G}) - 1$. Since G is friendly, the vertex $\psi(v)$ is on the shortest path between u and $\psi(u)$, which has length $\mathsf{diam}(\mathsf{G})$ (by definition of ψ). This implies that $\{\psi(u), \psi(v)\} \in \mathsf{E}(\mathsf{G})$, as needed.

Since G is generously vertex-transitive, there is an automorphism φ of G such that $\varphi(\boldsymbol{\alpha}) = \boldsymbol{\gamma}$ and $\varphi(\boldsymbol{\gamma}) = \boldsymbol{\alpha}$. Since φ preserves distances, it follows that $\varphi(\boldsymbol{\beta}) = \boldsymbol{\delta}$ and $\varphi(\boldsymbol{\delta}) = \boldsymbol{\beta}$.

Note that each pair of players is symmetric for the profile $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta} \rangle$ due to some automorphism from ψ, φ, $\varphi\psi$ and $\psi\varphi$. Hence, the profile $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta} \rangle$ is symmetric. By Observation 3.3, it follows that for each player $i \in [4]$, $\mathsf{U}_i(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta} \rangle) = \frac{1}{4}\,|\mathsf{V}(\mathsf{G})|$.

We continue to prove that the profile $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta} \rangle$ is a Nash equilibrium; since it is symmetric, we only have to prove that one of the players cannot improve by switching. So, assume that player 3 switches to vertex $\widehat{\boldsymbol{\gamma}}$. Consider the Voronoi game $\langle \mathsf{T}_d, [3] \rangle$ with players 1, 2 and 3. Since the pair $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is antipodal, the *Two-Guards Theorem for Bipartite, Odd-Diameter Friendly Graphs* (Corollary 5.3) implies that $\mathsf{U}_3(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \widehat{\boldsymbol{\gamma}} \rangle) = \frac{1}{4}\,|\mathsf{V}(\mathsf{G})|$. Clearly, $\mathsf{U}_3(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \widehat{\boldsymbol{\gamma}}, \boldsymbol{\delta} \rangle) \leq \mathsf{U}_3(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \widehat{\boldsymbol{\gamma}} \rangle)$. It follows that $\mathsf{U}_3(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \widehat{\boldsymbol{\gamma}}, \boldsymbol{\delta} \rangle) \leq \frac{1}{4}\,|\mathsf{V}(\mathsf{G})|$. Hence, $\mathsf{U}_3(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \widehat{\boldsymbol{\gamma}}, \boldsymbol{\delta} \rangle) \leq \mathsf{U}_3(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta} \rangle)$, as needed. ■

7 Three Players

For the case $\kappa = 3$, we shall consider some special profiles. A profile $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle$ is ***linear*** if $\mathsf{dist}(\boldsymbol{\alpha}, \boldsymbol{\beta}) + \mathsf{dist}(\boldsymbol{\beta}, \boldsymbol{\gamma}) = \mathsf{dist}(\boldsymbol{\alpha}, \boldsymbol{\gamma})$; then, $\boldsymbol{\beta}$ is called the ***middle vertex*** and player 2 is called the ***middle player***.

Tori with Odd Diameter. We show:

Theorem 7.1 *Consider the 2-dimensional torus T_2 with odd diameter $\sum_{j \in [2]} p_j$. Then, the Voronoi game $\langle \mathsf{T}_2, [3] \rangle$ has a Nash equilibrium.*

Proof: Assume, without loss of generality, that $p_1 > p_2$. Set $\boldsymbol{\alpha} = \langle 0, 0 \rangle$, $\boldsymbol{\beta} = \langle 1, 0 \rangle$ and $\boldsymbol{\gamma} = \langle p_1, p_2 \rangle$. We will prove that the profile $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle$ is a Nash equilibrium. Note that $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ are an antipodal pair of vertices. By Lemma 2.1 and the *Two-Guards Theorem for Bipartite, Odd-Diameter Friendly Graphs* (Corollary 5.3), $\mathsf{U}_2(\langle \boldsymbol{\alpha}, \boldsymbol{\delta}, \boldsymbol{\gamma} \rangle) = \frac{1}{4} \prod_{j \in [d]} (2p_j)$ for any vertex $\boldsymbol{\delta} \in \mathsf{V}(\mathsf{T}_d)$; thus, we only have to prove that neither player 1 nor 3 can improve her utility by switching. The claim will follow from the following two technical claims:

Lemma 7.2 (Player 3 Cannot Improve) *It holds that:* (1) $\mathsf{U}_3(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle) = 2p_1p_2 - p_2$. (2) *For each vertex $\widehat{\boldsymbol{\gamma}} \in \mathsf{V}(\mathsf{T}_2)$, $\mathsf{U}_3(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \widehat{\boldsymbol{\gamma}} \rangle) \leq 2p_1p_2 - p_2$.*

Lemma 7.3 (Player 1 Cannot Improve) *It holds that:* (1) $\mathsf{U}_1(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle) = p_1p_2 + p_2$. (2) *For each vertex $\widehat{\boldsymbol{\alpha}} \in \mathsf{V}(\mathsf{T}_2)$, $\mathsf{U}_1(\langle \widehat{\boldsymbol{\alpha}}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle) \leq p_1p_2 + p_2$.*

For Lemma 7.3, the proof of (1) will use Lemma 7.2; the proof of (2) will use ideas from the proof of the *Two-Guards Theorem for Friendly Graphs* (Theorem 5.2). ■

Hypercubes. We finally consider the Voronoi game $\langle \mathsf{H}_d, [3] \rangle$. Consider a profile $\mathbf{s} = \langle \boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_\kappa \rangle$ for the Voronoi game $\langle \mathsf{H}_d, [\kappa] \rangle$. Say that dimension $j \in [d]$ is ***irrelevant*** for the profile $\mathbf{s}$ if bit j is the same in all binary words $\boldsymbol{\alpha}_i$, with $i \in [\kappa]$. Denote as $\mathsf{irr}(\mathbf{s})$ the number of irrelevant dimensions for $\mathbf{s}$; clearly, $0 \leq \mathsf{irr}(\mathbf{s}) \leq d$. The profile $\mathbf{s}$ is ***irreducible*** if it has no irrelevant dimension. Clearly, an antipodal profile is irreducible. We continue with three preliminary observations.

Observation 7.4 *Consider an antipodal pair of vertices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ for the hypercube H_d. Then, for any vertex $\boldsymbol{\gamma} \in \mathsf{V}(\mathsf{H}_d)$, the profile $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle$ is linear.*

Observation 7.5 *Consider an irreducible profile $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle$ for the Voronoi game $\langle \mathsf{H}_d, [3] \rangle$. Then,* $\mathsf{dist}(\boldsymbol{\alpha}, \boldsymbol{\beta}) + \mathsf{dist}(\boldsymbol{\beta}, \boldsymbol{\gamma}) + \mathsf{dist}(\boldsymbol{\alpha}, \boldsymbol{\gamma}) = 2d$.

Observation 7.6 *Consider an irreducible profile $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle$ for the Voronoi game $\langle \mathsf{H}_d, [3] \rangle$. Then, there is an equivalent profile $\langle 0^d, 1^{p+q}0^r, 1^p0^q1^r \rangle$, for some triple of integers $p, q, r \in \mathsf{N}$.*

We first determine the utility of an arbitrary player in an irreducible profile. For each index $i \in \{0, 1\}$, define the combinatorial function $\mathsf{M}_i : \mathsf{N} \times \mathsf{N} \to \mathsf{N}$ with

$$\mathsf{M}_i(x, t) \quad = \quad \begin{cases} \sum_{j=\frac{x+1}{2}-t}^{\frac{x-1}{2}} \binom{x}{j} + \frac{i}{2}\binom{x}{\frac{x-1}{2}-t}, & \text{if } x \text{ is odd} \\ \sum_{j=\frac{x}{2}+1-t}^{\frac{x}{2}-1} \binom{x}{j} + \frac{1-i}{2}\binom{x}{\frac{x}{2}-t} + \frac{1}{2}\binom{x}{\frac{x}{2}}, & \text{if } x \text{ is even} \end{cases}.$$

We show:

Theorem 7.7 *Fix integers $x, y, z \in \mathsf{N}$ with $x + y + z = d$, and consider the irreducible profile $\mathbf{s} = \langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle$ with $\boldsymbol{\alpha} = 0^d$, $\boldsymbol{\beta} = 1^x 0^y 1^z$ and $\boldsymbol{\gamma} = 1^{x+y} 0^z$. Then,*

$$\mathsf{U}_2(\mathbf{s}) = \frac{1}{4}\,2^d + \begin{cases} 2\sum_{t\in[\frac{z}{2}]} \binom{z}{\frac{z}{2}-t} \mathsf{M}_0(x,t)\mathsf{M}_0(y,t)\,, & \text{if (x or y is odd) and z is even} \\ \frac{1}{12}\binom{x}{\frac{x}{2}}\binom{y}{\frac{y}{2}}\binom{z}{\frac{z}{2}} + \frac{1}{6}\sum_{t\in[\frac{z}{2}]}\binom{x}{\frac{x}{2}-t}\binom{y}{\frac{y}{2}-t}\binom{z}{\frac{z}{2}-t} & \\ +2\sum_{t\in[\frac{z}{2}]}\binom{z}{\frac{z}{2}-t}\mathsf{M}_0(x,t)\mathsf{M}_0(y,t)\,, & \text{if x, y and z are even} \\ 2\sum_{t=0}^{\frac{z-1}{2}}\binom{z}{\frac{z-1}{2}-t}\mathsf{M}_1(x,t)\mathsf{M}_1(y,t)\,, & \text{if (x or y is even) and z is odd} \\ \frac{1}{6}\sum_{t=0}^{z-1}\binom{x}{\frac{x-1}{2}-t}\binom{y}{\frac{y-1}{2}-t}\binom{z}{\frac{z-1}{2}-t} & \\ +2\sum_{t=0}^{\frac{z-1}{2}}\binom{z}{\frac{z-1}{2}-t}\mathsf{M}_1(x,t)\mathsf{M}_1(y,t) & \text{if x, y and z are odd} \end{cases}$$

Theorem 7.7 immediately implies:

Corollary 7.8 *Consider an antipodal profile $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle$ for the Voronoi game $\langle \mathsf{H}_d, [3] \rangle$, with $\mathsf{dist}(\boldsymbol{\alpha}, \boldsymbol{\gamma}) = d$, $\mathsf{dist}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = p$ and $\mathsf{dist}(\boldsymbol{\beta}, \boldsymbol{\gamma}) = q$. Then,*

$$\mathsf{U}_2(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle) = \frac{1}{4}\,2^d + \begin{cases} 0 & \text{if p or q is odd} \\ \frac{1}{12}\binom{p}{\frac{p}{2}}\binom{q}{\frac{q}{2}}, & \text{if p and q are even} \end{cases}.$$

We now show:

Theorem 7.9 *For any odd integer d, the Voronoi game $\langle \mathsf{H}_d, [3] \rangle$ has a Nash equilibrium. Specifically, an antipodal profile $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle$ with $\mathsf{dist}_{\mathsf{H}_d}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = 1$ is a Nash equilibrium.*

