The Increase of the Instability of Networks due to Quasi-Static Link Capacities ⋆

D. Koukopoulos [a,∗], M. Mavronicolas [b], P. Spirakis [c]

[a] *RACTI, 61 Riga Fereou, 26221 Patras, Greece.*

[b] *Department of Computer Science, University of Cyprus, 1678 Nicosia, Cyprus.*

[c] *Department of Computer Engineering and Informatics, University of Patras and RACTI, 61 Riga Fereou, 26221 Patras, Greece.*

Abstract

In this work, we study the impact of the dynamic changing of the network link capacities on the stability properties of *packet-switched* networks. Especially, we consider the *Adversarial, Quasi-Static Queueing Theory* model, where each link capacity may take on only two possible (integer) values, namely 1 and $C > 1$ under a (w, ρ)-*adversary*. We show that allowing the dynamic changing of the link capacities of a network with just ten nodes that uses the LIS (*Longest-in-System*) protocol for contention-resolution results in instability at rates $\rho > \sqrt{2} - 1$ for large enough values of C. The combination of dynamically changing link capacities with compositions of contention-resolution protocols on network queues suffices to drop the instability bound of a network to a substantially low value. We show that the composition of LIS with any of SIS (*Shortest-in-System*), NTS (*Nearest-to-Source*) and FTG (*Furthest-to-Go*) protocols is unstable at rates $\rho > \sqrt{2} - 1$ for large enough values of C. We prove that the instability bound of the network subgraphs that are forbidden for stability is affected by the dynamic changing of the link capacities presenting improved instability bounds for all the *directed subgraphs* that are known to be *forbidden* for stability on networks running a certain greedy protocol.

Key words: Adversarial Queueing Theory; Network Stability; Greedy Protocols

⋆ Part of this work has been appeared in the Proceedings of the 10th International Colloquium on Structural Information and Communication Complexity, Umea, Sweden, 2003, pp. 179–194. This work has been partially supported by the 6th Framework Programme of the European Union under contract number 001907 (DELIS).

∗ Corresponding author. Telephone: +302641034473, Fax:+302641058075.
Email addresses: koukopou@ceid.upatras.gr (D. Koukopoulos), mavronic@ucy.ac.cy (M. Mavronicolas), spirakis@cti.gr (P. Spirakis).

1 Introduction

1.1 Motivation and Framework

Objectives. We are interested in the behavior of *packet-switched networks* in which packets arrive dynamically at the *nodes* and they are routed in discrete time steps across the *links*. Recent years have witnessed a vast amount of work on analyzing packet-switched networks under *non-probabilistic* assumptions (rather than stochastic ones); we work within a model of *worst-case* continuous packet arrivals, originally proposed by Borodin *et al.* [4] and termed *Adversarial Queueing Theory* to reflect the assumption of an *adversarial* way of packet generation and path determination.

A major issue that arises in such a setting is that of *stability*– will the number of packets in the network remain bounded at all times? The answer to this question may depend on the *rate* of injecting packets into the network, the *capacity* of the links, which is the rate at which a link forwards outgoing packets, and the *protocol* that is used to resolve the conflict when more than one packet wants to cross a given link in a single time step. The underlying goal of our study is to establish the stability properties of networks and protocols when packets are injected by an adversary (rather than by an oblivious randomized process) and the link capacities are chosen by the same adversary in a dynamic way.

Model of Quasi-Static Capacities. Most studies of packet-switched networks assume that one packet can cross a network link (an edge) in a single time step. This assumption is well motivated when we assume that all network links are identical. However, a packet-switched network can contain different types of links, which is common especially in large-scale networks like Internet. Then, it is well motivated to assign a capacity to each link. Furthermore, if each link capacity takes on values in the two-valued set of integers $\{1, C\}$ for $C > 1$, C takes on large values and each value remains fixed for a long time, then we can consider approximately as a link failure the assigning of unit capacity to a link, while the assigning of capacity C to a link can be considered as the proper service rate. Therefore, the study of the stability behavior of networks and protocols under our model of quasi-static capacities can be considered as an approximation of the fault-tolerance of a network where links can temporarily fail (zero capacity).

In this work, we consider the impact on the stability behavior of protocols and networks if the adversary besides the packet injections in paths which it determines, it also can set the capacities of network edges in each time step. This subfield of study was initiated by Borodin *et al.* in [5]. Note that we

Table 1
Greedy protocols considered in this paper. (**US** stands for universally stable)

Protocol name	Which packet it advances:	**US**
SIS (*Shortest-In-System*)	The most recently injected packet	√ [1]
LIS (*Longest-In-System*)	The least recently injected packet	√ [1]
FTG (*Furthest-To-Go*)	The furthest packet from its destination	√ [1]
NTS (*Nearest-To-Source*)	The nearest packet to its origin	√ [1]
NTG-U-LIS (*Nearest-To-Go-Using-LIS*)	The nearest packet to its destination or the same as LIS for tie-breaking	**X** [2]

continue to assume uniform packet sizes.

