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THE PRICE OF ANARCHY FOR RESTRICTED PARALLEL LINKS *

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Abstract

In the model of *restricted parallel links,* n *users* must be routed on m parallel *links* under the restriction that the link for each user be chosen from a certain set of *allowed links* for the user. In a *(pure) Nash equilibrium,* no user may improve its own Individual Cost (*latency*) by unilaterally switching to another link from its set of allowed links. The *Price of Anarchy* is a widely adopted measure of the worst-case loss (relative to optimum) in system performance (*maximum latency*) incurred in a Nash equilibrium.

In this work, we present a comprehensive collection of bounds on Price of Anarchy for the model of restricted parallel links. Specifically, we prove:

- For the case of *identical users* and *identical links,* the Price of Anarchy is $\Omega\left(\frac{\lg m}{\lg \lg m}\right)$.
- For the case of identical users, the Price of Anarchy is $O\left(\frac{\lg n}{\lg \lg n}\right)$.
- For the case of identical links, the Price of Anarchy is $O\left(\frac{\lg m}{\lg \lg m}\right)$, which is asymptotically tight.
- For the most general case of *arbitrary users* and *related links,* the Price of Anarchy is at least $m-1$ and less than m.

The shown bounds reveal the dependence of the Price of Anarchy on n and m for all possible assumptions on users and links.

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1. Introduction

1.1. Framework

In the model of *restricted parallel links,* n non-cooperative users must route their unsplittable *traffics* (or *weights*) on m parallel *links* from a *source* node to a *sink* node. Each link has a *capacity* and the *latency* through a link is the ratio of the total traffic assigned to the link over its capacity. Since the latencies incurred on different links due to the same traffic are ordered by their capacities, we say that links are *related*.

A distinguishing feature of the model of restricted parallel links is the restriction that the link for each user must be chosen from a certain set of *allowed links* for the user. This restriction corresponds to an important special case of the *job scheduling problem* on *unrelated machines,* where only a subset of the users are allowed to use a machines but otherwise machines are related. On the other hand, the model is a generalization of the well studied KP model for *selfish routing* [11] that permits considering a set of allowed links for each user.

In a *(pure) Nash equilibrium* [13,14], each user is minimizing its *Individual Cost,* which is the latency on the link it chooses. So, a Nash equilibrium represents a stable state of the system in which no user has an incentive to unilaterally switch links. There is also a global objective function called *Social Cost* [11], which is the *makespan* (maximum latency); however, users do not adhere to it. The *Price of Anarchy* (or *Coordination Ratio*) [11,15] is the worst-case ratio of the Social Cost of a Nash equilibrium over the Social Cost of an optimal assignment. The Price of Anarchy is a measure of the worst-case system performance loss (relative to optimum) in a Nash equilibrium; it has been studied very intensively in the last few years.

1.2. Contribution

We present a comprehensive collection of bounds on Price of Anarchy for the model of restricted parallel links; we only consider pure Nash equilibria. Some of the bounds apply to the special case of *identical users* (resp., *identical links*) where all weights (resp., capacities) equal 1.

- For the simplest case of identical users and identical links, we present a counterexample to prove that the Price of Anarchy is $\Omega\left(\frac{\lg m}{\lg \lg m}\right)$ (Theorem 3.1).
- We then consider the case of identical users for which we prove that the Price of Anarchy is $O\left(\frac{\lg n}{\lg \lg n}\right)$ (Theorem 4.2). The proof establishes that a number of users significantly larger than the Price of Anarchy is necessary; we then employ a careful analysis to establish the claimed upper bound on the Price of Anarchy.
- For the case of identical links, we prove that the Price of Anarchy is $O\left(\frac{\lg m}{\lg \lg m}\right)$ (Theorem 5.3). The proof uses the same techniques as those for the case of identical users.

- We finally consider the most general case of arbitrary users and related links, for which we prove that the Price of Anarchy is at least $m-1$ (Theorem 6.5) and less than m (Theorem 6.6). For the lower bound, we present a counterexample. For the upper bound, we establish that a number of links larger than the Price of Anarchy is necessary; we then employ a careful analysis to establish the claimed upper bound on the Price of Anarchy.

The shown bounds shed light on the dependence of the Price of Anarchy on n and m for all possible assumptions on users and links. Moreover, our bounds imply a separation with respect to Price of Anarchy between the general case of arbitrary users and related links and each of the two special cases of identical users and identical links, respectively.

1.3. Related Work

Independently of our work, Awerbuch *et al.* [2] also have studied the model of restricted parallel links. Awerbuch *et al.* [2] focused on the case of arbitrary users and identical links, for which they proved that the Price of Anarchy is $O\left(\frac{\lg m}{\lg \lg m}\right)$ for pure Nash equilibria (cf. Theorem 5.3) and $\Theta\left(\frac{\lg m}{\lg \lg \lg m}\right)$ for all (mixed) Nash equilibria.

Tight bounds on the Price of Anarchy for the KP model [11] were proven in [4,10] for the case of identical links ($\Theta\left(\frac{\lg m}{\lg \lg m}\right)$ for all Nash equilibria), and in [4] for the case or related links ($\Theta\left(\min\left\{\frac{\lg m}{\lg \lg m}, \lg \frac{c_1}{c_m}\right\}\right)$ for pure Nash equilibria and $\Theta\left(\frac{\lg m}{\lg \lg \lg m}\right)$ for all Nash equilibria). For other bounds on Price of Anarchy for the KP model and its variants, see [6,7,8,9,12].

Suri *et al.* [17] studied a variant of the model of restricted parallel links where the Social Cost is the *total latency,* as opposed to maximum latency. (A corresponding variant of the KP model has been studied already in [9].) For this variant, Suri *et al.* [17] prove some *constant* bounds on Price of Anarchy. Two recent papers [1,3] already generalize the results of Suri *et al.* [17] to some more general classes of *(network) congestion games* [16].