Proof: Fix such an antipodal profile $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle$. By Observation 7.4, $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle$ is a linear profile (and $\boldsymbol{\beta}$ is the middle vertex); so, $\mathsf{dist}(\boldsymbol{\beta}, \boldsymbol{\gamma}) = d-1$. Since d is odd, Corollary 5.3 implies that for any vertex $\boldsymbol{\chi} \in \mathsf{V}(\mathsf{H}_d)$, $\mathsf{U}_2(\langle \boldsymbol{\alpha}, \boldsymbol{\chi}, \boldsymbol{\gamma} \rangle) = \frac{1}{4}\,2^d$; so, player 2 cannot improve her utility $\mathsf{U}_2(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ by switching to a vertex $\widehat{\boldsymbol{\beta}}$. So, in order to prove that the profile $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle$ is a Nash equilibrium, we only need to consider players 1 and 3. By Observation 7.6 (and its proof), there is an equivalent profile $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle$ with $\boldsymbol{\alpha} = 0^d$, $\boldsymbol{\beta} = 1^{p+q}0^r$ and $\boldsymbol{\gamma} = 1^p 0^q 1^r$, where $p + q = \mathsf{dist}_{\mathsf{H}_d}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = 1$, $p + r = \mathsf{dist}_{\mathsf{H}_d}(\boldsymbol{\alpha}, \boldsymbol{\gamma}) = d$, and $q + r = \mathsf{dist}_{\mathsf{H}_d}(\boldsymbol{\alpha}, \boldsymbol{\gamma}) = d - 1$. It follows that $p = 1$, $q = 0$ and $r = d - 1$, so that $\boldsymbol{\alpha} = 0^d$, $\boldsymbol{\beta} = 10^{d-1}$ and $\boldsymbol{\gamma} = 1^d$. By Observation 3.1, it suffices to determine the utilities $\mathsf{U}_i(\langle 0^d, 10^{d-1}, 1^d \rangle)$ with $i \in [3]$ and prove that the profile $\langle 0^d, 10^{d-1}, 1^d \rangle$ is a Nash equilibrium. We shall determine the utilities $\mathsf{U}_1(\langle 0^d, 10^{d-1}, 1^d \rangle)$ and $\mathsf{U}_3(\langle 0^d, 10^{d-1}, 1^d \rangle)$.

The utility $\mathsf{U}_1(\langle 0^d, 10^{d-1}, 1^d \rangle)$: Consider the automorphism $\Phi : \mathsf{V}(\mathsf{H}_d) \to \mathsf{V}(\mathsf{H}_d)$ such that for each vertex $\boldsymbol{\chi} \in \mathsf{V}(\mathsf{H}_d)$, $\Phi(\boldsymbol{\chi}) = \langle \chi_2, \ldots, \chi_d, \overline{\chi_1} \rangle$. Then, $\Phi(0^d) = 0^{d-1}1$, $\Phi(10^{d-1}) = 0^d$ and $\Phi(1^d) = 1^{d-1}0$. Since Φ is an automorphism, the profiles $\langle 0^d, 10^{d-1}, 1^d \rangle$ and $\langle 0^{d-1}1, 0^d, 1^{d-1}0 \rangle$ are equivalent. Hence, Observation 3.1 implies that $\mathsf{U}_1(\langle 0^d, 10^{d-1}, 1^d \rangle) = \mathsf{U}_1(\langle 0^{d-1}1, 0^d, 1^{d-1}0 \rangle) = \mathsf{U}_2(\langle 0^d, 0^{d-1}1, 1^{d-1}0 \rangle)$. Hence, we shall determine $\mathsf{U}_2(\langle 0^d, 0^{d-1}1, 1^{d-1}0 \rangle)$.

Apply Theorem 7.7 with $x = 0$, $y = d - 1$ and $z = 1$ to get that

$$\begin{aligned} \mathsf{U}_1(\langle 0^d, 10^{d-1}, 1^d \rangle) &= \frac{1}{4}\,2^d + 2\,\mathsf{M}_1(x,0)\mathsf{M}_1(y,0) \\ &= \frac{1}{4}\,2^d + \frac{1}{2}\binom{d-1}{\frac{d-1}{2}}. \end{aligned}$$

The utility $\mathsf{U}_3(\langle 0^d, 10^{d-1}, 1^d \rangle)$: Clearly,

$$\begin{aligned} & \mathsf{U}_3(\langle 0^d, 10^{d-1}, 1^d \rangle) \\ =\ & 2^d - \mathsf{U}_1(\langle 0^d, 10^{d-1}, 1^d \rangle) - \mathsf{U}_2(\langle 0^d, 10^{d-1}, 1^d \rangle) \\ =\ & \frac{1}{2}\,2^d - \frac{1}{2}\binom{d-1}{\frac{d-1}{2}} \\ \geq\ & \frac{1}{2}\,2^d - \frac{1}{2}\,2^{d-1} \qquad \left(\text{since } \binom{d-1}{\frac{d-1}{2}} \leq 2^{d-1}\right) \\ =\ & \frac{1}{4}\,2^d\,. \end{aligned}$$

We continue to prove:

Lemma 7.10 *The profile $\langle 0^d, 10^{d-1}, 1^d \rangle$ is a Nash equilibrium:*

The proof is now complete. ■

8 Epilogue

This work opens up an intriguing research agenda on the study of Nash equilibria for the Voronoi game $\langle \mathsf{G}, [\kappa] \rangle$ where G is transitive and $\kappa \geq 2$. A lot of interesting problems remain open; we conclude with some of them.

The full power of *Two-Guards*-like theorems is yet to be realized. Are there similar theorems when either G comes from some broader class encompassing the friendly graphs, or $\kappa > 3$? On a more concrete level, it is very interesting to generalize Theorem 7.7 and find combinatorial formulas for the three players' utilities when $\mathsf{G} = \mathsf{T}_d$; this may enable generalizing Theorem 7.9 to the torus T_d (with odd d). It is also interesting to study the *uniqueness* of Nash equilibria; in particular, we know no *non-antipodal* Nash equilibrium on some friendly graph G with $\kappa \geq 3$.

Beyond Theorems 6.1 and 6.2, nothing is known for the case $\kappa \geq 4$. (For example, we do not even know if these resuts can be extended to broader classes encompassing frindly graphs while still bypassing explicit enumeration.) This offers a very wide research avenue. In particular, we invite the reader to prove or disprove the following conjectures: (1) The game $\langle \mathsf{H}_d, [\kappa] \rangle$ with odd diameter d has a Nash equilibrium, whatever κ is. (2) The game $\langle \mathsf{H}_d, [\kappa] \rangle$ with even κ has a Nash equilibrium, whatever d is. (3) The game $\langle \mathsf{H}_d, [\kappa] \rangle$ with even d and odd κ has *no* Nash equilibrium.

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A Proofs from Section 2

A.1 Lemma 2.2

We start with Condition (F.1). Consider any pair of vertices S and T (with $|S| = |T| = k$). Clearly, $|S \setminus T| = |T \setminus S|$. Consider a permutation φ of $\mathcal{U}$ with $\varphi(S \setminus T) = T \setminus S$ and $\varphi(T \setminus S) = S \setminus S$, $\varphi(S \cap T) = S \cap T$ and $\varphi(\overline{S \cup T}) = \overline{S \cup T}$. Clearly, $\varphi(S) = T$ and $\varphi(T) = S$. Since φ is an automorphism of $\mathsf{J}(\nu, k, \ell)$, the claim follows. Condition (F.1) follows.
We continue with Condition (F.2). Consider a pair of antipodal vertices S and $\overline{S}$ in $\mathsf{J}(2k, k, k-1)$, and an arbitrary vertex T. Consider the path π_1 resulting by switching one element at a time from $S \setminus T$ to $T \setminus S$; consider also the path π_2 resulting by switching one element at a time from $S \cap T$ to $\overline{S} \setminus T$. Clearly, π_1 is a shortest path from S to T and π_2 is a shortest path from T to $\overline{S}$. So, $\pi_1\pi_2$ is a path from S to $\overline{S}$ of length k. Note that every path from S to $\overline{S}$ has length at least k, since only element may switch at a time and all k elements in S have to switch. Hence, $\pi_1\pi_2$ is a shortest path from S to $\overline{S}$, and Condition (F.2) follows.

B Proofs from Section 4

B.1 Corollary 4.3

Fix $\boldsymbol{\alpha}$ to be the vertex 1 in the cycle $1, 2, 3$ replacing vertex 0^3 in H_3; fix $\boldsymbol{\beta}$ to be the vertex 2 in the cycle $1, 2, 3$ replacing vertex 010 in H_3. Locate players 1 and 2 at vertices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively. We encourage the reader to verify that $|\mathsf{Vor}_1(\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle)| = 13$ and $|\mathsf{Vor}_2(\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle)| = 14$, with $|\mathsf{Vor}_1(\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle) \cap \mathsf{Vor}_2(\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle)| = 3$. It follows that $\mathsf{U}_1(\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle) = 10 + \frac{3}{2} = \frac{23}{2}$ while $\mathsf{U}_1(\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle) = 11 + \frac{3}{2} = \frac{25}{2}$. Hence, Proposition 4.2 implies that the Voronoi game $\langle \mathsf{CCC}_3, [2] \rangle$ has no Nash equilibrium.

C Proofs from Section 7 — Tori

Recall that for the torus T_d, for each dimension $j \in [d]$,

$$\mathsf{dist}_j(\alpha_j, \beta_j) \quad = \quad \min \{ |\alpha_j - \beta_j| \bmod 2p_j, (2p_j - |\alpha_j - \beta_j|) \bmod 2p_j \} \; .$$

C.1 Lemma 7.2

We shall prove (2); then, (1) will follow. Fix a vertex $\widehat{\boldsymbol{\gamma}} = \langle \widehat{\gamma_1}, \widehat{\gamma_2} \rangle \in \mathsf{V}(\mathsf{T}_2)$; we shall derive an upper bound for $\mathsf{U}_3(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \widehat{\boldsymbol{\gamma}} \rangle)$. Clearly, $\mathsf{U}_3(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \widehat{\boldsymbol{\gamma}} \rangle) = \mathsf{U}_1(\langle \widehat{\boldsymbol{\gamma}}, \boldsymbol{\beta}, \boldsymbol{\alpha} \rangle)$. Assume, without loss of generality, that $1 \leq \widehat{\gamma_1} \leq p_1$ and $0 \leq \widehat{\gamma_2} \leq p_2$. Otherwise, we provide automorphisms for T_2 preserving the utility $\mathsf{U}_3(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \widehat{\boldsymbol{\gamma}} \rangle)$:

- If $\widehat{\gamma_2} > p_2$, then consider the automorphism $\varphi : \mathsf{V}(\mathsf{T}_d) \rightarrow \mathsf{V}(\mathsf{T}_d)$ with $\varphi(\boldsymbol{\chi}) = \langle \varphi_1(\chi_1), \varphi_2(\chi_2) \rangle$ for each vertex $\boldsymbol{\chi} \in \mathsf{V}(\mathsf{T}_d)$, where $\varphi_1(\chi_1) = \chi_1$ and $\varphi_2(\chi_2) = (2p_2 - \chi_2) \bmod 2p_2$. Note that $\varphi(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$, $\varphi(\boldsymbol{\beta}) = \boldsymbol{\beta}$ and $\varphi(\widehat{\boldsymbol{\gamma}}) = \langle \widehat{\gamma_1}, \varphi_2(\widehat{\gamma_2}) \rangle$ with $0 \leq \varphi_2(\widehat{\gamma_2}) \leq p_2$. Thus, the profiles $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \widehat{\boldsymbol{\gamma}} \rangle$ and $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \varphi(\widehat{\boldsymbol{\gamma}}) \rangle$ are equivalent. Hence, Observation 3.1 implies that $\mathsf{U}_3(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \widehat{\boldsymbol{\gamma}} \rangle) = \mathsf{U}_3(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \varphi(\widehat{\boldsymbol{\gamma}}) \rangle)$.