Stability. Roughly speaking, a protocol P is *stable* [4] on a network $\mathcal{G}$ against an adversary $\mathcal{A}$ of rate ρ if there is a constant B (which may depend on $\mathcal{G}$ and $\mathcal{A}$) such that the number of packets in the system is bounded at all times by B. On the other hand, *a protocol* P *is universally stable* [4] if it is stable against every adversary of rate less than 1 and on every network. We also say that *a network* $\mathcal{G}$ *is universally stable* [4] if every greedy protocol is stable against every adversary of rate less than 1 on $\mathcal{G}$. We say *forbidden subgraphs* for network stability [2,8] any graph that is obtained by replacing any edge of the graphs $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4, \mathcal{U}_1$ and $\mathcal{U}_2$ (see Figure 4) by disjoint directed paths.

Greedy Protocols. We consider five *greedy* contention-resolution protocols– ones that always advance a packet across a queue (but one packet at each discrete time step) whenever there resides at least one packet in the queue (Table 1).

1.2 Contribution

We define here the *weakest* possible adversary of dynamically changing network *link capacities* in the context of Adversarial Queueing Theory where the adversary may set link capacities to any of two integer values 1 and C ($C > 1$ is a parameter called high capacity).[1] Moreover, once a link capacity takes on a value, the value stays fixed for a continuous time period proportional to the number of packets in the system at the time of setting the capacity to the value. We call this the *Adversarial, Quasi-Static Queueing Theory* model. In this framework, we consider four protocols LIS, SIS, FTG, NTS; all four were shown universally stable in the model of Adversarial Queueing Theory.

[1] In the classical Adversarial Queueing Theory only one capacity value is available to the adversary.

Table 2
Instability bounds of forbidden subgraphs in AQM vs. AQSQM. We denote **AQM** the Adversarial Queueing Theory Model, **AQSQM** the Adversarial Quasi-Static Queueing Theory Model and *s.p.* the simple path.

	Apply to:	Instability (AQM)	Instability (AQSQM)
$\mathcal{S}_1$	pure s.p.	$\rho > 0.87055$ [2, Lemma 12]	$\rho > 0.8191$ [Thm. 5.1]
$\mathcal{S}_2$	pure s.p.	$\rho > 0.84089$ [2, Lemma 13]	$\rho > 0.8191$ [Thm. 5.2]
$\mathcal{S}_3$	pure s.p.	$\rho > 0.84089$ [2, Lemma 14]	$\rho > 0.8191$ [Thm. 5.3]
$\mathcal{S}_4$	pure s.p.	$\rho > 0.84089$ [2, Lemma 15]	$\rho > 0.8191$ [Thm. 5.4]
$\mathcal{U}_1$	not pure s.p.	$\rho > 0.84089$ [2, Theorem 3]	$\rho > 0.794$ [Thm. 5.5]
$\mathcal{U}_2$	not pure s.p.	$\rho > 0.84089$ [2, Lemma 9]	$\rho > 0.754$ [Thm. 5.6]

- We construct a simple LIS network of only 10 nodes that is unstable at rates $\rho > \sqrt{2} - 1$ for large enough values of C (Theorem 3.1). This result is the first one that presents an instability bound on the injection rate less than 1/2 for a small-size network. Till now instability bounds of 1/2 or less have been proved only on parameterized networks. To show this, we use an adversarial construction that sets properly the capacities of various networks links to unit for specified time intervals in order to accumulate packets.
- We consider networks where different protocols may run on their nodes (heterogeneous networks, Internet). Thus, we prove that the composition of the LIS protocol with any of SIS, NTS and FTG is unstable at rates $\rho > \sqrt{2}-1$ (for large enough values of C) (Theorems 4.1, 4.2 and 4.3). To show this, we provide interesting combinatorial constructions of networks where we specify the contention-resolution protocol to be used to each queue.
- We examine the impact on network stability of dynamically changing network link capacities presenting bounds on injection rate that guarantee instability for all the *directed subgraphs* that are known to be *forbidden* for stability. Through involved adversarial constructions we improve the state-of-the-art instability bound that is induced by certain known forbidden subgraphs on networks running a certain greedy protocol (Theorems 5.1, 5.2, 5.3, 5.4, 5.5 and 5.6). More specifically, we improve the instability bound of the six simple subgraphs in Figure 4 that have been proved in [2] to be forbidden subgraphs for the universal stability of networks. For purpose of completeness and comparison, we summarize, in Table 2, all results that are shown in this work and in [2] concerning instability bounds on the injection rate for the forbidden subgraphs ($\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{S}_4$, $\mathcal{U}_1$, $\mathcal{U}_2$).

Adversarial Queueing Model. Adversarial Queueing Theory was developed by Borodin et al. [4] as a more realistic model that replaces traditional stochastic assumptions in Queueing Theory [6] by more robust, worst-case ones. It received a lot of interest and attention in the study of stability and instability issues (see, e.g., [1,2,7,9,11,13]). The universal stability of various natural greedy protocols (SIS, LIS, NTS and FTG) was established by Andrews *et al.* [1]. Also, several greedy protocols such as NTG (Nearest-To-Go) have been proved unstable at arbitrarily small rates of injection in [13].

Stability in Heterogeneous Networks. The subfield of study of the stability properties of compositions of universally stable protocols was introduced by Koukopoulos *et al.* in [9,11,10] where lower bounds of 0.683, 0.519 and 0.5 on the injection rates that guarantee instability for the composition pairs LIS-SIS, LIS-NTS and LIS-FTG were presented.