Elsässer *et al.* [5] studied a further restriction of the model of restricted parallel links, called *interaction graphs,* where all sets of allowed links for the users have size 2. The results of Elsässer *et al.* [5] for their model include bounds on Price of Anarchy. In particular, Elsässer *et al.* [5, Theorem 3] prove that $\Omega\left(\frac{\lg m}{\lg \lg m}\right)$ is still a lower bound on Price of Anarchy for the case of identical users and identical links in the more restricted model of interaction graphs.

1.4. Organization

Section 2 summarizes the model of restricted parallel links. Section 3 considers the case of identical users and identical links. The case of identical users and related links is considered in Section 4. The symmetric case of arbitrary users and identical links is considered in Section 5. Section 6 considers the case of arbitrary users and related links. We conclude in Section 7.

2. Restricted Parallel Links

Throughout, denote for each positive integer m, $[m] = \{1, \ldots, m\}$; take that $[0] = \emptyset$. For any integer $k \geq 1$, the *Gamma Function* Γ is defined by $\Gamma(k+1) = k!$. Both Γ and its inverse Γ^{-1} are monotone increasing. It is well known that for any integer $k \geq 1$, $\Gamma^{-1}(k) = \Theta\left(\frac{\lg k}{\lg \lg k}\right)$.

We consider a network consisting of a set of m *parallel links* $1, \ldots, m$ from a *source* node to a *sink* node. Each of n *users* $1, \ldots, n$ wishes to route a particular amount of traffic along a (non-fixed) link from source to sink. Assume throughout that $m \geq 2$ and $n \geq 2$.

Denote w_i the *weight* of user $i \in [n]$. Assume, without loss of generality, that $w_1 \geq \ldots \geq w_n$, and denote $W = \sum_{i \in [n]} w_i$. The *weight vector* $\mathbf{w}$ is the tuple of all weights. In the case of *identical users,* all weights equal 1; weights vary arbitrarily in the case of *arbitrary users.*

Denote $c_j > 0$ the *capacity* of link $j \in [m]$. The *capacity vector* $\mathbf{c}$ is the tuple of all capacities. Denote $C = \sum_{j \in [m]} c_j$. The *latency* for weight w through link $j \in [m]$ is $\frac{w}{c_j}$. In the case of *identical links,* all capacities equal 1; capacities vary arbitrarily in the case of *related links.* An *instance* is a pair $\langle \mathbf{w}, \mathbf{c} \rangle$.

Associated with each user $i \in [n]$ is a *strategy set* $\mathcal{L}_i \subseteq [m]$, as the set of *allowed links* for user i; thus, user i can only be assigned to a link from $\mathcal{L}_i$. So, a *strategy* for user i is some link from its set of allowed links $\mathcal{L}_i$. Denote $\mathcal{L} = \times_{i \in [n]} \mathcal{L}_i$; clearly, $\mathcal{L} \subseteq [m]^n$. An *assignment* $\mathbf{L} = \langle l_1, \ldots, l_n \rangle \in \mathcal{L}$ is a tuple of strategies, one for each user.

Fix now an assignment $\mathbf{L}$. The *load* $\Delta_j(\mathbf{L})$ on link j is the sum of weights of users assigned to link j; thus, $\Delta_j(\mathbf{L}) = \sum_{k: l_k = j} w_k$. The *latency* $\Lambda_j(\mathbf{L})$ on link j is given by $\Lambda_j(\mathbf{L}) = \dfrac{\Delta_j(\mathbf{L})}{c_j}$. The *Individual Cost* $\mathsf{IC}_i(\mathbf{L})$ of user $i \in [n]$ in assignment $\mathbf{L}$ is the latency of the link it chooses; that is, $\mathsf{IC}_i(\mathbf{L}) = \Lambda_{l_i}(\mathbf{L})$.

Associated with an instance $\langle \mathbf{w}, \mathbf{c} \rangle$ and an assignment $\mathbf{L}$ is the *Social Cost* [11, Section 2], denoted $\mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{L})$, which is the maximum, over all links, latency due to the load through the link; so, $\mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{L}) = \max_{j \in [m]} \Lambda_j(\mathbf{L})$. Associated with an instance $\langle \mathbf{w}, \mathbf{c} \rangle$ is the *Optimum* [11, Section 2], denoted $\mathsf{OPT}(\mathbf{w}, \mathbf{c})$, which is the least possible, over all assignments, of the maximum, over all links, latency due to the load through the link; so, $\mathsf{OPT}(\mathbf{w}, \mathbf{c}) = \min_{\mathbf{L} \in \mathcal{L}} \mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{L}) = \min_{\mathbf{L} \in \mathcal{L}} \max_{j \in [m]} \Lambda_j(\mathbf{L})$.

Say that a user $i \in [n]$ is *satisfied* in assignment $\mathbf{L}$ if for all links $j \in \mathcal{L}_i$, $\mathsf{IC}_i(\mathbf{L}) \leq \dfrac{\Delta_j(\mathbf{L}) + w_i}{c_j}$; so, a satisfied user cannot decrease its Individual Cost by switching to a different allowed link. Say that $\mathbf{L}$ is a *Nash equilibrium* [14] if all users are satisfied in $\mathbf{L}$.