- If $\widehat{\gamma_1} = 0$ or $\widehat{\gamma_1} > p_1$, then consider the automorphism $\varphi : \mathsf{V}(\mathsf{T}_2) \rightarrow \mathsf{V}(\mathsf{T}_2)$ with $\varphi(\boldsymbol{\chi}) = \langle \varphi_1(\chi_1), \varphi_2(\chi_2) \rangle$ for each vertex $\boldsymbol{\chi} \in \mathsf{V}(\mathsf{T}_2)$, where $\varphi_1(\chi_1) = (1 - \chi_1) \bmod 2p_1$ and $\varphi_2(\chi_2) = \chi_2$. Note that $\varphi(\boldsymbol{\alpha}) = \boldsymbol{\beta}$, $\varphi(\boldsymbol{\beta}) = \boldsymbol{\alpha}$ and $\varphi(\widehat{\boldsymbol{\gamma}}) = \langle \varphi_1(\widehat{\gamma_1}), \widehat{\gamma_2} \rangle$ with $1 \leq \varphi_1(\widehat{\gamma_1}) \leq p_1$. Thus, the profiles $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \widehat{\boldsymbol{\gamma}} \rangle$ and $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \varphi(\widehat{\boldsymbol{\gamma}}) \rangle$ are equivalent. Hence, Observation 3.1 implies that $\mathsf{U}_3(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \widehat{\boldsymbol{\gamma}} \rangle) = \mathsf{U}_3(\langle \boldsymbol{\beta}, \boldsymbol{\alpha}, \varphi(\widehat{\boldsymbol{\gamma}}) \rangle) = \mathsf{U}_3(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \varphi(\widehat{\boldsymbol{\gamma}}) \rangle)$, as needed.

Consider now the torus $\widehat{\mathsf{T}}_2[2p_1 - 1, 2p_2]$ derived from the torus $\mathsf{T}_2[2p_1, 2p_2]$ by eliminating row 0; so,

$$\mathsf{V}(\widehat{\mathsf{T}}_2) \quad = \quad \{1, \ldots, 2p_1 - 1\} \times \{0, \ldots, 2p_2 - 1\}$$

and

$$\mathsf{E}(\widehat{\mathsf{T}}_2) \quad = \quad (\mathsf{E}(\mathsf{T}_2) \cap \mathsf{V}(\widehat{\mathsf{T}}_2)^2) \cup \{((1,\chi_2),(2p_1-1,\chi_2)) \mid 0 \leq \chi_2 \leq 2p_1-1\}\,.$$

Note that for every vertex $\boldsymbol{\chi}$ in $\widehat{\mathsf{T}}_2$,

$$\mathsf{dist}_{\widehat{\mathsf{T}}_2}(\boldsymbol{\chi},\boldsymbol{\beta}) \quad = \quad \min\{\mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi},\boldsymbol{\beta}),\mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi},\boldsymbol{\alpha})\}$$

and

$$\mathsf{dist}_{\widehat{\mathsf{T}}_2}(\boldsymbol{\chi},\widehat{\boldsymbol{\gamma}}) \quad = \quad \begin{cases} \mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi},\widehat{\boldsymbol{\gamma}})\,, & \text{if } 1 \leq \chi_1 < \widehat{\gamma}_1 + p_1 \\ \mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi},\widehat{\boldsymbol{\gamma}}) - 1\,, & \text{if } \widehat{\gamma}_1 + p_1 \leq \chi_1 < 2p_1 \end{cases}\,.$$

Consider now the Voronoi game $\langle \widehat{\mathsf{T}}_2, [2]\rangle$, with utility functions $\widehat{\mathsf{U}}_1$ and $\widehat{\mathsf{U}}_2$ for players 1 and 2, respectively. Since $\widehat{\mathsf{T}}_2$ is vertex-transitive, Lemma 4.1 implies that for each profile $\widehat{\mathbf{s}}$, $\widehat{\mathsf{U}}_1(\widehat{\mathbf{s}}) = \widehat{\mathsf{U}}_1(\widehat{\mathbf{s}}) = \frac{1}{2}\,|\mathsf{V}(\widehat{\mathsf{T}}_2)| = (2p_1-1)p_2$. Fix now $\widehat{\mathbf{s}} = \langle \widehat{\boldsymbol{\gamma}},\boldsymbol{\beta}\rangle$ and $\mathbf{s} = \langle \widehat{\boldsymbol{\gamma}},\boldsymbol{\beta},\boldsymbol{\alpha}\rangle$.

We shall evaluate $\mathsf{U}_1(\langle \widehat{\boldsymbol{\gamma}},\boldsymbol{\beta},\boldsymbol{\alpha}\rangle)$ by evaluating the difference to $\widehat{\mathsf{U}}_1(\langle \widehat{\boldsymbol{\gamma}},\boldsymbol{\beta},\rangle)$ due to *(i)* the set of vertices $\{\langle 0,\chi_2\rangle \mid 0 \leq \chi_2 \leq 2p_2-1\}$ (which are not present in $\widehat{\mathsf{T}}_2$), and *(ii)* player 3 (located at $\boldsymbol{\alpha}$). So, write

$$\mathsf{U}_1(\langle \widehat{\boldsymbol{\gamma}},\boldsymbol{\beta},\boldsymbol{\alpha}\rangle) \quad = \quad \widehat{\mathsf{U}}_1(\langle \widehat{\boldsymbol{\gamma}},\boldsymbol{\beta},\rangle) + \sigma_1 - \sigma_2\,,$$

where:

- $\sigma_1 \geq 0$ is the amount added to $\mathsf{U}_1(\langle \widehat{\boldsymbol{\gamma}},\boldsymbol{\beta},\boldsymbol{\alpha}\rangle)$ due to vertices in the set $\{\langle 0,\chi_2\rangle \mid 0 \leq \chi_2 \leq 2p_2-1\}$; this amount is a *loss* for player 1 in the Voronoi game $\langle \widehat{\mathsf{T}}_2,[2]\rangle$.
- $\sigma_2 \geq 0$ is the amount subtracted from $\mathsf{U}_1(\langle \widehat{\boldsymbol{\gamma}},\boldsymbol{\beta},\boldsymbol{\alpha}\rangle)$ due to vertices $\boldsymbol{\chi} = \langle \chi_1,\chi_2\rangle$ with $\widehat{\gamma}_1 + p_1 \leq \chi_1 < 2p_1$; this amount is a *win* for player 1 in the Voronoi game $\langle \widehat{\mathsf{T}}_2,[2]\rangle$. (Note that there are vertices in this set which were closest to player 3 (located at $\boldsymbol{\alpha} = \langle 0,0\rangle$) in the profile $\langle \widehat{\boldsymbol{\gamma}},\boldsymbol{\beta},\boldsymbol{\alpha}\rangle$, but closest to player 1 in the profile $\langle \widehat{\boldsymbol{\gamma}},\boldsymbol{\beta}\rangle$.)

(Note that all vertices $\boldsymbol{\chi} = \langle \chi_1,\chi_2\rangle$ with $1 \leq \chi_1 < \widehat{\gamma}_1 + p_1$ are not closest to player 1; hence, they contribute the same amount to the utilities of player 1 in the profiles $\langle \widehat{\boldsymbol{\gamma}},\boldsymbol{\beta},\boldsymbol{\alpha}\rangle$ and $\langle \widehat{\boldsymbol{\gamma}},\boldsymbol{\beta}\rangle$), respectively.)
For the rest of the proof, we shall estimate σ_1 and σ_2. We proceed by case analysis on the relation between $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$.

1. Assume first that $\widehat{\gamma}_1 > \widehat{\gamma}_2$. We shall prove that $\mathsf{U}_1(\langle \widehat{\boldsymbol{\gamma}},\boldsymbol{\beta},\boldsymbol{\alpha}\rangle) \leq (2p_1-1)p_2$, with $\mathsf{U}_1(\langle \widehat{\boldsymbol{\gamma}},\boldsymbol{\beta},\boldsymbol{\alpha}\rangle) = (2p_1-1)p_2$ in the special case where $\widehat{\gamma}_1 = p_1$.

   We first prove that $\sigma_1 = 0$. Consider any vertex $\langle 0,\chi_2\rangle$ with $0 \leq \chi_2 \leq 2p_2-1$. Clearly,

$$\begin{array}{lcl} & \mathsf{dist}_{\mathsf{T}_2}(\langle 0,\chi_2\rangle,\boldsymbol{\gamma}) & \\ = & \widehat{\gamma}_1 + \mathsf{dist}_{\mathsf{T}_2}(\langle 0,\chi_2\rangle,\langle 0,\widehat{\gamma}_2\rangle) & (\text{since } 1 \leq \widehat{\gamma}_1 \leq p_1) \\ > & \widehat{\gamma}_2 + \mathsf{dist}_{\mathsf{T}_2}(\langle 0,\chi_2\rangle,\langle 0,\widehat{\gamma}_2\rangle) & (\text{since } \widehat{\gamma}_1 > \widehat{\gamma}_2) \\ = & \mathsf{dist}_{\mathsf{T}_2}(\langle 0,0\rangle,\langle 0,\widehat{\gamma}_2\rangle) + \mathsf{dist}_{\mathsf{T}_2}(\langle 0,\chi_2\rangle,\langle 0,\widehat{\gamma}_2\rangle) & (\text{since } 0 \leq \widehat{\gamma}_2 \leq p_2) \\ \geq & \mathsf{dist}_{\mathsf{T}_2}(\langle 0,0\rangle,\langle 0,\chi_2\rangle) & (\text{by the triangle inequality}) \\ = & \mathsf{dist}_{\mathsf{T}_2}(\langle 0,\chi_2\rangle,\boldsymbol{\alpha})\,. & \end{array}$$

   This implies that $\langle 0,\chi_2\rangle \notin \mathsf{Vor}_1(\langle \widehat{\boldsymbol{\gamma}},\boldsymbol{\beta},\boldsymbol{\alpha}\rangle)$. Since $\langle 0,\chi_2\rangle$ was chosen arbitrarily, it follows that $\sigma_1 = 0$. Hence,

$$\begin{array}{rcl} \mathsf{U}_1(\langle \widehat{\boldsymbol{\gamma}},\boldsymbol{\beta},\boldsymbol{\alpha}\rangle) & = & \widehat{\mathsf{U}}_1(\langle \widehat{\boldsymbol{\gamma}},\boldsymbol{\beta},\rangle) - \sigma_2 \\ & \leq & \widehat{\mathsf{U}}_1(\langle \widehat{\boldsymbol{\gamma}},\boldsymbol{\beta},\rangle) \\ & = & (2p_1-1)p_2\,. \end{array}$$

   Note that for $\widehat{\gamma}_1 = p_1$, there is no vertex $\boldsymbol{\chi} = \langle \chi_1,\chi_2\rangle$ with $\widehat{\gamma}_1 + p_1 \leq \chi_1 < 2p_1$; so, in this case, $\sigma_2 = 0$ and

$$\begin{array}{rcl} \mathsf{U}_1(\langle \widehat{\boldsymbol{\gamma}},\boldsymbol{\beta},\boldsymbol{\alpha}\rangle) & = & \widehat{\mathsf{U}}_1(\langle \widehat{\boldsymbol{\gamma}},\boldsymbol{\beta},\rangle) \\ & = & (2p_1-1)p_2\,. \end{array}$$

2. Assume now that $\widehat{\gamma}_1 = \widehat{\gamma}_2$. We shall prove that $\mathsf{U}_1(\langle \widehat{\boldsymbol{\gamma}}, \boldsymbol{\beta}, \boldsymbol{\alpha}\rangle) \leq (2p_1 - 1)p_2$. To do so, we shall calculate σ_1 and establish an upper bound on σ_2.