Instability of Forbidden Subgraphs. Alvarez et al. in [2, Theorems 8, 12] give a characterization for the universal stability of directed networks when the packets follow simple paths (paths do not contain repeated edges) that are pure (simple paths do not contain repeated vertices) and simple paths that are not pure (simple paths contain repeated vertices).[2] According to this characterization, a directed network graph is not pure simple path universally stable if and only if it does not contain as subgraphs any of the extensions of the subgraphs $\mathcal{U}_1$ or $\mathcal{U}_2$ [2, Theorem 8]; it is pure simple path universally stable if and only if it does not contain as subgraphs any of the extensions of the subgraphs $\mathcal{S}_1$ or $\mathcal{S}_2$ or $\mathcal{S}_3$ or $\mathcal{S}_4$ [2, Theorem 12] (see Figure 4).

Stability Issues in Dynamic Networks. Borodin *et al.* in [5] studied for the first time the impact on stability when the edges of a network can have capacities. They proved that the universal stability of networks is preserved under this varying context. Also, it was shown that many well-known universally stable protocols (SIS, NTS, FTG) do maintain their universal stability when the link capacity is changing dynamically, whereas the universal stability of LIS is not preserved. More specifically Borodin *et al.* in [5, Theorem 1] presented for the first time an instability bound of $\rho > C/(2C-1) > 0.5$ for the LIS protocol.

[2] Corresponding characterization for the stability of undirected networks was shown in [1, Theorem 3.16].

1.4 Road Map

The rest of this paper is organized as follows. Section 2 presents model definitions. Section 3 presents our instability bound for LIS. Section 4 demonstrates instability bounds for protocol compositions. Section 5 shows instability bounds for forbidden subgraphs. We conclude, in Section 6, with a discussion of our results and some open problems.

2 Definitions and Preliminaries

The model definitions are patterned after those in [4, Section 3], they are adjusted to reflect the fact that the edge capacities may vary arbitrarily as in [5, Section 2], but we address the weakest possible model of changing capacities. We consider that a routing network is modelled by a directed graph $\mathcal{G} = (V, E)$. Each node $u \in V$ represents a communication switch, and each edge $e \in E$ represents a link between two switches. In each node, there is a buffer (queue) associated with each outgoing link. Time proceeds in discrete time steps. Buffers store packets that are injected into the network with a route, which is a simple directed path in $\mathcal{G}$. A *packet* is an atomic entity that resides at a buffer at the end of any step. It must travel along paths in the network from its *source* to its *destination,* both of which are nodes in the network. When a packet is injected, it is placed in the buffer of the first link on its route. When a packet reaches its destination, we say that it is *absorbed.* During each step, a packet may be sent from its current node along one of the outgoing edges from that node. Edges can have different integer capacities, which may or may not vary over time. Denote $\mathsf{C}_e(t)$ the *capacity* of the edge e at time step t. That is, we assume that the edge e is capable of simultaneously transmitting up to $\mathsf{C}_e(t)$ packets at time t.

Let $C > 1$ be an integer parameter. We demand that $\forall e$ and $\forall t$ $C_e(t) \in \{1, C\}$ (i.e. each edge capacity can get only two values, high and low). We also demand for each edge e that $C_e(t)$ stays at some value for a continuous period of time at least equal to $f(\rho, C)s$ time steps, where s is the number of packets in the system at the time of setting the link capacity to the value and $f(\rho, C)$ is a function of the injection rate ρ of the adversary in the network and the high link capacity C. We call this the *Adversarial, Quasi-Static Queueing Theory Model.* This model is the weakest possible of the models that are implied by [5].

Any packets that wish to travel along an edge e at a particular time step, but they are not sent, they wait in a queue for the edge e. The *delay* of a packet is the number of steps that are spent by the packet, while waiting in queues. At each step, an *adversary* generates a set of requests. A *request* is a *path* specifying the route that will be followed by a packet. [3] We say that the adversary generates a set of packets when it generates a set of requested paths. Also, we say that a packet p *requires* an edge e at time t if the edge e lies on the path from its position to its destination at time t. We restrict our study to the case of *non-adaptive* routing, where the path that is traversed by each packet is fixed at the time of injection, so that we are able to focus on queueing rather than routing aspects of the problem. (See [3] for an extension of the adversarial model to the case of *adaptive* routing.) There are no computational restrictions on how the adversary chooses its requests at any given time step.

Fix any arbitrary positive integer $w \geq 1$. For any edge e of the network and any sequence of w consecutive time steps, define $N(w, e)$ to be the number of paths that are injected by the adversary during the time interval of w consecutive time steps requiring to traverse the edge e. For any constant ρ, $0 < \rho \leq 1$, a (w, ρ)-*adversary* is an adversary that injects packets subject to the following *load condition*: For every edge e and for every sequence τ of w consecutive time steps, $N(\tau, e) \leq \rho \sum_{t \in \tau} \mathsf{C_e}(t)$. We say that a (w, ρ)-adversary injects packets at rate ρ with *window size* w. The assumption that $\rho \leq 1$ ensures that it is not necessary a priori that some edge of the network is congested (that happens when $\rho > 1$).