The *Price of Anarchy* [11,15] (also known as *Coordination Ratio* [11, Section 2]), denoted PoA, is the *worst-case* ratio $\dfrac{\mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{L})}{\mathsf{OPT}(\mathbf{w}, \mathbf{c})}$ over all instances $\langle \mathbf{w}, \mathbf{c} \rangle$ and Nash equilibria $\mathbf{L}$; thus,

$$\mathsf{PoA} \quad = \quad \max_{\langle \mathbf{w}, \mathbf{c} \rangle, \mathbf{L}} \frac{\mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{L})}{\mathsf{OPT}(\mathbf{w}, \mathbf{c})} .$$

3. Identical Users and Identical Links

We prove:

Theorem 3.1 *Consider the case of identical users and identical links. Then,*

$$\mathsf{PoA} \quad = \quad \Omega\left(\frac{\lg m}{\lg \lg m}\right) .$$

Proof: Consider an instance $\langle \mathbf{w}, \mathbf{c} \rangle$ with n users and m links. We construct the strategy sets of the users as follows. Fix some sufficiently large integer p (to be determined later).

- Partition the set of links into $p+1$ disjoint subsets $\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_p$ with:
  - $|\mathcal{M}_0| = 1$.
  - For each integer l, where $1 \leq l \leq p$, $|\mathcal{M}_l| = (p-1) \cdot \prod_{j \in [l-1]} (p-j)$.

  Note that since $|\mathcal{M}_0| \leq |\mathcal{M}_1| < \ldots < |\mathcal{M}_p|$ the partition implies that $m < (p+1) \cdot |\mathcal{M}_p| = (p+1)(p-1)(p-1)! < (p+1)! = \Gamma(p+2)$. So, $p > \Gamma^{-1}(m) - 2$.
- Partition the set of users into p disjoint subsets $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_{p-1}$ with:
  - For each integer k, where $0 \leq k \leq p-1$, $|\mathcal{U}_k| = (p-k) \cdot |\mathcal{M}_k|$.
  - The strategy set of each user in $\mathcal{U}_k$ is $\mathcal{M}_k \cup \mathcal{M}_{k+1}$.

We now construct a Nash equilibrium $\mathbf{L}$ and an optimal assignment $\mathbf{Q}$ such that $\mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{L}) = p$ and $\mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) = 1$.

- Construct an assignment $\mathbf{L}$ as follows.
  - All p users from the set $\mathcal{U}_0$ are assigned to the single link in $\mathcal{M}_0$.
  - For each integer k, where $1 \leq k \leq p-1$, $p-k$ users from $\mathcal{U}_k$ are assigned to each link in $\mathcal{M}_k$. (Note that no user is assigned to any link in $\mathcal{M}_p$.)

  By the construction of $\mathbf{L}$, the latency on each link in the set $\mathcal{M}_l$, where $0 \leq l \leq p$, is $p-l$. Thus, for each integer l, where $0 \leq l \leq p-1$, no user assigned to a link in the set $\mathcal{M}_l$ can decrease its Individual Cost by switching either to a different link from the set $\mathcal{M}_l$ or to a link from the set $\mathcal{M}_{l+1}$. So, all users are satisfied in $\mathbf{L}$ and $\mathbf{L}$ is a Nash equilibrium with

$$\begin{aligned} \mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{L}) &= \max_{j \in [m]} \Lambda_j(\mathbf{L}) \\ &= \max_{0 \leq l \leq p} (p-l) \\ &= p . \end{aligned}$$

- Note that $|\mathcal{M}_0| + |\mathcal{M}_1| = p$ and $|\mathcal{U}_0| = p$. Note also that for each integer k, $1 \leq k \leq p-1$,

$$\begin{aligned} |\mathcal{U}_k| &= (p-k) \cdot |\mathcal{M}_k| \\ &= (p-k)(p-1) \cdot \prod_{j \in [k-1]} (p-j) \\ &= (p-1) \cdot \prod_{j \in [k]} (p-j) \\ &= |\mathcal{M}_{k+1}|. \end{aligned}$$

  So, it is possible to assign each user in $\mathcal{U}_0$ to a distinct link in $\mathcal{M}_0 \cup \mathcal{M}_1$, and to assign each user in $\mathcal{U}_k$, where $1 \leq k \leq p-1$, to a distinct link in $\mathcal{M}_{k+1}$. Call $\mathbf{Q}$ the resulting assignment. Then, $\mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) = 1$ and $\mathbf{Q}$ is optimal. So, $\mathsf{OPT}(\mathbf{w}, \mathbf{c}) = 1$.

It follows that

$$\begin{aligned} \mathsf{PoA} &\geq \frac{\mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{L})}{\mathsf{OPT}(\mathbf{w}, \mathbf{c})} \\ &= p \\ &> \Gamma^{-1}(m) - 2 \\ &= \Omega\left(\frac{\lg m}{\lg \lg m}\right), \end{aligned}$$

as needed. □

Theorem 3.1 implies that $\Omega\left(\frac{\lg m}{\lg \lg m}\right)$ is a lower bound on the Price of Anarchy for the more general cases of arbitrary users or related links (or of both).

4. Identical Users

We prove:

Theorem 4.2 *Consider the case of identical users. Then,*

$$\mathsf{PoA} = O\left(\frac{\lg n}{\lg \lg n}\right).$$

Proof: Consider any arbitrary instance $\langle \mathbf{w}, \mathbf{c} \rangle$ with an associated Nash equilibrium $\mathbf{L}$ such that

$$p \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \leq \mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{L}) < (p+1) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c})$$

for some integer $p \in \mathbb{N}$, and an optimal assignment $\mathbf{Q}$. To prove an upper bound on the Price of Anarchy, it suffices to prove an upper bound on $p+1$. To do so, we will prove a lower bound (as a function of p) on the number of users that are necessary for such a Nash equilibrium $\mathbf{L}$. We will then use this lower bound to prove an upper bound of $O\left(\frac{\lg n}{\lg \lg n}\right)$ on $p+1$. We continue with the details of the formal proof.