   To calculate σ_1, consider a vertex $\langle 0, \chi_2\rangle$ with $0 \leq \chi_2 \leq 2p_2 - 1$. We proceed by case analysis.

   (a) Assume first that $0 \leq \chi_2 < \widehat{\gamma}_2$. Since $\widehat{\gamma}_2 \leq p_2$, it follows that $\chi_2 < p_2$. Then,

$$\begin{array}{rlr} & \mathsf{dist}_{\mathsf{T}_2}(\langle 0, \chi_2\rangle, \boldsymbol{\alpha}) & \\ = & \mathsf{dist}(0, \chi_2) & \\ = & \chi_2 & (\text{since } \chi_2 < p_2)\,, \end{array}$$

   while

$$\begin{array}{rlr} & \mathsf{dist}_{\mathsf{T}_2}(\langle 0, \chi_2\rangle, \widehat{\boldsymbol{\gamma}}) & \\ = & \mathsf{dist}(0, \widehat{\gamma}_1) + \mathsf{dist}(\chi_2, \widehat{\gamma}_2) & \\ = & \widehat{\gamma}_1 + \widehat{\gamma}_2 - \chi_2 & (\text{since } 0 \leq \chi_2 < \widehat{\gamma}_2) \\ = & 2\widehat{\gamma}_2 - \chi_2 & (\text{since } \widehat{\gamma}_1 = \widehat{\gamma}_2)\,. \end{array}$$

   Since $\chi_2 < \widehat{\gamma}_2$, $\chi_2 < 2\widehat{\gamma}_2 - \chi_2$. This implies that $\mathsf{dist}_{\mathsf{T}_2}(\langle 0, \chi_2\rangle, \boldsymbol{\alpha}) < \mathsf{dist}_{\mathsf{T}_2}(\langle 0, \chi_2\rangle, \widehat{\boldsymbol{\gamma}})$. Hence, $\langle 0, \chi_2\rangle \notin \mathsf{Vor}_1(\langle \widehat{\boldsymbol{\gamma}}, \boldsymbol{\beta}, \boldsymbol{\alpha}\rangle)$.

   (b) Assume now that $\widehat{\gamma}_2 \leq \chi_2 \leq p_2$. Then,

$$\begin{array}{rlr} & \mathsf{dist}_{\mathsf{T}_2}(\langle 0, \chi_2\rangle, \widehat{\boldsymbol{\gamma}}) & \\ = & \mathsf{dist}(0, \widehat{\gamma}_1) + \mathsf{dist}(\chi_2, \widehat{\gamma}_2) & \\ = & \widehat{\gamma}_1 + (\chi_2 - \widehat{\gamma}_2) & (\text{since } 1 \leq \widehat{\gamma}_1 \leq p_1) \\ = & \chi_2 & (\text{since } \widehat{\gamma}_1 = \widehat{\gamma}_2) \\ = & \mathsf{dist}_{\mathsf{T}_2}(\langle 0, \chi_2\rangle, \boldsymbol{\alpha}) & (\text{since } \chi_2 \leq p_2)\,. \end{array}$$

   Hence, the vertex $\langle 0, \chi_2\rangle$ contributes $\frac{1}{2}$ to each of the utilities of players 1 and 3 in the profile $\langle \widehat{\boldsymbol{\gamma}}, \boldsymbol{\beta}, \boldsymbol{\alpha}\rangle$.

   (c) Assume finally that $p_2 < \chi_2 \leq 2p_2 - 1$. Since $\widehat{\gamma}_2 \leq p_2$, it follows that $\widehat{\gamma}_2 < \chi_2$. Then,

$$\begin{array}{rlr} & \mathsf{dist}_{\mathsf{T}_2}(\langle 0, \chi_2\rangle, \boldsymbol{\alpha}) & \\ = & \mathsf{dist}(0, \chi_2) & \\ = & 2p_2 - \chi_2 & (\text{since } p_2 < \chi_2 \leq 2p_2 - 1)\,, \end{array}$$

   while

$$\begin{array}{rlr} & \mathsf{dist}_{\mathsf{T}_2}(\langle 0, \chi_2\rangle, \widehat{\boldsymbol{\gamma}}) & \\ = & \mathsf{dist}(0, \widehat{\gamma}_1) + \mathsf{dist}(\chi_2, \widehat{\gamma}_2) & \\ = & \widehat{\gamma}_1 + \min\{\chi_2 - \widehat{\gamma}_2, 2p_2 - (\chi_2 - \widehat{\gamma}_2)\} & (\text{since } \widehat{\gamma}_2 < \chi_2)\,. \end{array}$$

   - Assume first that $2p_2 - (\chi_2 - \widehat{\gamma}_2) \leq \chi_2 - \widehat{\gamma}_2$. Then,

$$\begin{array}{rcl} \mathsf{dist}_{\mathsf{T}_2}(\langle 0, \chi_2\rangle, \widehat{\boldsymbol{\gamma}}) & = & \widehat{\gamma}_1 + 2p_2 - (\chi_2 - \widehat{\gamma}_2) \\ & > & 2p_2 - \chi_2 \\ & = & \mathsf{dist}_{\mathsf{T}_2}(\langle 0, \chi_2\rangle, \boldsymbol{\alpha})\,. \end{array}$$

   - Assume now that $2p_2 - (\chi_2 - \widehat{\gamma}_2) > \chi_2 - \widehat{\gamma}_2$, or $p_2 > \chi_2 - \widehat{\gamma}_2$. Then,

$$\begin{array}{rlr} & \mathsf{dist}_{\mathsf{T}_2}(\langle 0, \chi_2\rangle, \widehat{\boldsymbol{\gamma}}) & \\ = & \widehat{\gamma}_1 + \chi_2 - \widehat{\gamma}_2 & \\ = & \chi_2 & (\text{since } \widehat{\gamma}_1 = \widehat{\gamma}_2) \\ > & 2p_2 - \chi_2 & (\text{since } p_2 < \chi_2) \\ = & \mathsf{dist}_{\mathsf{T}_2}(\langle 0, \chi_2\rangle, \boldsymbol{\alpha})\,. & \end{array}$$

So, in all cases, $\mathsf{dist}_{\mathsf{T}_2}(\langle 0, \chi_2\rangle, \boldsymbol{\alpha}) < \mathsf{dist}_{\mathsf{T}_2}(\langle 0, \chi_2\rangle, \widehat{\boldsymbol{\gamma}})$. Hence, $\langle 0, \chi_2\rangle \notin \mathsf{Vor}_1(\langle \widehat{\boldsymbol{\gamma}}, \boldsymbol{\beta}, \boldsymbol{\alpha}\rangle)$.

It follows that

$$\begin{aligned}\sigma_1 &= \frac{1}{2}\left|\{\langle 0, \chi_2\rangle \mid \widehat{\gamma}_2 \leq \chi_2 \leq p_2\}\right| \\ &= \frac{1}{2}\left(p_2 - \widehat{\gamma}_2 + 1\right).\end{aligned}$$

We now prove a lower bound on σ_2. We only consider vertices $\langle 2p_2 - 1, \chi_2\rangle$ with $\widehat{\gamma}_2 \leq \chi_2 \leq p_2$. The rest of the proof employs some similar technical arguments; it is omitted.

3. Assume finally that $\widehat{\gamma}_1 < \widehat{\gamma}_2$. We shall prove that $\mathsf{U}_1(\langle \widehat{\boldsymbol{\gamma}}, \boldsymbol{\beta}, \boldsymbol{\alpha}\rangle) \leq (2p_1 - 1)p_2$. The proof is similar to the one for the case $\widehat{\gamma}_1 > \widehat{\gamma}_2$; it is omitted.

C.2 Lemma 7.3

For (1), note that

$$\begin{aligned}&\mathsf{U}_2(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\rangle) \\ =\ & |\mathsf{V}(\mathsf{T}_2)| - \mathsf{U}_2(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\rangle) - \mathsf{U}_3(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\rangle) \\ =\ & 4p_1p_2 - p_1p_2 - (2p_1 - 1)p_2 \qquad \text{(by Corollary 5.3 and Lemma 7.2)} \\ =\ & p_1p_2 + p_2\,,\end{aligned}$$

as needed. For (2), fix a vertex $\widehat{\boldsymbol{\alpha}} \in \mathsf{V}(\mathsf{T}_2)$. Consider the automorphism $\phi : \mathsf{V}(\mathsf{T}_2) \rightarrow \mathsf{V}(\mathsf{T}_2)$ such that for each vertex $\boldsymbol{\chi} \in \mathsf{V}(\mathsf{T}_2)$, $\phi(\boldsymbol{\chi}) = \langle (\chi_1 - 1) \bmod 2p_1, \chi_2\rangle$. Set $\boldsymbol{\alpha}_1 = \phi(\boldsymbol{\beta})$, $\widehat{\boldsymbol{\beta}}_1 = \phi(\boldsymbol{\gamma})$ and $\boldsymbol{\gamma}_1 = \phi(\widehat{\boldsymbol{\alpha}})$; denote $\boldsymbol{\gamma}_1 = \langle \gamma_1, \gamma_2\rangle$. So, ϕ transforms the profile $\langle \widehat{\boldsymbol{\alpha}}, \boldsymbol{\beta}, \boldsymbol{\gamma}\rangle$ into the profile $\langle \phi(\widehat{\boldsymbol{\alpha}}), \phi(\boldsymbol{\beta}), \phi(\boldsymbol{\gamma})\rangle = \langle \boldsymbol{\gamma}_1, \boldsymbol{\alpha}_1, \widehat{\boldsymbol{\beta}}_1\rangle$. It follows that the profiles $\langle \widehat{\boldsymbol{\alpha}}, \boldsymbol{\beta}, \boldsymbol{\gamma}\rangle$ and $\langle \phi(\widehat{\boldsymbol{\alpha}}), \phi(\boldsymbol{\beta}), \phi(\boldsymbol{\gamma})\rangle = \langle \boldsymbol{\gamma}_1, \boldsymbol{\alpha}_1, \widehat{\boldsymbol{\beta}}_1\rangle$ are equivalent. Hence, Observation 3.1 implies that $\mathsf{U}_1(\langle \phi(\widehat{\boldsymbol{\alpha}}), \phi(\boldsymbol{\beta}), \phi(\boldsymbol{\gamma})\rangle) = \mathsf{U}_1(\langle \boldsymbol{\gamma}_1, \boldsymbol{\alpha}_1, \widehat{\boldsymbol{\beta}}_1\rangle) = \mathsf{U}_3(\langle \boldsymbol{\alpha}_1, \widehat{\boldsymbol{\beta}}_1, \boldsymbol{\gamma}_1\rangle)$. We observe that $\boldsymbol{\alpha}_1 = \langle 0, 0\rangle = \boldsymbol{\alpha}$ and $\widehat{\boldsymbol{\beta}}_1 = \langle p_1 - 1, p_2\rangle$.