In order to formalize the behavior of a network under the adversarial, quasi-static queueing theory model, we use the notions of *system* and *system configuration*. A triple of the form $\langle \mathcal{G}, \mathcal{A}, \mathsf{P} \rangle$ where $\mathcal{G}$ is a network, $\mathcal{A}$ is an adversary and P is the used protocol on the network queues is called a system. The execution of the system proceeds in global time steps numbered $0, 1, \ldots$. Each time-step is divided in two sub-steps. In the first sub-step, one packet is sent from each non-empty buffer over its corresponding link. In the second sub-step, packets are received by the nodes at the other end of the links; they are absorbed (eliminated) if that node is their destination, and otherwise they are placed in the buffer of the next link on their respective routes. New packets are injected in the second sub-step.

In every time step t, the current configuration C^t of a system $\langle \mathcal{G}, \mathcal{A}, \mathsf{P} \rangle$ is a collection of sets $\{S_e^t : e \epsilon \mathcal{G}\}$, such that S_e^t is the set of packets waiting in the queue of the edge e at the end of step t. If the current system configuration is C^t, we obtain the system configuration C^{t+1} for the next time step as follows: (i) Addition of new packets to some of the sets S_e^t, each of which has an assigned path in $\mathcal{G}$, and (ii) for each non-empty set S_e^t deletion of a single

[3] In this work, it is assumed, as it is common in packet routing, that all paths are simple paths with no overlapping edges. However, in Section 5 we consider two different kinds of simple paths: simple paths that do not contain repeated vertices (pure simple paths) and simple paths that contain repeated vertices (not pure simple paths).

packet $p \epsilon S_e^t$ and its insertion into the set S_f^{t+1} where f is the edge following e on its assigned path (if e is the last edge on the path of p, then p is not inserted into any set.) A time evolution of the system for a $((w, \rho))$-adversary is a sequence of such configurations $C^1, C^2, \ldots$. An execution of the adversary's construction on a system $\langle \mathcal{G}, \mathcal{A}, \mathsf{P} \rangle$ determines the time evolution of the system configuration.

A *contention-resolution* protocol specifies, for each pair of an edge e and a time step, which packet among those waiting at the tail of the edge e will be moved along the edge e. A *greedy* contention-resolution protocol always specifies some packet to move along the edge e if there are packets waiting to use the edge e. In this work, we restrict attention to deterministic, greedy contention-resolution protocols. In particular, we consider:

- SIS (*Shortest-in-System*) gives priority to the most recently injected packet into the network;
- LIS (*Longest-in-System*) gives priority to the least recently injected packet into the network;
- FTG (*Furthest-to-Go*) gives priority to the packet that has to traverse the larger number of edges to its destination;
- NTS (*Nearest-to-Source*) gives priority to the packet that has traversed the smallest number of edges from its origin;
- NTG-U-LIS (*Nearest-To-Go-Using-LIS*) gives priority to the nearest packet to its destination or the least recently injected packet for tie-breaking.

All these contention-resolution protocols require some tie-breaking rule in order to be unambiguously defined. In this work, we can assume any well-determined tie breaking rule for the adversary.

In the adversarial constructions we study here for proving instability, we split time into *phases*. In each phase, we study the evolution of the *system configuration* by considering corresponding *time rounds*. For each phase, we inductively prove that the number of packets of a specific subset of queues in the system increases in order to guarantee instability. This inductive argument can be applied repeatedly, thus showing instability.

Furthermore, we assume that there is a sufficiently large number of packets s_0 in the initial system configuration. This will imply instability results for networks with an *empty* initial configuration, as it was established by Andrews *et al.* [1, Lemma 2.9]. For simplicity, and in a way similar to that in [1] and in works following it, we omit floors and ceilings from our analysis, and we, sometimes, count time steps and packets only roughly. This may only result to loosing small additive constants, while it implies a gain in clarity.

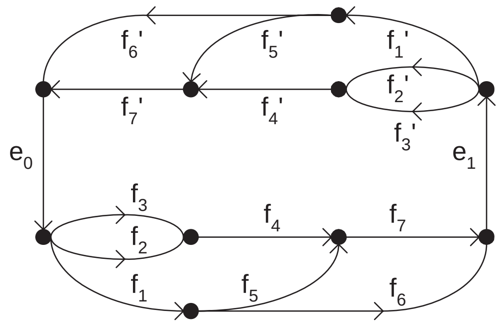

Fig. 1. The network $\mathcal{N}$.

3 Instability Bound for LIS

In this section, we present a lower bound on the injection rate that guarantees instability for the LIS protocol on the network $\mathcal{N}$ (see Figure 1). We show:

Theorem 3.1 *Let $\rho' = \sqrt{2} - 1 + \epsilon$ with $0 < \epsilon \leq 3/2 - \sqrt{2}$ and $C > 1$ where C is a particular function of ρ'. For the network $\mathcal{N}$ there is an adversary $\mathcal{A}$ of rate ρ that can change the link capacities of $\mathcal{N}$ between the two integer values 1 and C such that the system $\langle \mathcal{N}, \mathcal{A}, \mathsf{LIS} \rangle$ is unstable for every $\rho > \rho'$. When $C \to \infty$ the system $\langle \mathcal{N}, \mathcal{A}, \mathsf{LIS} \rangle$ is unstable for $\rho > \sqrt{2} - 1$.*

PROOF. The construction of the adversary $\mathcal{A}$ is broken into phases.

Inductive Hypothesis: At the beginning of phase j (suppose j is even), there are s_j packets that are queued in the queues f_1', f_4', f_5', f_7' (in total) requiring to traverse the edges e_0, f_2, f_4.