Consider now a link $j \in [m]$ with $c_j < \frac{1}{\mathsf{OPT}(\mathbf{w}, \mathbf{c})}$. Note that in the optimal assignment $\mathbf{Q}$, no user is assigned to link j (since otherwise $\frac{1}{c_j} \leq \Lambda_j(\mathbf{Q}) \leq \mathsf{OPT}(\mathbf{w}, \mathbf{c})$, or $c_j \geq \frac{1}{\mathsf{OPT}(\mathbf{w}, \mathbf{c})}$). If, in addition, $\Lambda_j(\mathbf{L}) < \mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{L})$, then link j can be eliminated (together with all users assigned to it in $\mathbf{L}$) with no change to $\mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{L})$ and no increase to $\mathsf{OPT}(\mathbf{w}, \mathbf{c})$. So, assume, without loss of generality, that for each link $j \in [m]$, either $c_j \geq \frac{1}{\mathsf{OPT}(\mathbf{w}, \mathbf{c})}$ or $\Lambda_j(\mathbf{L}) = \mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{L})$.

Define $\mathcal{M}_0$ as the set of links $j \in [m]$ with latency

$$\Lambda_j(\mathbf{L}) \quad \geq \quad p \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \ .$$

Clearly, $\mathcal{M}_0 \neq \emptyset$. By definition of latency, this implies that

$$\sum_{j \in \mathcal{M}_0} \Delta_j(\mathbf{L}) \quad \geq \quad p \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_0} c_j \ .$$

We prove an inductive claim:

Lemma 4.1 *For each $l \in [p-1]$, there is a set of links $\mathcal{M}_l$ with $\mathcal{M}_l \cap (\mathcal{M}_0 \cup \ldots \cup \mathcal{M}_{l-1}) = \emptyset$ such that:*

(1) $\sum_{j \in \mathcal{M}_l} c_j \geq (p-1) \cdot \prod_{j \in [l-1]} (p-j) \cdot \sum_{j \in \mathcal{M}_0} c_j$.

(2) *For each link $j \in \mathcal{M}_l$, $\Lambda_j(\mathbf{L}) \geq (p-l) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c})$.*

(3) $\sum_{j \in \mathcal{M}_l} \Delta_j(\mathbf{L}) \geq (p-1) \cdot \prod_{j \in [l]} (p-j) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_0} c_j$.

(4) *There are at least $(p-1) \cdot \prod_{j \in [l]} (p-j) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_0} c_j$ users assigned by $\mathbf{L}$ to links in $\mathcal{M}_0 \cup \ldots \cup \mathcal{M}_l$ whose strategy sets include links outside $\mathcal{M}_0 \cup \ldots \cup \mathcal{M}_l$.*

Proof: By (strong) induction on l. For the basis case, let $l = 1$. Recall that $\sum_{j \in \mathcal{M}_0} \Delta_j(\mathbf{L}) \geq p \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_0} c_j$. In the optimal assignment $\mathbf{Q}$, $\Lambda_j(\mathbf{Q}) \leq \mathsf{OPT}(\mathbf{w}, \mathbf{c})$ for each link $j \in [m]$. By definition of latency, this implies that $\sum_{j \in \mathcal{M}_0} \Delta_j(\mathbf{Q}) \leq \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_0} c_j$. It follows that there are at least

$$p \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_0} c_j - \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_0} c_j \quad = \quad (p-1) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_0} c_j$$

excess users assigned by $\mathbf{L}$ to links in $\mathcal{M}_0$ whose strategy sets include links outside $\mathcal{M}_0$.

> Define $\mathcal{M}_1$ as the set of all links outside $\mathcal{M}_0$ that are included in the strategy sets of such excess users; so $\mathcal{M}_1 \cap \mathcal{M}_0 = \emptyset$.

Clearly, in $\mathbf{Q}$, all these excess users are assigned to links in $\mathcal{M}_1$, so that

$$\sum_{j \in \mathcal{M}_1} \Delta_j(\mathbf{Q}) \quad \geq \quad (p-1) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_0} c_j \ .$$

We now prove the four claimed properties for the set $\mathcal{M}_1$.

- Clearly,

$$\begin{aligned}\sum_{j\in\mathcal{M}_1} c_j &= \sum_{j\in\mathcal{M}_1} \frac{\Delta_j(\mathbf{Q})}{\Lambda_j(\mathbf{Q})} \\ &\geq \frac{\sum_{j\in\mathcal{M}_1}\Delta_j(\mathbf{Q})}{\mathsf{OPT}(\mathbf{w},\mathbf{c})} \\ &\geq \frac{(p-1)\cdot\mathsf{OPT}(\mathbf{w},\mathbf{c})\cdot\sum_{j\in\mathcal{M}_0} c_j}{\mathsf{OPT}(\mathbf{w},\mathbf{c})} \\ &= (p-1)\cdot\sum_{j\in\mathcal{M}_0} c_j\,,\end{aligned}$$

  which proves (1).

- To prove (2), consider any link $j \in \mathcal{M}_1$. Since $j \notin \mathcal{M}_0$, it follows that $\Lambda_j(\mathbf{L}) < p \cdot \mathsf{OPT}(\mathbf{w},\mathbf{c})$. Since $\mathsf{SC}(\mathbf{w},\mathbf{c},\mathbf{L}) \geq p \cdot \mathsf{OPT}(\mathbf{w},\mathbf{c})$, this implies that $\Lambda_j(\mathbf{L}) < \mathsf{SC}(\mathbf{w},\mathbf{c},\mathbf{L})$. Therefore, $c_j \geq \frac{1}{\mathsf{OPT}(\mathbf{w},\mathbf{c})}$.