Furthermore, denote $\boldsymbol{\beta}_1 = \langle p_1, p_2\rangle$. Note that $\boldsymbol{\beta}_1 = \boldsymbol{\gamma}$. (We introduced this redundancy in notation only because $\boldsymbol{\beta}_1$ makes more explicit the reference to player 2.) Clearly, $\boldsymbol{\alpha}_1$ and $\boldsymbol{\beta}_1$ are an antipodal pair of vertices in T_2 with $\mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\alpha}_1, \boldsymbol{\beta}_1) = \sum_{j\in[2]} p_j$. Recall that $\mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\alpha}_1, \boldsymbol{\gamma}_1) + \mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\beta}_1, \boldsymbol{\gamma}_1) = \mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)$. Since $\sum_{j\in[2]} p_j$ is odd, it follows that $\mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\alpha}_1, \boldsymbol{\gamma}_1)$ and $\mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\beta}_1, \boldsymbol{\gamma}_1)$ have different parity. Furthermore, the *Two-Guards Theorem for Bipartite, Odd-Diameter Friendly Graphs* (Corollary 5.3) implies that $\mathsf{U}_3(\langle \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \boldsymbol{\gamma}_1\rangle) = \frac{1}{4}(2p_1)(2p_2) = p_1p_2$.

Loosely speaking, we shall evaluate the influence of the location of player 2 ($\boldsymbol{\beta}_1$ and $\widehat{\beta}_1$, respectively) on the utility of player 3 ($\mathsf{U}_3(\langle \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \boldsymbol{\gamma}_1\rangle)$ and $\mathsf{U}_3(\langle \boldsymbol{\alpha}_1, \widehat{\boldsymbol{\beta}}_1\boldsymbol{\gamma}_1\rangle)$, respectively). For each index $\ell \in \{0, 1, 2\}$, consider the sets $\mathcal{A}_\ell = \mathcal{A}_\ell(\langle \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \boldsymbol{\gamma}_1\rangle)$, $\mathcal{B}_\ell = \mathcal{B}_\ell(\langle \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \boldsymbol{\gamma}_1\rangle)$ and $\widehat{\mathcal{B}}_\ell = \mathcal{B}_\ell(\langle \boldsymbol{\alpha}_1, \widehat{\boldsymbol{\beta}}_1, \boldsymbol{\gamma}_1\rangle)$.

Recall the automorphism Ψ from Section 3, which is induced by the antipodal pair of vertices $\boldsymbol{\alpha} = \langle 0, 0\rangle$ and $\boldsymbol{\gamma} = \langle p_1, p_2\rangle$; so, $\Psi(\boldsymbol{\alpha}) = \boldsymbol{\gamma}$ and $\Psi(\boldsymbol{\gamma}) = \boldsymbol{\alpha}$. We shall distinguish the two cases $0 \leq \gamma_1 < p_1$ and $p_1 \leq \gamma_1 < 2p_1 - 1$. Define $\gamma = \gamma_1$ if $0 \leq \gamma_1 < p_1$, and $\gamma = \gamma_1 - p_1$ if $p_1 \leq \gamma_1 \leq 2p_1 - 1$; so, in all cases, $0 \leq \gamma < p_1$. Induced by γ is a partition $\mathsf{V}(\mathsf{T}_2)$ into the four sets

$$\begin{aligned}\mathcal{I}_1 &= \{\boldsymbol{\chi} \in \mathsf{V}(\mathsf{T}_2) \mid 0 \leq \chi_1 \leq \gamma\}, \\ \mathcal{I}_2 &= \{\boldsymbol{\chi} \in \mathsf{V}(\mathsf{T}_2) \mid \gamma < \chi_1 < p_1\}, \\ \mathcal{I}_3 &= \{\boldsymbol{\chi} \in \mathsf{V}(\mathsf{T}_2) \mid p_1 \leq \chi_1 \leq p_1 + \gamma\}\end{aligned}$$

and

$$\mathcal{I}_4 = \{\boldsymbol{\chi} \in \mathsf{V}(\mathsf{T}_2) \mid p_1 + \gamma < \chi_1 < 2p_1\}.$$

We observe:

Observation C.1 *For each vertex $\boldsymbol{\chi} \in \mathsf{V}(\mathsf{T}_2)$, the following hold:*

(1) *Assume that* $\gamma = \gamma_1$. *Then,* $\boldsymbol{\chi} \in \mathcal{I}_1$ *if and only if* $\Psi(\boldsymbol{\chi}) \in \mathcal{I}_3$.

(2) *Assume that* $\gamma = \gamma_1 - p_1$. *Then,* $\boldsymbol{\chi} \in \mathcal{I}_2$ *if and only if* $\Psi(\boldsymbol{\chi}) \in \mathcal{I}_4$.

Note that for each vertex $\boldsymbol{\chi} \in \mathsf{V}(\mathsf{T}_2)$,

$$\mathsf{dist}(\boldsymbol{\chi}, \widehat{\boldsymbol{\beta}}_1) = \begin{cases} \mathsf{dist}(\boldsymbol{\chi}, \boldsymbol{\beta}_1) - 1\,, & \text{if } \boldsymbol{\chi} \in \mathcal{I}_1 \cup \mathcal{I}_2 \\ \mathsf{dist}(\boldsymbol{\chi}, \boldsymbol{\beta}_1) + 1\,, & \text{if } \boldsymbol{\chi} \in \mathcal{I}_3 \cup \mathcal{I}_4 \end{cases}$$

So, for each vertex $\boldsymbol{\chi} \in \mathsf{V}(\mathsf{T}_2)$, the distances $\mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \widehat{\boldsymbol{\beta}}_1)$ and $\mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \boldsymbol{\beta}_1)$ have different parities. We proceed by case analysis on the parity of $\mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\beta}_1, \boldsymbol{\gamma}_1)$.

1. Assume first that $\mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\beta}_1, \boldsymbol{\gamma}_1)$ is even. Then, clearly, $\mathcal{A}_1(\langle \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \boldsymbol{\gamma}_1 \rangle) = \widehat{\mathcal{A}}_1(\langle \boldsymbol{\alpha}_1, \widehat{\boldsymbol{\beta}}_1, \boldsymbol{\gamma}_1 \rangle) = \widehat{\mathcal{B}}_1(\langle \boldsymbol{\alpha}_1, \widehat{\boldsymbol{\beta}}_1, \boldsymbol{\gamma}_1 \rangle) = \emptyset$. Hence,

$$\begin{aligned} \mathsf{U}_3(\langle \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \boldsymbol{\gamma}_1 \rangle) &= |\mathcal{A}_0 \cap \mathcal{B}_0| + \frac{1}{2}\, |\mathcal{A}_0 \cap \mathcal{B}_1| \\ &= |\mathcal{A}_0 \cap \mathcal{B}_0| + \frac{1}{2}\, |\mathcal{A}_0 \cap \mathcal{B}_1 \cap (\mathcal{I}_1 \cup \mathcal{I}_2)| + \frac{1}{2}\, |\mathcal{A}_0 \cap \mathcal{B}_1 \cap (\mathcal{I}_3 \cup \mathcal{I}_4)| \end{aligned}$$

and

$$\mathsf{U}_3(\langle \boldsymbol{\alpha}_1, \widehat{\boldsymbol{\beta}}_1, \boldsymbol{\gamma}_1 \rangle) = |\mathcal{A}_0 \cap \widehat{\mathcal{B}}_0|\,.$$

Recall that $\mathsf{U}_3(\langle \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \boldsymbol{\gamma}_1 \rangle) = p_1 p_2$. We continue to evaluate $\mathsf{U}_3(\langle \boldsymbol{\alpha}_1, \widehat{\boldsymbol{\beta}}_1, \boldsymbol{\gamma}_1 \rangle)$. By the definition of $\boldsymbol{\beta}_1$ and $\widehat{\boldsymbol{\beta}}_1$, it follows that

$$\begin{aligned} \widehat{\mathcal{B}}_0 &= \mathcal{B}_0 \cup \{\boldsymbol{\chi} \in \mathcal{B}_1 \mid \mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \boldsymbol{\gamma}_1) < \mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \widehat{\boldsymbol{\beta}}_1)\} \\ &= \mathcal{B}_0 \cup \{\boldsymbol{\chi} \in \mathcal{B}_1 \cap (\mathcal{I}_1 \cup \mathcal{I}_2) \mid \mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \boldsymbol{\gamma}_1) < \mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \widehat{\boldsymbol{\beta}}_1)\} \\ &\quad \cup \{\boldsymbol{\chi} \in \mathcal{B}_1 \cap (\mathcal{I}_3 \cup \mathcal{I}_4) \mid \mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \boldsymbol{\gamma}_1) < \mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \widehat{\boldsymbol{\beta}}_1)\}\,. \end{aligned}$$

Note that for every vertex $\boldsymbol{\chi} \in \mathcal{B}_1 \cap (\mathcal{I}_1 \cup \mathcal{I}_2)$, $\mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \boldsymbol{\gamma}_1) > \mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \widehat{\boldsymbol{\beta}}_1)$; for every vertex $\boldsymbol{\chi} \in \mathcal{B}_1 \cap (\mathcal{I}_3 \cup \mathcal{I}_4)$, $\mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \boldsymbol{\gamma}_1) < \mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \widehat{\boldsymbol{\beta}}_1)$. It follows that

$$\mathcal{A}_0 \cap \widehat{\mathcal{B}}_0 = (\mathcal{A}_0 \cap \mathcal{B}_0) \cup (\mathcal{A}_0 \cap \mathcal{B}_1 \cap (\mathcal{I}_3 \cup \mathcal{I}_4))\,.$$

Since $\mathcal{B}_0$ and $\mathcal{B}_1$ are disjoint, this implies that

$$|\mathcal{A}_0 \cap \widehat{\mathcal{B}}_0| = |\mathcal{A}_0 \cap \mathcal{B}_0| + |\mathcal{A}_0 \cap \mathcal{B}_1 \cap (\mathcal{I}_3 \cup \mathcal{I}_4)|\,.$$

This implies that

$$\begin{aligned} &\mathsf{U}_3(\langle \boldsymbol{\alpha}_1, \widehat{\boldsymbol{\beta}}_1, \boldsymbol{\gamma}_1 \rangle) \\ &= \mathsf{U}_3(\langle \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \boldsymbol{\gamma}_1 \rangle) - \frac{1}{2}\, |\mathcal{A}_0 \cap \mathcal{B}_1 \cap (\mathcal{I}_1 \cup \mathcal{I}_2)| - \frac{1}{2}\, |\mathcal{A}_0 \cap \mathcal{B}_1 \cap (\mathcal{I}_3 \cup \mathcal{I}_4)| + |\mathcal{A}_0 \cap \mathcal{B}_1 \cap (\mathcal{I}_3 \cup \mathcal{I}_4)| \\ &= p_1 p_2 + \frac{1}{2}\, |\mathcal{A}_0 \cap \mathcal{B}_1 \cap (\mathcal{I}_3 \cup \mathcal{I}_4)| - \frac{1}{2}\, |\mathcal{A}_0 \cap \mathcal{B}_1 \cap (\mathcal{I}_1 \cup \mathcal{I}_2)|\,. \end{aligned}$$

We proceed by case analysis.