Induction Step: At the beginning of phase $j+1$, there will be $s_{j+1} > s_j$ packets that will be queued in the queues f_1, f_4, f_5, f_7 (in total) requiring to traverse the edges e_1, f_2', f_4'.

We will construct an adversary $\mathcal{A}$ such that the induction step will hold. The main ideas of the construction of $\mathcal{A}$ are (a) the accurate tuning of the duration of each round of every phase j (as a function of the high capacity C, the injection rate ρ and the number of packets in the system at the beginning of phase j, s_j) to maximize the growth of the packet population in the system, (b) the careful setting of the capacities of some edges to 1 for specified time intervals in order to accumulate packets, and (c) the careful injection of packets in order to guarantee that the load condition is satisfied. When we inject packets into different network queues simultaneously, we choose to assign them paths that do not overlap in order to preserve the load condition.

Proving that the induction step holds, we ensure that the inductive hypothesis will hold at the beginning of phase $j+1$ for the symmetric edges with an increased value of s_j, $s_{j+1} > s_j$. By the symmetry of the network, repeating the phase construction an unbounded number of times, we will create an unbounded number of packets in the network.

From the inductive hypothesis, initially, there are s_j packets (that constitute the set of packets S) in the queues $f_1^{'}, f_4^{'}, f_5^{'}, f_7^{'}$ requiring to traverse the edges e_0, f_2, f_4. In order to prove the induction step, it is assumed that the set S has a large enough number of $|S| = s_j$ packets in the initial system configuration.

During phase j the adversary plays three rounds of injections as follows:

- **Round 1:** It lasts $|T_1| = s_j/C$ time steps.

  Adversary's behavior. During this round the edges $f_1^{'}, f_2^{'}, f_4^{'}, f_5^{'}, f_7^{'}, e_0, f_1, f_5, f_7, e_1$ have high capacity C, while all the other edges have unit capacity. The adversary injects a set X of $|X| = \rho C|T_1|$ packets in the queue e_0 wanting to traverse the edges $e_0, f_1, f_5, f_7, e_1, f_2^{'}, f_4^{'}$ and a set S_1 of $|S_1| = \rho|T_1|$ packets in the queue f_2 wanting to traverse the edges f_2, f_4. These injections satisfy the load condition because the edges $e_0, f_1, f_5, f_7, e_1, f_2^{'}, f_4^{'}$ have high capacity C and the edges f_2, f_4 have unit capacity during this round, and the injection paths of the different packet sets do not have overlapped edges.

  Evolution of the system configuration. The packets of the set S delay the packets of the set X in the queue e_0 and the packets of the set S_1 in the queue f_2 because they are longer time in the system than the packets of the sets X and S_1. At the same time, the packets of the set S are delayed in f_2 due to the unit capacity of the edge f_2. At the end of this round, the remaining packets of the set S in f_2 are $|S^{'}| = |S| - |T_1|$ packets. The packets of the set S that manage to traverse the edge f_2 continue traversing their remaining path and they are absorbed. Therefore, the number of packets in the queue f_2 at the end of this round requiring to traverse the edges f_2, f_4 is a set S_2 of $|S_2| = |S^{'}| + |S_1|$ packets.

- **Round 2:** It lasts $|T_2| = |S_2|/C$ time steps.

  Adversary's behavior. During this round the edges $f_2, f_4, f_7, e_0, e_1, f_2^{'}, f_4^{'}$ have high capacity C, while all the other edges have unit capacity. The adversary injects a set Y of $|Y| = \rho C|T_2|$ packets in the queue f_4 requiring to traverse the edges $f_4, f_7, e_1, f_2^{'}, f_4^{'}$. These packet injections satisfy the load condition because the assigned path consists of edges that have high capacity C during this round.

  Evolution of the system configuration. The packets of the set Y are delayed by the packets of the set S_2 in the queue f_4 because the packets of the set S_2 are longer time in the system than the packets of the set Y. The packets of the set S_2 traverse the edge f_4 and they are absorbed. At the same time, the packets of the set X are delayed in the queue f_1 due to its unit capacity.

Therefore, the remaining packets of the set X in the queue f_1 is a set $|X^{'}|$ of $|X^{'}| = |X| - |T_2|$ packets.

- **Round 3:** It lasts $|T_3| = |X^{'}|/C$ time steps.

  Adversary's behavior. During this round the edges $f_1, f_6, e_1, f_2^{'}, f_4^{'}$ have high capacity C, while all the other edges have unit capacity. The adversary injects a set Z of $|Z| = \rho C|T_3|$ packets in the queue f_1 requiring to traverse the edges $f_1, f_6, e_1, f_2^{'}, f_4^{'}$. These packet injections satisfy the load condition because the assigned path consists of edges that have high capacity C during this round.

  Evolution of the system configuration. The packets of the set $X^{'}$ delay the packets of the set Z in the queue f_1 because they are longer time in the system than the packets of the set Z. At the same time, the packets of the set $X^{'}$ are delayed in f_5 due to the unit capacity of the edge f_5. Therefore, the remaining packets of the set $X^{'}$ in the queue f_5 is a set $|X^{''}|$ of $|X^{''}| = |X^{'}| - |T_3|$ packets. Moreover, the packets of the set Y are delayed in f_4 due to the unit capacity of the edge f_4 during this round. Therefore, the remaining packets of the set Y in the queue f_4 is a set $|Y^{'}|$ of $|Y^{'}| = |Y| - |T_3|$ packets.