  Consider any link $j' \in \mathcal{M}_0$ to which $\mathbf{L}$ assigns some excess user. Since $\mathbf{L}$ is a Nash equilibrium,

$$\begin{aligned}\Lambda_{j'}(\mathbf{L}) &\leq \Lambda_j(\mathbf{L}) + \frac{1}{c_j} \\ &\leq \Lambda_j(\mathbf{L}) + \mathsf{OPT}(\mathbf{w},\mathbf{c})\ .\end{aligned}$$

  However, by definition of the set $\mathcal{M}_0$,

$$\Lambda_{j'}(\mathbf{L}) \quad \geq \quad p\cdot\mathsf{OPT}(\mathbf{w},\mathbf{c})\ .$$

  It follows that

$$\Lambda_j(\mathbf{L}) \quad \geq \quad (p-1)\cdot\mathsf{OPT}(\mathbf{w},\mathbf{c})\,,$$

  and the proof of (2) is now complete.

- To prove (3), we use (2) and (1) to derive that

$$\begin{aligned}\sum_{j\in\mathcal{M}_1}\Delta_j(\mathbf{L}) &= \sum_{j\in\mathcal{M}_1}\Lambda_j(\mathbf{L})\cdot c_j \\ &\geq (p-1)\cdot\mathsf{OPT}(\mathbf{w},\mathbf{c})\cdot\sum_{j\in\mathcal{M}_1} c_j \\ &\geq (p-1)\cdot\mathsf{OPT}(\mathbf{w},\mathbf{c})\cdot(p-1)\cdot\sum_{j\in\mathcal{M}_0} c_j \\ &= (p-1)^2\cdot\mathsf{OPT}(\mathbf{w},\mathbf{c})\cdot\sum_{j\in\mathcal{M}_0} c_j\,,\end{aligned}$$

  as needed for proving (3).

- Recall first that in the optimal assignment $\mathbf{Q}$, $\Lambda_j(\mathbf{Q}) \leq \mathsf{OPT}(\mathbf{w}, \mathbf{c})$ for each link $j \in [m]$. By definition of latency, this implies that $\sum_{j \in \mathcal{M}_1} \Delta_j(\mathbf{Q}) \leq \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_1} c_j$.

  Clearly, the number of users assigned by $\mathbf{L}$ to links in $\mathcal{M}_0 \cup \mathcal{M}_1$ whose strategy sets include links outside $\mathcal{M}_0 \cup \mathcal{M}_1$ is at least

$$
\begin{aligned}
& \sum_{r \in \{0\} \cup [1]} \sum_{j \in \mathcal{M}_r} (\Delta_j(\mathbf{L}) - \Delta_j(\mathbf{Q})) \\
= \ & \sum_{j \in \mathcal{M}_0} \Delta_j(\mathbf{L}) - \sum_{j \in \mathcal{M}_0} \Delta_j(\mathbf{Q}) + \sum_{j \in \mathcal{M}_1} \Delta_j(\mathbf{L}) - \sum_{j \in \mathcal{M}_1} \Delta_j(\mathbf{Q}) \\
\geq \ & p \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_0} c_j - \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_0} c_j \\
& + (p-1) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_1} c_j - \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_1} c_j \\
= \ & (p-1) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_0} c_j + (p-2) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_1} c_j \\
\geq \ & (p-1) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_0} c_j + (p-2) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot (p-1) \cdot \sum_{j \in \mathcal{M}_0} c_j \\
= \ & ((p-1) + (p-2)(p-1)) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_0} c_j \\
= \ & (p-1)^2 \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_0} c_j
\end{aligned}
$$

  as needed for proving (4).

The proof of the basis case is now complete.

Assume inductively that for some integer $l \geq 2$, the claim holds for all integers not exceeding $(l-1)$. We will prove the claim for l.

By induction hypothesis (condition (4)), there are at least $(p-1) \cdot \prod_{j \in [l-1]} (p-j) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_0} c_j$ excess users assigned by $\mathbf{L}$ to links in $\mathcal{M}_0 \cup \ldots \cup \mathcal{M}_{l-1}$ whose strategy sets include links outside $\mathcal{M}_0 \cup \ldots \cup \mathcal{M}_{l-1}$.

> Define $\mathcal{M}_l$ as the set of all links outside $\mathcal{M}_0 \cup \ldots \cup \mathcal{M}_{l-1}$ that are included in the strategy sets of such excess users; so, $\mathcal{M}_l \cap (\mathcal{M}_0 \cup \ldots \cup \mathcal{M}_{l-1}) = \emptyset$.

Clearly, in $\mathbf{Q}$, all these excess users are assigned to links in $\mathcal{M}_l$, so that

$$
\sum_{j \in \mathcal{M}_l} \Delta_j(\mathbf{Q}) \quad \geq \quad (p-1) \cdot \prod_{j \in [l-1]} (p-j) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_0} c_j \, .
$$

We now prove the four claimed properties for the set $\mathcal{M}_l$.

- Clearly,

$$\begin{aligned}\sum_{j\in\mathcal{M}_l} c_j &= \sum_{j\in\mathcal{M}_l} \frac{\Delta_j(\mathbf{Q})}{\Lambda_j(\mathbf{Q})}\\ &\geq \frac{\sum_{j\in\mathcal{M}_l}\Delta_j(\mathbf{Q})}{\mathsf{OPT}(\mathbf{w},\mathbf{c})}\\ &\geq \frac{(p-1)\cdot\prod_{j\in[l-1]}(p-j)\cdot\mathsf{OPT}(\mathbf{w},\mathbf{c})\cdot\sum_{j\in\mathcal{M}_0}c_j}{\mathsf{OPT}(\mathbf{w},\mathbf{c})}\\ &= (p-1)\cdot\prod_{j\in[l-1]}(p-j)\cdot\sum_{j\in\mathcal{M}_0}c_j\,,\end{aligned}$$

which proves (1).