(a) Assume first that $\gamma = \gamma_1$. Then, by Lemma 5.1 (Conditions (C.1) and (C.3)) and Observation C.1 (Condition (C.1)), it follows that $|\mathcal{A}_0 \cap \mathcal{B}_1 \cap \mathcal{I}_3| = |\mathcal{A}_0 \cap \mathcal{B}_1 \cap \mathcal{I}_1|$. Hence,

$$\begin{aligned} \mathsf{U}_3(\langle \boldsymbol{\alpha}_1, \widehat{\boldsymbol{\beta}}_1, \boldsymbol{\gamma}_1 \rangle) &= p_1 p_2 + \frac{1}{2}\, |\mathcal{A}_0 \cap \mathcal{B}_1 \cap \mathcal{I}_4| - \frac{1}{2}\, |\mathcal{A}_0 \cap \mathcal{B}_1 \cap \mathcal{I}_2| \\ &\leq p_1 p_2 + \frac{1}{2}\, |\mathcal{A}_0 \cap \mathcal{B}_1 \cap \mathcal{I}_4|\,. \end{aligned}$$

— Fix a vertex $\boldsymbol{\chi} \in \mathcal{B}_1 \cap \mathcal{I}_4$. Since $\gamma = \gamma_1$,

$$\begin{aligned} \mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \boldsymbol{\gamma}) &= \mathsf{dist}_1(\chi_1, \gamma_1) + \mathsf{dist}_2(\chi_2, \gamma_2) \\ &= 2p_1 - 1 - \chi_1 + \gamma_1 + \mathsf{dist}_2(\chi_2, \gamma_2) \end{aligned}$$

and

$$\begin{aligned} \mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \boldsymbol{\beta}_1) &= \mathsf{dist}_1(\chi_1, p_1) + \mathsf{dist}_2(\chi_2, p_2) \\ &= \chi_1 - p_1 + \mathsf{dist}_2(\chi_2, p_2) \,. \end{aligned}$$

Since $\boldsymbol{\chi} \in \mathcal{B}_1$, $\mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \boldsymbol{\gamma}) = \mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \boldsymbol{\beta}_1)$, so that

$$\chi_1 = \frac{1}{2} \left(3p_1 - 1 + \gamma_1 + \mathsf{dist}_2(\chi_2, \gamma_2) - \mathsf{dist}_2(\chi_2, p_2)\right) .$$

It follows that for every χ_2 with $0 \leq \chi_2 < 2p_2 - 1$, there is at most one χ_1 with $p_1 + \gamma < \chi_1 < 2p_1$; hence, there is at most one vertex $\boldsymbol{\chi} \in \mathcal{I}_4$, which implies that there is at most one vertex $\boldsymbol{\chi} \in \mathcal{B}_1 \cap \mathcal{I}_4$. It follows that $|\mathcal{B}_1 \cap \mathcal{I}_4| \leq 2p_2$, and the claim follows.

(b) Assume now that $\gamma_1 = \gamma + p_1$. Then, by Lemma 5.1 (Conditions (C.1) and (C.3)) and Observation C.1 (Condition (C.2)), it follows that $|\mathcal{A}_0 \cap \mathcal{B}_1 \cap \mathcal{I}_2| = |\mathcal{A}_0 \cap \mathcal{B}_1 \cap \mathcal{I}_4|$. Hence,

$$\begin{aligned} \mathsf{U}_3(\langle \boldsymbol{\alpha}_1, \widehat{\boldsymbol{\beta}}_1, \boldsymbol{\gamma}_1 \rangle) &= p_1 p_2 + \frac{1}{2} |\mathcal{A}_0 \cap \mathcal{B}_1 \cap \mathcal{I}_3| - \frac{1}{2} |\mathcal{A}_0 \cap \mathcal{B}_1 \cap \mathcal{I}_1| \\ &\leq p_1 p_2 + \frac{1}{2} |\mathcal{A}_0 \cap \mathcal{B}_1 \cap \mathcal{I}_3| \,. \end{aligned}$$

— Fix a vertex $\boldsymbol{\chi} \in \mathcal{B}_1 \cap \mathcal{I}_3$. Since $\gamma = \gamma_1 - p_1$,

$$\begin{aligned} \mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \boldsymbol{\gamma}) &= \mathsf{dist}_1(\chi_1, \gamma_1) + \mathsf{dist}_2(\chi_2, \gamma_2) \\ &= \gamma_1 - \chi_1 + \mathsf{dist}_2(\chi_2, \gamma_2) \end{aligned}$$

and

$$\begin{aligned} \mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \boldsymbol{\beta}_1) &= \mathsf{dist}_1(\chi_1, p_1) + \mathsf{dist}_2(\chi_2, p_2) \\ &= \chi_1 - p_1 + \mathsf{dist}_2(\chi_2, p_2) \,. \end{aligned}$$

Since $\boldsymbol{\chi} \in \mathcal{B}_1$, $\mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \boldsymbol{\gamma}) = \mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \boldsymbol{\beta}_1)$, so that

$$\chi_1 = \frac{1}{2} \left(p_1 + \gamma_1 + \mathsf{dist}_2(\chi_2, \gamma_2) - \mathsf{dist}_2(\chi_2, p_2)\right) .$$

It follows that for every χ_2 with $0 \leq \chi_2 < 2p_2 - 1$, there is at most one χ_1 with $p_1 \leq \chi_1 \leq p_1 + \gamma$; hence, there is at most one vertex $\boldsymbol{\chi} \in \mathcal{I}_3$, which implies that there is at most one vertex $\boldsymbol{\chi} \in \mathcal{B}_1 \cap \mathcal{I}_3$. It follows that $|\mathcal{B}_1 \cap \mathcal{I}_3| \leq 2p_2$, and the claim follows.

2. Assume now that $\mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\beta}_1, \boldsymbol{\gamma}_1)$ is odd. Then, $\mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\alpha}_1, \boldsymbol{\gamma}_1)$ and $\mathsf{dist}_{\mathsf{T}_2}(\widehat{\boldsymbol{\beta}}_1, \boldsymbol{\gamma}_1)$ are both even. Since $\mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\beta}_1, \boldsymbol{\gamma}_1)$ is odd, $\mathcal{B}_1 = \emptyset$. Clearly,

$$\mathcal{B}_0 = \widehat{\mathcal{B}_0} \cup \widehat{\mathcal{B}}_1 \cap (\mathcal{I}_1 \cup \mathcal{I}_2)$$

and

$$\mathcal{B}_2 = \widehat{\mathcal{B}}_2 \cup \widehat{\mathcal{B}}_1 \cap (\mathcal{I}_3 \cup \mathcal{I}_4) \,.$$

Hence,

$$\begin{aligned} \mathsf{U}_3(\langle \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \boldsymbol{\gamma}_1 \rangle) &= |\mathcal{A}_0 \cap \mathcal{B}_0| + \frac{1}{2} |\mathcal{A}_1 \cap \mathcal{B}_0| \\ &= |\mathcal{A}_0 \cap \widehat{\mathcal{B}}_0| + |\mathcal{A}_0 \cap \widehat{\mathcal{B}}_1 \cap (\mathcal{I}_1 \cup \mathcal{I}_2)| + \frac{1}{2} |\mathcal{A}_1 \cap \widehat{\mathcal{B}}_0| + \frac{1}{2} |\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1 \cap (\mathcal{I}_1 \cup \mathcal{I}_2)| \,. \end{aligned}$$

Since

$$\mathsf{U}_3(\langle \boldsymbol{\alpha}_1, \widehat{\boldsymbol{\beta}}_1, \boldsymbol{\gamma}_1\rangle) = |\mathcal{A}_0 \cap \widehat{\mathcal{B}}_0| + \frac{1}{2}|\mathcal{A}_1 \cap \widehat{\mathcal{B}}_0| + \frac{1}{3}|\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1|,$$

it follows that

$$\begin{aligned}
\mathsf{U}_3(\langle \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \boldsymbol{\gamma}_1\rangle) &= \mathsf{U}_3(\langle \boldsymbol{\alpha}_1, \widehat{\boldsymbol{\beta}}_1, \boldsymbol{\gamma}_1\rangle) - \frac{1}{2}|\mathcal{A}_0 \cap \widehat{\mathcal{B}}_1 \cap (\mathcal{I}_3 \cup \mathcal{I}_4)| + \frac{1}{2}|\mathcal{A}_0 \cap \hat{\mathcal{B}}_1 \cap (\mathcal{I}_1 \cup \mathcal{I}_2)| \\
&\quad -\frac{1}{3}|\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1| + \frac{1}{2}|\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1 \cap (\mathcal{I}_1 \cup \mathcal{I}_2)| \\
&= \mathsf{U}_3(\langle \boldsymbol{\alpha}_1, \widehat{\boldsymbol{\beta}}_1, \boldsymbol{\gamma}_1\rangle) - \frac{1}{2}|\mathcal{A}_0 \cap \widehat{\mathcal{B}}_1 \cap (\mathcal{I}_3 \cup \mathcal{I}_4)| + \frac{1}{2}|\mathcal{A}_0 \cap \hat{\mathcal{B}}_1 \cap (\mathcal{I}_1 \cup \mathcal{I}_2)| \\
&\quad +\frac{1}{6}|\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1 \cap (\mathcal{I}_1 \cup \mathcal{I}_2)| - \frac{1}{3}|\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1 \cap (\mathcal{I}_3 \cup \mathcal{I}_4)|.
\end{aligned}$$

We proceed by case analysis.

(a) Assume first that $\gamma = \gamma_1$. Then, by Lemma 5.1 (Conditions (C.1) and (C.3)) and Observation C.1 (Condition (C.1)), it follows that $|\mathcal{A}_0 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_1| = |\mathcal{A}_0 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_3|$ and $|\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_1| = |\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_3|$. Hence,

$$\begin{aligned}
\mathsf{U}_3(\langle \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \boldsymbol{\gamma}_1\rangle) &= \mathsf{U}_3(\langle \boldsymbol{\alpha}_1, \widehat{\boldsymbol{\beta}}_1, \boldsymbol{\gamma}_1\rangle) - \frac{1}{2}|\mathcal{A}_0 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_4| + \frac{1}{2}|\mathcal{A}_0 \cap \hat{\mathcal{B}}_1 \cap \mathcal{I}_2| \\
&\quad +\frac{1}{6}|\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_2| - \frac{1}{3}|\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_4| - \frac{1}{6}|\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_3|.
\end{aligned}$$

which implies that

$$\begin{aligned}
\mathsf{U}_3(\langle \boldsymbol{\alpha}_1, \widehat{\boldsymbol{\beta}}_1, \boldsymbol{\gamma}_1\rangle) &= p_1 p_2 + \frac{1}{2}|\mathcal{A}_0 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_4| - \frac{1}{2}|\mathcal{A}_0 \cap \hat{\mathcal{B}}_1 \cap \mathcal{I}_2| \\
&\quad -\frac{1}{6}|\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_2| + \frac{1}{3}|\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_4| + \frac{1}{6}|\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_3|.
\end{aligned}$$

We observe that $\widehat{\mathcal{B}}_1 \cap \mathcal{I}_3 = \emptyset$. This implies that

$$\begin{aligned}
&\mathsf{U}_3(\langle \boldsymbol{\alpha}_1, \widehat{\boldsymbol{\beta}}_1, \boldsymbol{\gamma}_1\rangle) \\
= \;& p_1 p_2 + \frac{1}{2}|\mathcal{A}_0 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_4| - \frac{1}{2}|\mathcal{A}_0 \cap \hat{\mathcal{B}}_1 \cap \mathcal{I}_2| - \frac{1}{6}|\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_2| + \frac{1}{3}|\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_4|| \\
\leq \;& p_1 p_2 + \frac{1}{2}|\mathcal{A}_0 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_4| + \frac{1}{2}|\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_4|| \\
\leq \;& p_1 p_2 + \frac{1}{2}|\widehat{\mathcal{B}}_1 \cap \mathcal{I}_4|.
\end{aligned}$$

Fix a vertex $\boldsymbol{\chi} \in \widehat{\mathcal{B}}_1 \cap \mathcal{I}_4$. Since $\gamma = \gamma_1$,

$$\begin{aligned}
\mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \boldsymbol{\gamma}) &= \mathsf{dist}_1(\chi_1, \gamma_1) + \mathsf{dist}_2(\chi_2, \gamma_2) \\
&= 2p_1 - 1 - \chi_1 + \gamma_1 + \mathsf{dist}_2(\chi_2, \gamma_2)
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \widehat{\boldsymbol{\beta}}_1) &= \mathsf{dist}_1(\chi_1, p_1 - 1) + \mathsf{dist}_2(\chi_2, p_2) \\
&= \chi_1 - p_1 + 1 + \mathsf{dist}_2(\chi_2, p_2).
\end{aligned}$$

Since $\boldsymbol{\chi} \in \mathcal{B}_1$, $\mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \boldsymbol{\gamma}) = \mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \boldsymbol{\beta}_1)$, so that

$$\chi_1 = \frac{1}{2}\left(3p_1 - 2 + \gamma_1 + \mathsf{dist}_2(\chi_2, \gamma_2) - \mathsf{dist}_2(\chi_2, p_2)\right).$$

It follows that for every χ_2 with $0 \leq \chi_2 < 2p_2 - 1$, there is at most one χ_1 with $p_1 + \gamma < \chi_1 < 2p_1$; hence, there is at most one vertex $\boldsymbol{\chi} \in \mathcal{I}_4$, which implies that there is at most ob'ne vertex $\boldsymbol{\chi} \in \widehat{\mathcal{B}}_1 \cap \mathcal{I}_4$. It follows that $|\widehat{\mathcal{B}}_1 \cap \mathcal{I}_4| \leq 2p_2$, and the claim follows.