  Note that during this round $|K| = 2|T_3|$ packets arrive in the queue f_7 from the queues f_4, f_5. However, the edge f_7 has unit capacity and the duration of this round is $|T_3|$ time steps. Consequently, at the end of this round $|L| = |T_3|$ packets will remain in the queue f_7 requiring to traverse the edges $f_7, e_1, f_2^{'}, f_4^{'}$. Thus, the number of packets in the queues f_1, f_4, f_5, f_7 requiring to traverse the edges $e_1, f_2^{'}, f_4^{'}$ at the end of this round is $s_{j+1} = |X^{''}| + |Y^{'}| + |Z| + |L|$. Substituting the quantities $|X^{''}|, |Y^{'}|, |Z|$ and $|L|$, we take $s_{j+1} = \rho s_j - \frac{1+\rho}{C} s_j + \frac{2-\rho}{C^2} s_j + \frac{\rho-1}{C^3} s_j + \rho s_j + \frac{\rho^2-2\rho}{C} s_j + \frac{1}{C^2} s_j + \frac{\rho-1}{C^3} s_j + \rho^2 s_j - \frac{\rho}{C} s_j + \frac{\rho-\rho^2}{C^2} s_j + \frac{\rho}{C} s_j - \frac{1}{C^2} s_j + \frac{1-\rho^2}{C^3} s_j$.

  In order to have instability, we must have $s_{j+1} > s_j$, that is $\rho^2[1 + \frac{1}{C} - \frac{1}{C^2}] + \rho[2 - \frac{3}{C} + \frac{1}{C^3}] + [-\frac{1}{C} + \frac{2}{C^2} - \frac{1}{C^3}] > 1$. Initially, note that the following inequalities hold: (i) $1 + \frac{1}{C} - \frac{1}{C^2} < 1 + \frac{1}{C}$, (ii) $2 - \frac{3}{C} + \frac{1}{C^3} < 2$, (iii) $1 + \frac{1}{C} - \frac{2}{C^2} + \frac{1}{C^3} > 1 - \frac{2}{C^2}$. Therefore, the inequality becomes $\rho^2[1 + \frac{1}{C}] + 2\rho > 1 - \frac{2}{C^2}$. Thus, it suffices to be shown that $\rho^2[C^2 + C] + 2\rho C^2 > C^2 - 2$. This inequality holds for ρ larger than the largest root ρ_1 of the polynomial $\rho^2[C^2 + C] + 2\rho C^2 - (C^2 - 2)$. The largest root of the polynomial is $\rho_1 = \frac{-2C^2 + \sqrt{4C^4 + 4(C^2-2)(C^2+C)}}{2C(C+1)} = \frac{\sqrt{2C^2+C-2-\frac{2}{C}}}{C+1} - \frac{C}{C+1}$. But, $\frac{\sqrt{2C^2+C-2-\frac{2}{C}}}{C+1} < \sqrt{\frac{2C}{C+1}}$. Therefore, $\rho_1 < \sqrt{\frac{2C}{C+1}} - \frac{C}{C+1}$. Thus, it holds that $\rho > \sqrt{\frac{2C}{C+2}} - \frac{C}{C+2} > \rho_1$.

  When $C \to \infty$, it holds that $\frac{1}{C^k} \to 0$ for all $k \geq 1$. Then, the inequality $\rho^2[1+\frac{1}{C}-\frac{1}{C^2}]+\rho[2-\frac{3}{C}+\frac{1}{C^3}]+[-\frac{1}{C}+\frac{2}{C^2}-\frac{1}{C^3}] > 1$ becomes $\rho^2+2\rho-1 > 0$ which holds for $\rho > \sqrt{2}-1$. Note that if we have a sequence of equations $f_C(\rho)$ and there exists the limit $\lim_{C\to\infty} f_C(\rho) = f_\infty(\rho)$, then it holds fundamentally by the theory of function limits that if $\rho(C)$ is the root of $f_C(\rho) = 0$, then $\lim_{C\to\infty} \rho(C)$ is the root of $f_\infty(\rho)$. Therefore, for $\rho > \sqrt{2} - 1$ the

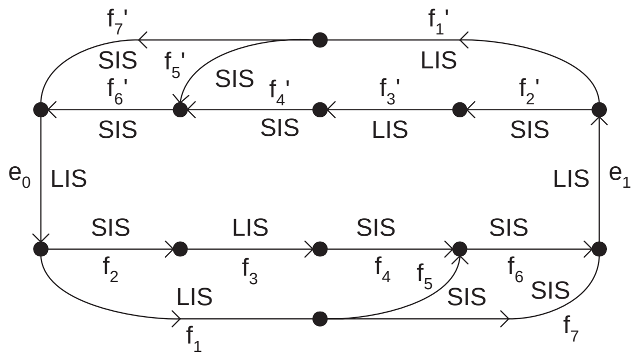

Fig. 2. The network $\mathcal{G}_1$.