- To prove (2), consider any link $j \in \mathcal{M}_l$. Since $j \notin \mathcal{M}_0$, it follows that $\Lambda_j(\mathbf{L}) < p \cdot \mathsf{OPT}(\mathbf{w},\mathbf{c})$. Since $\mathsf{SC}(\mathbf{w},\mathbf{c},\mathbf{L}) \geq p\cdot\mathsf{OPT}(\mathbf{w},\mathbf{c})$, this implies that $\Lambda_j(\mathbf{L}) < \mathsf{SC}(\mathbf{w},\mathbf{c},\mathbf{L})$. Therefore, $c_j \geq \frac{1}{\mathsf{OPT}(\mathbf{w},\mathbf{c})}$.

  Recall that in the optimal assignment $\mathbf{Q}$, $\Lambda_j(\mathbf{Q}) \leq \mathsf{OPT}(\mathbf{w},\mathbf{c})$ for each link $j \in [m]$. By definition of latency, this implies that $\sum_{j\in\mathcal{M}_{l-1}}\Delta_j(\mathbf{Q}) \leq \mathsf{OPT}(\mathbf{w},\mathbf{c})\cdot\sum_{j\in\mathcal{M}_{l-1}}c_j$. By induction hypothesis (condition (3)), $\sum_{j\in\mathcal{M}_{l-1}}\Delta_j(\mathbf{L}) \geq (p-1)\cdot\prod_{j\in[l-1]}(p-j)\cdot\mathsf{OPT}(\mathbf{w},\mathbf{c})\cdot\sum_{j\in\mathcal{M}_0}c_j$. It follows that there is some excess user assigned to some link $j' \in \mathcal{M}_{l-1}$. Since $\mathbf{L}$ is a Nash equilibrium,

$$\begin{aligned}\Lambda_{j'}(\mathbf{L}) &\leq \Lambda_j(\mathbf{L}) + \frac{1}{c_j}\\ &\leq \Lambda_j(\mathbf{L}) + \mathsf{OPT}(\mathbf{w},\mathbf{c})\ .\end{aligned}$$

  By induction hypothesis (condition (2)),

$$\begin{aligned}\Lambda_{j'}(\mathbf{L}) &\geq (p-(l-1))\cdot\mathsf{OPT}(\mathbf{w},\mathbf{c})\\ &= (p-l)\cdot\mathsf{OPT}(\mathbf{w},\mathbf{c}) + \mathsf{OPT}(\mathbf{w},\mathbf{c})\ .\end{aligned}$$

  It follows that

$$\Lambda_j(\mathbf{L}) \geq (p-l)\cdot\mathsf{OPT}(\mathbf{w},\mathbf{c})\,,$$

  and the proof of (2) is now complete.

- To prove (3), we use (2) and (1) to derive that

$$\begin{aligned}\sum_{j\in\mathcal{M}_l}\Delta_j(\mathbf{L}) &= \sum_{j\in\mathcal{M}_l}\Lambda_j(\mathbf{L})\cdot c_j\\ &\geq (p-l)\cdot\mathsf{OPT}(\mathbf{w},\mathbf{c})\cdot\sum_{j\in\mathcal{M}_l}c_j\\ &\geq (p-l)\cdot\mathsf{OPT}(\mathbf{w},\mathbf{c})\cdot(p-1)\cdot\prod_{j\in[l-1]}(p-j)\cdot\sum_{j\in\mathcal{M}_0}c_j\\ &= (p-1)\cdot\prod_{j\in[l]}(p-j)\cdot\mathsf{OPT}(\mathbf{w},\mathbf{c})\cdot\sum_{j\in\mathcal{M}_0}c_j\,,\end{aligned}$$

as needed for proving (3).

- Recall first that in the optimal assignment $\mathbf{Q}$, $\Lambda_j(\mathbf{Q}) \leq \mathsf{OPT}(\mathbf{w}, \mathbf{c})$ for each link $j \in [m]$. By definition of latency, this implies that $\sum_{j \in \mathcal{M}_l} \Delta_j(\mathbf{Q}) \leq \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_l} c_j$.

  Clearly, the number of users assigned by $\mathbf{L}$ to links in $\mathcal{M}_0 \cup \ldots \cup \mathcal{M}_l$ whose strategy sets include links outside $\mathcal{M}_0 \cup \ldots \mathcal{M}_l$ is at least

$$
\begin{aligned}
& \sum_{r \in \{0\} \cup [l]} \sum_{j \in \mathcal{M}_r} (\Delta_j(\mathbf{L}) - \Delta_j(\mathbf{Q})) \\
= \quad & \sum_{r \in [l-1]} \sum_{j \in \mathcal{M}_r} (\Delta_j(\mathbf{L}) - \Delta_j(\mathbf{Q})) + \sum_{j \in \mathcal{M}_l} \Delta_j(\mathbf{L}) - \sum_{j \in \mathcal{M}_l} \Delta_j(\mathbf{Q}) \\
\geq \quad & (p-1) \cdot \prod_{j \in [l-1]} (p-j) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_0} c_j \\
& + (p-l) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_l} c_j - \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_{l-1}} c_j \\
\geq \quad & (p-1) \cdot \prod_{j \in [l-1]} (p-j) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_0} c_j \\
& + (p-l-1) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_l} c_j \\
\geq \quad & (p-1) \cdot \prod_{j \in [l-1]} (p-j) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_0} c_j \\
& + (p-l-1) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot (p-1) \cdot \prod_{j \in [l-1]} (p-j) \cdot \sum_{j \in \mathcal{M}_0} c_j \\
= \quad & (1 + (p-l-1)) \cdot (p-1) \cdot \prod_{j \in [l-1]} (p-j) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_0} c_j \\
= \quad & (p-1) \cdot \prod_{j \in [l]} (p-j) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_0} c_j \, ,
\end{aligned}
$$

  as needed for proving (4).