(b) Assume now that $\gamma_1 = \gamma + p_1$. Then, by Lemma 5.1 (Conditions (C.1) and (C.3)) and Observation C.1 (Condition (C.2)), it follows that $|\mathcal{A}_0 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_2| = |\mathcal{A}_0 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_4|$ and $|\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_2| = |\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_4|$. Hence,

$$\begin{aligned}\mathsf{U}_3(\langle \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \boldsymbol{\gamma}_1 \rangle) &= \mathsf{U}_3(\langle \boldsymbol{\alpha}_1, \widehat{\boldsymbol{\beta}}_1, \boldsymbol{\gamma}_1 \rangle) - \frac{1}{2} |\mathcal{A}_0 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_3| + \frac{1}{2} |\mathcal{A}_0 \cap \hat{\mathcal{B}}_1 \cap \mathcal{I}_1| \\ &\quad + \frac{1}{6} |\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_1| - \frac{1}{3} |\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_3| - \frac{1}{6} |\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_4| .\end{aligned}$$

which implies that

$$\begin{aligned}\mathsf{U}_3(\langle \boldsymbol{\alpha}_1, \widehat{\boldsymbol{\beta}}_1, \boldsymbol{\gamma}_1 \rangle) &= p_1 p_2 + \frac{1}{2} |\mathcal{A}_0 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_3| - \frac{1}{2} |\mathcal{A}_0 \cap \hat{\mathcal{B}}_1 \cap \mathcal{I}_1| \\ &\quad - \frac{1}{6} |\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_1| + \frac{1}{3} |\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_3| + \frac{1}{6} |\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_4| .\end{aligned}$$

We observe that $\widehat{\mathcal{B}}_1 \cap \mathcal{I}_4 = \emptyset$. This implies that

$$\begin{aligned}&\mathsf{U}_3(\langle \boldsymbol{\alpha}_1, \widehat{\boldsymbol{\beta}}_1, \boldsymbol{\gamma}_1 \rangle) \\ = \;& p_1 p_2 + \frac{1}{2} |\mathcal{A}_0 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_3| - \frac{1}{2} |\mathcal{A}_0 \cap \hat{\mathcal{B}}_1 \cap \mathcal{I}_1| - \frac{1}{6} |\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_1| + \frac{1}{3} |\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_3| \\ \leq \;& p_1 p_2 + \frac{1}{2} |\mathcal{A}_0 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_3| + \frac{1}{2} |\mathcal{A}_1 \cap \widehat{\mathcal{B}}_1 \cap \mathcal{I}_3| \\ \leq \;& p_1 p_2 + \frac{1}{2} |\widehat{\mathcal{B}}_1 \cap \mathcal{I}_3| .\end{aligned}$$

Fix a vertex $\boldsymbol{\chi} \in \mathcal{B}_1 \cap \mathcal{I}_3$. Since $\gamma = \gamma_1 - p_1$,

$$\begin{aligned}\mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \boldsymbol{\gamma}) &= \mathsf{dist}_1(\chi_1, \gamma_1) + \mathsf{dist}_2(\chi_2, \gamma_2) \\ &= \gamma_1 - \chi_1 + \mathsf{dist}_2(\chi_2, \gamma_2)\end{aligned}$$

and

$$\begin{aligned}\mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \widehat{\boldsymbol{\beta}}_1) &= \mathsf{dist}_1(\chi_1, p_1 - 1) + \mathsf{dist}_2(\chi_2, p_2) \\ &= \chi_1 - p_1 + 1 + \mathsf{dist}_2(\chi_2, p_2) .\end{aligned}$$

Since $\boldsymbol{\chi} \in \mathcal{B}_1$, $\mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \boldsymbol{\gamma}) = \mathsf{dist}_{\mathsf{T}_2}(\boldsymbol{\chi}, \boldsymbol{\beta}_1)$, so that

$$\chi_1 = \frac{1}{2} \left(p_1 - 1 + \gamma_1 + \mathsf{dist}_2(\chi_2, \gamma_2) - \mathsf{dist}_2(\chi_2, p_2) \right) .$$

It follows that for every χ_2 with $0 \leq \chi_2 < 2p_2 - 1$, there is at most one χ_1 with $p_1 \leq \chi_1 \leq p_1 + \gamma$; hence, there is at most one vertex $\boldsymbol{\chi} \in \mathcal{I}_3$, which implies that there is at most one vertex $\boldsymbol{\chi} \in \widehat{\mathcal{B}}_1 \cap \mathcal{I}_3$. It follows that $|\widehat{\mathcal{B}}_1 \cap \mathcal{I}_3| \leq 2p_2$, and the claim follows.

The proof is now complete.

D Proofs from Section 7— Hypercubes

We shall denote as $1(\boldsymbol{\chi})$ the number of occurrences of 1 in the binary vector $\boldsymbol{\chi}$. We list an elementary inequality between binomial coefficients which will be used later.

Observation D.1 *For any pair of integers p, q with $2 \leq p \leq q$,*

$$\binom{p}{\frac{p - \mathsf{Odd}(p)}{2}} \cdot \binom{q}{\frac{q - \mathsf{Odd}(q)}{2}} < \binom{p-2}{\frac{p - 2 - \mathsf{Odd}(p)}{2}} \cdot \binom{q+2}{\frac{q + 2 - \mathsf{Odd}(q)}{2}} .$$

D.1 Observation 7.6

We start with a preliminary definition. Two profiles $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle$ and $\langle \widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{\gamma}} \rangle$ are ***distance-equivalent*** if $\mathsf{dist}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \mathsf{dist}(\widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}})$, $\mathsf{dist}(\boldsymbol{\alpha}, \boldsymbol{\gamma}) = \mathsf{dist}(\widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\gamma}})$ and $\mathsf{dist}(\boldsymbol{\beta}, \boldsymbol{\gamma}) = \mathsf{dist}(\widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{\gamma}})$. We observe:

Observation D.2 *Two distance-equivalent profiles are equivalent.*

Consider a triple of integers $p, q, r \in \mathsf{N}$ with $p + q = \mathsf{dist}_{\mathsf{H}_d}(\boldsymbol{\alpha}, \boldsymbol{\beta})$, $p + r = \mathsf{dist}_{\mathsf{H}_d}(\boldsymbol{\alpha}, \boldsymbol{\gamma})$ and $q + r = \mathsf{dist}_{\mathsf{H}_d}(\boldsymbol{\beta}, \boldsymbol{\gamma})$. Note that the profiles $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle$ and $\langle 0^d, 1^{p+q}0^r, 1^p0^q1^r \rangle$ are distance-equivalent; hence, by Observation D.2, they are equivalent.

D.2 Theorem 7.7

Define a combinatorial function F with arguments $a, b, r \in \mathsf{N}_0$, where $r \leq b$, as follows.

$$\begin{aligned}
\mathsf{F}(a, b, r) &= \begin{cases} \sum_{\ell = \frac{a+b+1}{2} - r}^{a} \binom{a}{\ell} & \text{if } a + b \text{ is odd} \\ \sum_{\ell = \frac{a+b}{2} + 1 - r}^{a} \binom{a}{\ell} + \frac{1}{2} \binom{a}{\frac{a+b}{2} - r} & \text{if } a + b \text{ is even} \end{cases} \\
&= \begin{cases} \sum_{\ell = 0}^{\frac{a-b-1}{2} + r} \binom{a}{\ell} & \text{if } a + b \text{ is odd} \\ \sum_{\ell = 0}^{\frac{a-b}{2} - 1 + r} \binom{a}{\ell} + \frac{1}{2} \binom{a}{\frac{a-b}{2} + r} & \text{if } a + b \text{ is even} \end{cases}
\end{aligned}$$

We prove:

Lemma D.3 *Fix a triple of integers $a, z, r \in \mathsf{N}_0$, with $r \leq z$.*

(1) *Assume that z is even, so that $r = \frac{z}{2} \pm t$ with $0 \leq t \leq \frac{z}{2}$. Then,*

$$\mathsf{F}(a, z, \frac{z}{2} \pm t) = \frac{1}{2} 2^a \pm \mathsf{M}_0(a, t).$$

(2) *Assume that z is odd, so that $r = \frac{z-1}{2} \pm t$ with $0 \leq t \leq \frac{z-1}{2}$. Then,*

$$\mathsf{F}(a, z, \frac{z-1}{2} \pm t) = \frac{1}{2} 2^a \pm \mathsf{M}_1(a, t).$$

Fix now an arbitrary vertex $\boldsymbol{\delta} \in \mathsf{V}(\mathsf{T}_d)$. Write $\boldsymbol{\delta} = \boldsymbol{\chi}\boldsymbol{\psi}\boldsymbol{\zeta}$, where $|\boldsymbol{\chi}| = x$, $|\boldsymbol{\psi}| = y$ and $|\boldsymbol{\zeta}| = z$. We shall consider the quantities $1(\boldsymbol{\chi})$, $1(\boldsymbol{\psi})$ and $1(\boldsymbol{\zeta})$. Note that

$$\mathsf{dist}(\boldsymbol{\delta}, \boldsymbol{\beta}) = x - 1(\boldsymbol{\chi}) + 1(\boldsymbol{\psi}) + z - 1(\boldsymbol{\zeta}),$$

$$\mathsf{dist}(\boldsymbol{\delta}, \boldsymbol{\alpha}) = 1(\boldsymbol{\chi}) + 1(\boldsymbol{\psi}) + 1(\boldsymbol{\zeta})$$

and

$$\mathsf{dist}(\boldsymbol{\delta}, \boldsymbol{\gamma}) = x - 1(\boldsymbol{\chi}) + y - 1(\boldsymbol{\psi}) + 1(\boldsymbol{\zeta}).$$