system is unstable. Analyzing the inequality $\rho' > \sqrt{\frac{2C}{C+1}} - \frac{C}{C+1}$, we take $C > \frac{-[2(\rho')^2+2\rho'-2]+\sqrt{4[(\rho')^2+\rho'-1]^2-4(\rho')^2[(\rho')^2+2\rho'-1]}}{2[(\rho')^2+2\rho'-1]} = 1$. If we replace ρ' with $\sqrt{2}-1+\epsilon$ at this inequality, we estimate a $C > 1$ such that $s_{j+1} > s_j$. This argument can be repeated for an infinite number of phases showing that the number of packets in the system increases forever for $\rho > \sqrt{2}-1$. □

4 Instability Bounds for Protocol Compositions

In this section, we demonstrate instability bounds for protocol compositions. First, we show an instability bound for the composition of LIS and SIS protocols on the network $\mathcal{G}_1$ (see Figure 2). The edges $e_0, e_1, f_1, f_1', f_3, f_3'$ of $\mathcal{G}_1$ use the LIS protocol, while the remaining edges of $\mathcal{G}_1$ use the SIS protocol.

Theorem 4.1 *Let $\rho' = \sqrt{2}-1+\epsilon$ with $0 < \epsilon \leq 3/2-\sqrt{2}$ and $C > 1$ where C is a particular function of ρ'. For the network $\mathcal{G}_1$ there is an adversary $\mathcal{A}_1$ of rate ρ that can change the link capacities of $\mathcal{G}_1$ between the two integer values 1 and C such that the system $\langle \mathcal{G}_1, \mathcal{A}_1, \mathsf{LIS}, \mathsf{SIS} \rangle$ is unstable for every $\rho > \rho'$. When $C \to \infty$ the system $\langle \mathcal{G}_1, \mathcal{A}_1, \mathsf{LIS}, \mathsf{SIS} \rangle$ is unstable for $\rho > \sqrt{2}-1$.*

PROOF. The construction of the adversary $\mathcal{A}_1$ is broken into phases.

Inductive Hypothesis: At the beginning of phase j (suppose j is even), there are s_j packets that are queued in the queues f_1', f_4', f_5', f_6' (in total) requiring to traverse the edges e_0, f_2, f_3, f_4.

Induction Step: At the beginning of phase $j+1$, there will be $s_{j+1} > s_j$ packets that will be queued in the queues f_1, f_4, f_5, f_6 (in total) requiring to traverse the edges e_1, f_2', f_3', f_4'.

We will construct an adversary $\mathcal{A}_1$ such that the induction step will hold. Proving that the induction step holds, we ensure that the inductive hypothesis will hold at the beginning of phase $j+1$ for the symmetric edges with an increased value of s_j, $s_{j+1} > s_j$. By the symmetry of the network, repeating the phase construction an unbounded number of times, we will create an unbounded number of packets in the network.

From the inductive hypothesis, initially, there are s_j packets (that constitute the set of packets S) in the queues $f_1^{'}, f_4^{'}, f_5^{'}$, $f_6^{'}$ requiring to traverse the edges e_0, f_2, f_3, f_4. In order to prove the induction step, it is assumed that the set S has a large enough number of $|S| = s_j$ packets in the initial system configuration.

During phase j, the adversary plays three rounds of injections as follows:

- **Round 1:** It lasts $|T_1| = s_j/C$ time steps.

  Adversary's behavior. During this round the edge f_2 has unit capacity, while all the other edges have high capacity C. Also, the adversary injects a set X of $|X| = \rho C|T_1|$ packets in the queue e_0 wanting to traverse the edges $e_0, f_1,\ f_5, f_6, e_1, f_2^{'}, f_3^{'}, f_4^{'}$ and a set S_1 of $|S_1| = \rho|T_1|$ packets in the queue f_2 wanting to traverse the edge f_2. These injections satisfy the load condition because the edges $e_0, f_1,\ f_5, f_6, e_1, f_2^{'}, f_3^{'}, f_4^{'}$ have high capacity C and the edge f_2 has unit capacity during this round, and the injection paths of the different packet sets do not have overlapped edges.

  Evolution of the system configuration. The packets of the set S delay the packets of the set X in the queue e_0 that uses the LIS protocol because they are longer time in the system than the packets of the set X. At the same time, the packets of the set S are delayed in the queue f_2 that uses the SIS protocol due to the packets of the set S_1, which are shorter time in the system than the packets of the set S, and the unit capacity of the edge f_2. At the end of this round, the remaining packets of the set S in f_2 is a set S_2 of $|S_2| = |S| - (|T_1| - |S_1|)$ packets. The packets of the set S that manage to traverse the edge f_2, traverse their remaining path and they are absorbed.

- **Round 2:** It lasts $|T_2| = |S_2|/C$ time steps.

  Adversary's behavior. During this round the edge f_1 has unit capacity, while all the other edges have high capacity C. Also, the adversary injects a set Y of $|Y| = \rho C|T_2|$ packets in the queue f_3 requiring to traverse the edges $f_3, f_4, f_6, e_1, f_2^{'},\ f_3^{'}, f_4^{'}$. These packet injections satisfy the load condition because the assigned path consists of edges that have high capacity C during this round.

  Evolution of the system configuration. The packets of the set S_2 delay the packets of the set Y in the queue f_3 that uses the LIS protocol because they are longer time in the system than the packets of the set Y. The packets of the set S_2 traverse the edge f_3 and they are absorbed. At the same time, the packets of the set X are delayed in the queue f_1 due to the unit capacity of the edge f_1. Thus, at the end of this round the remaining packets of the set X in the queue f_1 is a set $|X^{'}|$ of $|X^{'}| = |X| - |T_2|$ packets.