The proof of the inductive claim is now complete. □

We now prove an upper bound on $p+1$. Fix any link $j \in \mathcal{M}_0$. Clearly, $\Lambda_j(\mathbf{L}) \leq \mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{L}) < (p+1) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c})$. Recall that by definition of $\mathcal{M}_0$, $\Lambda_j(\mathbf{L}) \geq p \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) > 0$. This implies that $\Lambda_j(\mathbf{L}) \geq \frac{1}{c_j}$. It follows that $\frac{1}{c_j} < (p+1) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c})$. This implies that

$$
\mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_0} c_j \quad > \quad \frac{1}{p+1} \, .
$$

Assume, without loss of generality, that $p \geq 3$ (otherwise $k+1 \in O(1)$). Then, by Lemma 4.1 (condition (3)),

$$\begin{aligned}
n &\geq \sum_{j \in \mathcal{M}_{p-1}} \Delta_j(\mathbf{L}) + \sum_{j \in \mathcal{M}_{p-2}} \Delta_j(\mathbf{L}) \\
&\geq (p-1) \cdot \prod_{j \in [p-1]} (p-j) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_0} c_j \\
&\quad + (p-1) \cdot \prod_{j \in [p-2]} (p-j) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \cdot \sum_{j \in \mathcal{M}_0} c_j \\
&> 2 \cdot (p-1) \cdot (p-1)! \cdot \frac{1}{p+1} \\
&\geq (p-1)! \\
&= \Gamma(p)\,.
\end{aligned}$$

Hence

$$\begin{aligned}
p+1 &< \Gamma^{-1}(n) + 1 \\
&= O\left(\frac{\lg n}{\lg \lg n}\right),
\end{aligned}$$

as needed. □

We remark that Theorems 3.1 and 4.2 leave a gap between our bounds on Price of Anarchy for the case of identical users. Closing this gap remains an interesting open problem.

5. Identical Links

With a similar proof as in Theorem 4.2, we can prove an upper bound that matches asymptotically the lower bound shown in Theorem 3.1.

Theorem 5.3 *Consider the case of identical links. Then,*

$$\mathsf{PoA} = O\left(\frac{\lg m}{\lg \lg m}\right).$$

Theorems 4.2 and 5.3 together imply:

Theorem 5.4 *Consider the case of identical users and identical links. Then,*

$$\mathsf{PoA} = O\left(\frac{\lg \min\{m, n\}}{\lg \lg \min\{m, n\}}\right).$$

We remark that in the interesting cases where $n \geq m$, Theorems 3.1 and 5.4 provide asymptotically tight bounds on Price of Anarchy for the case of identical users and identical links.

6. The General Case

We first prove the lower bound:

Theorem 6.5 *Consider the case of arbitrary users and related links. Then,* $\mathsf{PoA} \geq m - 1$.

Proof: Consider an instance $\langle \mathbf{w}, \mathbf{c} \rangle$ as follows:

- For each link $j \in [m]$, the capacity c_j is

$$c_j = \frac{(m-1)!}{(j-1)!} .$$

- There are $n = m - 1$ users; the weight of user $i \in [m-1]$ is $w_i = c_i$.

Moreover, assume that for each user $i \in [m-1]$, the strategy set $\mathcal{L}_i$ is $\mathcal{L}_i = \{i, i+1\}$.

- Construct an assignment $\mathbf{L}$ as follows:

  Each user $i \in [m-1]$ is assigned to link $i+1$.

  We will argue that all users are satisfied in $\mathbf{L}$.

  – Note that the Individual Cost of each user $i \in [m-1] \setminus \{1\}$ is

$$\begin{aligned} \mathsf{IC}_i(\mathbf{L}) &= \Lambda_{i+1}(\mathbf{L}) \\ &= \frac{w_i}{c_{i+1}} \\ &= \frac{c_i}{c_{i+1}} \\ &= i . \end{aligned}$$

  On the other hand,

$$\begin{aligned} \frac{\Delta_i + w_i}{c_i} &= \frac{w_{i-1} + w_i}{c_i} \\ &= \frac{c_{i-1} + c_i}{c_i} \\ &= \frac{c_{i-1}}{c_i} + 1 \\ &= (i-1) + 1 \\ &= i . \end{aligned}$$

  It follows that user $i \in [m-1] \setminus \{1\}$ is satisfied in $\mathbf{L}$.

  – Consider now user 1. Since $c_1 = c_2$ and there are no users assigned to link 1, user 1 cannot decrease its Individual Cost by switching from link 2 to link 1. So, user 1 is also satisfied in $\mathbf{L}$.

  It follows that $\mathbf{L}$ is a Nash equilibrium. Clearly,

$$\begin{aligned} \mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{L}) &= \max_{j \in [m]} \Lambda_j(\mathbf{L}) \\ &= \max_{j \in [m] \setminus \{1\}} (j-1) \\ &= m - 1 . \end{aligned}$$

- Construct now an assignment $\mathbf{Q}$ as follows:

  Each user $i \in [m-1]$ is assigned to link i.

  Clearly, for each link $j \in [m-1]$, $\Lambda_j(\mathbf{L}) = \frac{w_j}{c_j} = 1$ and $\Lambda_m(\mathbf{L}) = 0$. So, $\mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) = 1$. Thus, $\mathsf{OPT}(\mathbf{w}, \mathbf{c}) \leq 1$.