Hence,

$$\begin{aligned}
\mathsf{dist}(\boldsymbol{\delta}, \boldsymbol{\beta}) \leq \mathsf{dist}(\boldsymbol{\delta}, \boldsymbol{\alpha}) &\Leftrightarrow x - 1(\boldsymbol{\chi}) + 1(\boldsymbol{\psi}) + z - 1(\boldsymbol{\zeta}) \leq 1(\boldsymbol{\chi}) + 1(\boldsymbol{\psi}) + 1(\boldsymbol{\zeta}) \\
&\Leftrightarrow 1(\boldsymbol{\chi}) \geq \frac{x + y}{2} - 1(\boldsymbol{\zeta})
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{dist}(\boldsymbol{\delta}, \boldsymbol{\beta}) \leq \mathsf{dist}(\boldsymbol{\delta}, \boldsymbol{\gamma}) &\Leftrightarrow x - 1(\boldsymbol{\chi}) + 1(\boldsymbol{\psi}) + z - 1(\boldsymbol{\zeta}) \leq x - 1(\boldsymbol{\chi}) + y - 1(\boldsymbol{\psi}) + 1(\boldsymbol{\zeta}) \\
&\Leftrightarrow 1(\boldsymbol{\psi}) \leq \frac{y - z}{2} + 1(\boldsymbol{\zeta}).
\end{aligned}$$

Denote $\mathbf{r} = \langle x, y, z \rangle$. For each integer $r \in \mathsf{N}_0$ with $0 \leq r \leq z$, define $\Pi(\mathbf{r}, r)$ as the contribution to $\mathsf{U}_2(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle)$ of all vertices $\boldsymbol{\delta} \in \mathsf{Vor}_2(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle)$ with $1(\boldsymbol{\zeta}) = r$; roughly speaking, each vertex $\boldsymbol{\delta} \in \mathsf{Vor}_2(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle)$ with $1(\boldsymbol{\zeta}) = r$ is counted for $\mathsf{U}_2(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle)$ with weight $\Pi(\mathbf{r}, r)$. So,

$$\mathsf{U}_2(\langle \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle) = \sum_{0 \leq r \leq z} \binom{z}{r} \Pi(\mathbf{r}, r) .$$

We continue to derive a formula for $\Pi(\mathbf{r}, r)$ in terms of the function F.

Lemma D.4 *For each vector $\mathbf{r}$ and integer $r \in \mathsf{N}_0$ with $0 \leq r \leq z$,*

$$\Pi(\mathbf{r}, r) = \begin{cases} \mathsf{F}(x, z, r) \cdot \mathsf{F}(y, z, r) & \text{if } x + z \text{ or } y - z \text{ is odd} \\ \mathsf{F}(x, z, r) \cdot \mathsf{F}(y, z, r) + \frac{1}{12} \binom{x}{\frac{x+z}{2} - r} \binom{y}{\frac{y-z}{2} - r} & \text{if } x + z \text{ and } y - z \text{ are even} \end{cases} .$$

The proof is completed now using Lemma D.4 and standard combinatorial identities and properties of the binomial coefficients.

D.3 Corollary 7.8

By Observation 7.6, assume that $\boldsymbol{\alpha} = 0^d$, $\boldsymbol{\beta} = 0^q 1^p$ and $\boldsymbol{\gamma} = 1^d$. The claim follows now from Theorem 7.7 by setting $x = q$, $y = p$ and $z = 0$.

D.4 Lemma 7.10

We first introduce some notation. Denote as $\widetilde{\mathbf{s}} = \langle \widetilde{\boldsymbol{\alpha}}_1, \ldots, \widetilde{\boldsymbol{\alpha}}_\kappa \rangle$ the profile derived from $\mathbf{s}$ by eliminating from each vertex $\widetilde{\boldsymbol{\alpha}}_i$, $i \in [\kappa]$, all irrelevant dimensions; so, for each player $i \in [\kappa]$, $\widetilde{\boldsymbol{\alpha}}_i \in \mathsf{V}(\mathsf{H}_{d-\mathsf{irr}(\mathbf{s})})$, so that $\widetilde{\mathbf{s}}$ is a profile for the Voronoi game $\langle \mathsf{H}_{d-\mathsf{irr}(\mathbf{s})}, [\kappa] \rangle$. We observe:

Observation D.5 *Consider a profile $\mathbf{s}$ for the Voronoi game $\langle \mathsf{H}_d, [\kappa] \rangle$. Then, for each player $i \in [\kappa]$, $\mathsf{U}_i(\mathbf{s}) = \mathsf{U}_i(\widetilde{\mathbf{s}}) \cdot 2^{\mathsf{irr}(\mathbf{s})}$.*

Recall that for each player $i \in [3]$, $\mathsf{U}_i(\langle 0^d, 10^{d-1}, 1^d \rangle) \geq \frac{1}{4} 2^d$. Recall also that we only need to consider deviations by players 1 and 3. Note that if player 1 (resp., player 3) deviates to colocate with either player 2 or player 3 (resp., player 1), she will receive utility no more than $\frac{1}{4} 2^d$. It follows that we only need to examine the case where there is no colocation in the profile resulting from the deviation of a player. We proceed by case analysis.

1. Player 1 deviates: Fix any vertex $\widehat{\boldsymbol{\alpha}} \in \mathsf{V} \setminus \{\boldsymbol{\beta}, \boldsymbol{\gamma}\}$, and consider the profile $\langle \widehat{\boldsymbol{\alpha}}, 10^{d-1}, 1^d \rangle$. Assume that $\widehat{\boldsymbol{\alpha}} = \alpha \boldsymbol{\delta}$, where $\alpha \in \{0, 1\}$ and $\boldsymbol{\delta} \in \{0, 1\}^{d-1}$. Set $\boldsymbol{\delta} = 0^p 1^q$ for a pair of integers $p, q \in \mathsf{N}_0$ with $p + q = d - 1$. Since $d - 1$ is even, it follows that either p and q are odd or p and q are even. We proceed by case analysis on $\alpha \in \{0, 1\}$.

   (a) Assume first that $\alpha = 1$. Then, *(i)* the profile $\langle \widehat{\boldsymbol{\alpha}}, 10^{d-1}, 1^d \rangle$ is linear, and *(ii)* dimension 1 becomes irrelevant, so that $\mathsf{irr}\left(\langle \widehat{\boldsymbol{\alpha}}, 10^{d-1}, 1^d \rangle\right) = 1$ and $\langle \boldsymbol{\delta}, 0^{d-1}, 1^{d-1} \rangle$ is an irreducible profile for the Voronoi game $\langle \mathsf{H}_{d-1}, [3]\rangle$. It follows that

$$\begin{array}{lll}
& \mathsf{U}_1(\langle \widehat{\boldsymbol{\alpha}}, 10^{d-1}, 1^d\rangle) & \\
= & \mathsf{U}_2(\langle 10^{d-1}, \widehat{\boldsymbol{\alpha}}, 1^d\rangle) & \\
= & 2 \cdot \mathsf{U}_2(\langle 0^{d-1}, \boldsymbol{\delta}, 1^{d-1}\rangle) & \text{(by Observation D.5)} \\
= & 2 \cdot \left(\frac{1}{4}\, 2^{d-1} + \begin{cases} 0\,, & \text{if } p \text{ and } q \text{ are odd} \\ \frac{1}{12}\binom{p}{\frac{p}{2}}\binom{q}{\frac{q}{2}}, & \text{if } p \text{ and } q \text{ are even} \end{cases} \right) & \text{(by Corollary 7.8)} \\
= & \frac{1}{4}\, 2^{d} + \begin{cases} 0\,, & \text{if } p \text{ and } q \text{ are odd} \\ \frac{1}{6}\binom{p}{\frac{p}{2}}\binom{q}{\frac{q}{2}}, & \text{if } p \text{ and } q \text{ are even} \end{cases} & \\
< & \frac{1}{4}\, 2^{d} + \begin{cases} 0\,, & \text{if } p \text{ and } q \text{ are odd} \\ \frac{1}{6}\binom{d-1}{\frac{d-1}{2}} & \text{if } p \text{ and } q \text{ are even} \end{cases} & \text{(by Observation D.1)} \\
< & \mathsf{U}_1(\langle 0^d, 10^{d-1}, 1^d\rangle)\,. &
\end{array}$$

(b) Assume now that $\alpha = 0$. Then, the profile $\langle 00^p1^q, 10^{d-1}, 1^d\rangle$ is equivalent to the irreducible profile $\langle 1^q0^p1, 0^d, 1^{p+q}0\rangle$. Hence, Observation 3.1 implies that $\mathsf{U}_1(\langle 00^p1^q, 10^{d-1}, 1^d\rangle) = \mathsf{U}_1(\langle 1^q0^p1, 0^d, 1^{p+q}0\rangle) = \mathsf{U}_2(\langle 0^d, 1^q0^p1, 1^{p+q}0\rangle)$. We now apply Theorem 7.7 with $x = q$, $y = p$ and $z = 1$ to determine $\mathsf{U}_2(\langle 0^d, 1^q0^p1, 1^{p+q}0\rangle)$. We proceed by case analysis on the parity of p and q.

i. p and q are even: Then,

$$\begin{array}{lll}
& \mathsf{U}_1(\langle 00^p1^q, 10^{d-1}, 1^d\rangle) & \\
= & \mathsf{U}_2(\langle 0^d, 1^q0^p1, 1^{p+q}0\rangle) & \\
= & \frac{1}{4}\, 2^d + 2\, \mathsf{M}_1(p,0)\, \mathsf{M}_1(q,0) & \text{(by Theorem 7.7)} \\
= & \frac{1}{4}\, 2^d + 2\left(\frac{1}{2}\binom{p}{\frac{p}{2}} \cdot \frac{1}{2}\binom{q}{\frac{q}{2}}\right) & \\
= & \frac{1}{4}\, 2^d + \frac{1}{2}\binom{p}{\frac{p}{2}} \cdot \binom{q}{\frac{q}{2}} & \\
< & \frac{1}{4}\, 2^d + \frac{1}{2}\binom{d-1}{\frac{d-1}{2}} & \text{(by Observation D.1)} \\
= & \mathsf{U}_1(\langle 0^d, 10^{d-1}, 1^d\rangle)\,. &
\end{array}$$

ii. p and q are odd: Then,

$$\begin{array}{lll}
& \mathsf{U}_1(\langle 00^p1^q, 10^{d-1}, 1^d\rangle) & \\
= & \mathsf{U}_2(\langle 0^d, 1^q0^p1, 1^{p+q}0\rangle) & \\
= & \frac{1}{4}\, 2^d + \frac{1}{6}\binom{p}{\frac{p-1}{2}}\binom{q}{\frac{q-1}{2}} + 2\, \mathsf{M}_1(p,0)\, \mathsf{M}_1(q,0) & \text{(by Theorem 7.7)} \\
= & \frac{1}{4}\, 2^d + \frac{1}{6}\binom{p}{\frac{p-1}{2}}\binom{q}{\frac{q-1}{2}} + 2\left(\frac{1}{2}\binom{p}{\frac{p-1}{2}} \cdot \frac{1}{2}\binom{q}{\frac{q-1}{2}}\right) & \\
= & \frac{1}{4}\, 2^d + \frac{2}{3}\binom{p}{\frac{p-1}{2}} \cdot \binom{q}{\frac{q-1}{2}} & \\
< & \frac{1}{4}\, 2^d + \frac{2}{3}\binom{1}{0}\binom{d-2}{\frac{d-3}{2}} & \text{(by Observation D.1)} \\
< & \frac{1}{4}\, 2^d + \frac{2}{3} \cdot \frac{1}{2}\binom{d-1}{\frac{d-1}{2}} & \text{(since } \binom{m-1}{\frac{m-2}{2}} = \frac{1}{2}\binom{m}{\frac{m}{2}} \text{ for any odd } m) \\
= & \mathsf{U}_1(\langle 0^d, 10^{d-1}, 1^d\rangle)\,. &
\end{array}$$

2. Player 3 deviates: Similar to the case where player 1 deviates; it is omitted.