It follows that

$$\begin{aligned} \mathsf{PoA} &\geq \frac{\mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{L})}{\mathsf{OPT}(\mathbf{w}, \mathbf{c})} \\ &\geq \frac{m-1}{1} \\ &= m-1\,, \end{aligned}$$

as needed. □

We now prove the upper bound:

Theorem 6.6 *Consider the case of arbitrary users and related links. Then,* $\mathsf{PoA} < m$.

Proof: Consider any arbitrary instance $\langle \mathbf{w}, \mathbf{c} \rangle$ with an associated Nash equilibrium $\mathbf{L}$ such that

$$p \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \quad \leq \quad \mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{L}) \quad < \quad (p+1) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c})$$

for some integer $p \in \mathbb{N}$, and an optimal assignment $\mathbf{Q}$. To prove an upper bound on the Price of Anarchy, it suffices to prove an upper bound on $p+1$. To do so, we will prove a lower bound (as a function of p) on the number of links that are necessary for such a Nash equilibrium $\mathbf{L}$. We will then use this lower bound to prove an upper bound of m on $p+1$. We continue with the details of the formal proof.

We prove an inductive claim:

Lemma 6.2 *For each integer $i \in [p]$, there exists a distinct link $l_i \in [m]$ with latency $\Lambda_{l_i}(\mathbf{L}) \geq (p-i+1) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c})$.*

Proof: By (strong) induction on i. For the basis case, let $i = 1$. Since $\mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{L}) \geq p \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c})$, there is a link $l_1 \in [m]$ with latency $\Lambda_{l_1}(\mathbf{L}) \geq p \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c})$, as needed.

Assume inductively that for some integer $i \geq 2$ the claim holds for all integers not exceeding $(i-1)$. We will prove the claim for i. By induction hypothesis, there exist $i-1$ distinct links $l_1, \ldots, l_{i-1}$ with

$$\Lambda_{l_j}(\mathbf{L}) \quad \geq \quad (p-j+1) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c})\ ,$$

for each integer $j \in [i-1]$. Since $j \leq i-1$ and $i \leq p$, it follows that $j \leq p-1$. So, $p-j+1 \geq 2$. It follows that for each integer $j \in [i-1]$,

$$\Lambda_{l_j}(\mathbf{L}) \quad > \quad \mathsf{OPT}(\mathbf{w}, \mathbf{c})\ ,$$

Since $\mathsf{OPT}(\mathbf{w}, \mathbf{c}) = \mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) \geq \Lambda_{l_j}(\mathbf{Q})$ for each integer $j \in [i-1]$, it follows that for each integer $j \in [i-1]$, $\Lambda_{l_j}(\mathbf{L}) > \Lambda_{l_j}(\mathbf{Q})$. So, $\sum_{j \in [i-1]} \Lambda_{l_j}(\mathbf{L}) > \sum_{j \in [i-1]} \Lambda_{l_j}(\mathbf{Q})$. It follows that there is some user i_0 assigned by $\mathbf{L}$ to some link in the set $\{l_1, \ldots, l_{i-1}\}$

that is assigned by $\mathbf{Q}$ to some link $l_i \notin \{l_1, \ldots, l_{i-1}\}$ (otherwise, $\sum_{j \in [i-1]} \Lambda_{l_j}(\mathbf{Q}) \geq \sum_{j \in [i-1]} \Lambda_{l_j}(\mathbf{L})$). Thus, l_i is an allowed link for user i_0.

Since $\mathbf{L}$ is a Nash equilibrium, user i_0 has no incentive to switch from its link l_j, where $j \in [i-1]$, to link l_i. Since user i_0 is assigned to link l_i in $\mathbf{Q}$, the additional latency on link l_i in $\mathbf{L}$ due to user i_0 switching to link l_i is at most the latency on link l_i in $\mathbf{Q}$; since $\mathbf{Q}$ is optimal, this additional latency is at most $\mathsf{OPT}(\mathbf{w}, \mathbf{c})$. It follows that

$$\Lambda_{l_j}(\mathbf{L}) \quad \leq \quad \Lambda_{l_i}(\mathbf{L}) + \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \ .$$

By induction hypothesis,

$$\begin{aligned} \Lambda_{l_j}(\mathbf{L}) &\geq (p - j + 1) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \\ &\geq (p - (i-1) + 1) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \\ &= (p - i + 1) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) + \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \ . \end{aligned}$$

It follows that

$$\Lambda_{l_i}(\mathbf{L}) \quad \geq \quad (p - i + 1) \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}) \ .$$

The proof of the inductive claim is now complete. □

Lemma 6.2 implies that for $\mathbf{L}$, there are p distinct links with latency larger than $\mathsf{OPT}(\mathbf{w}, \mathbf{c})$. Since $\sum_{j \in [m]} \Lambda_j(\mathbf{L}) = \sum_{j \in [m]} \Lambda_j(\mathbf{Q})$ and $\Lambda_j(\mathbf{Q}) \leq \mathsf{OPT}(\mathbf{w}, \mathbf{c})$ for each $j \in [m]$, it follows that there is some other link with latency smaller than $\mathsf{OPT}(\mathbf{w}, \mathbf{c})$. So, $p \leq m - 1$ or $p + 1 \leq m$, as needed. □

7. Epilogue

We presented a comprehensive collection of lower and upper bounds on Price of Anarchy for the model of restricted parallel links, where we considered only pure Nash equilibria. The case of identical users is the only case for which we do not yet know tight bounds. Most important, what are tight bounds on Price of Anarchy (for the general case of arbitrary users and related links) when *all* (mixed) Nash equilibria are considered? Deriving such tight bounds remains an important open problem.

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