- **Round 3:** It lasts $|T_3| = |X^{'}|/C$ time steps.

  Adversary's behavior. During this round the edges f_4, f_5, f_6 have unit capacity, while all the other edges have high capacity C. Also, the adversary injects a set Z of $|Z| = \rho C|T_3|$ packets in the queue f_1 requiring to traverse the edges $f_1, f_7, e_1, f_2^{'}, f_3^{'}, f_4^{'}$. Also, it injects a set S_3 of $|S_3| = \rho|T_3|$ packets in the queue f_4 wanting to traverse the edge f_4, a set S_4 of $|S_4| = \rho|T_3|$ packets in the queue f_5 wanting to traverse the edge f_5 and a set S_5 of $|S_5| = \rho|T_3|$ packets in the queue f_6 wanting to traverse the edge f_6. These injections satisfy the load condition because the edges $f_1, f_7, e_1, f_2^{'}, f_3^{'}, f_4^{'}$ have high capacity C and the edges f_4, f_5, f_6 have unit capacity during this round, and the injection paths of the different packet sets do not have overlapped edges.

  Evolution of the system configuration. The packets of the set $X^{'}$ delay the packets of the set Z in the queue f_1 that uses the LIS protocol because they are longer time in the system than the packets of the set Z. At the same time the packets of the set $X^{'}$ are delayed in the queue f_5 that uses the SIS protocol due to the unit capacity of the edge f_5 during this round and the packets of the set S_4 that are shorter time in the system than the packets of the set $X^{'}$. Therefore, the remaining packets of the set $X^{'}$ in the queue f_5 is a set $|X^{''}|$ of $|X^{''}| = |X^{'}| - (|T_3| - |S_4|)|$. Moreover, the packets of the set Y are delayed in the queue f_4 that uses the SIS protocol due to the unit capacity of the edge f_4 during this round and the packets of the set S_3 that are shorter time in the system than the packets of the set Y. Thus, the remaining packets of the set Y in the queue f_4 is a set $|Y^{'}|$ of $|Y^{'}| = |Y| - (|T_3| - |S_3|)$ packets.

  Note that during this round $|K| = 2|T_3| - |S_3| - |S_4|$ packets arrive in the queue f_6 from the queues f_4, f_5. However, the edge f_6 has unit capacity and uses the SIS protocol that gives priority to the packets of the set S_5. Furthermore, the duration of this round is $|T_3|$ time steps. Consequently, at the end of this round the number of packets that remain in the queue f_6 requiring to traverse the edges $f_6, e_1, f_2^{'}, f_3^{'}, f_4^{'}$ is $|L| = |K| + |S_5| - |T_3|$. Thus, the number of packets in the queues f_1, f_4, f_5, f_6 requiring to traverse the edges $e_1, f_2^{'}, f_3^{'}, f_4^{'}$ at the end of this round is $s_{j+1} = |X^{''}| + |Y^{'}| + |Z| + |L|$. Substituting the quantities $|X^{''}|, |Y^{'}|, |Z|$ and $|L|$, we take $s_{j+1} = \rho s_j + \frac{\rho^2-\rho-1}{C} s_j + 2\frac{1-\rho}{C^2} s_j + \frac{-\rho^2+2\rho-1}{C^3} s_j + \rho s_j + \frac{2\rho^2-2\rho}{C} s_j + \frac{1-\rho}{C^2} s_j + \frac{-\rho^2+2\rho-1}{C^3} s_j + \rho^2 s_j - \frac{\rho}{C} s_j + \frac{\rho-\rho^2}{C^2} s_j + \frac{\rho-\rho^2}{C} s_j + 2\frac{\rho-1}{C^2} s_j + \frac{\rho^2-2\rho+1}{C^3} s_j$.

  In order to have instability, we must have $s_{j+1} > s_j$, that is $\rho^2[1 + \frac{2}{C} - \frac{1}{C^2} - \frac{1}{C^3}] + \rho[2 - \frac{3}{C} - \frac{1}{C^2} + \frac{2}{C^3}] + [-\frac{1}{C} + \frac{2}{C^2} - \frac{1}{C^3}] > 1$. Initially, note that the following inequalities hold: (i) $1 + \frac{2}{C} - \frac{1}{C^2} - \frac{1}{C^3} < 1 + \frac{2}{C}$, (ii) $2 - \frac{3}{C} - \frac{1}{C^2} + \frac{2}{C^3} < 2$, (iii) $1 + \frac{1}{C} - \frac{2}{C^2} + \frac{1}{C^3} > 1 - \frac{2}{C^2}$. Therefore, the inequality becomes

$\rho^2[1+\frac{2}{C}]+2\rho > 1-\frac{2}{C^2}$. Thus, it suffices to be shown that $\rho^2[C^2+2C]+2\rho C^2 > C^2-2$. This inequality holds for ρ larger than the largest root ρ_1 of the polynomial $\rho^2[C^2+2C]+2\rho C^2-(C^2-2)$. The largest root of the polynomial is $\rho_1 = \frac{-2C^2+\sqrt{4C^4+4(C^2-2)(C^2+2C)}}{2C(C+2)} = \frac{\sqrt{2C^2+2C-2-\frac{4}{C}}}{C+2} - \frac{C}{C+2}$. But, $\frac{\sqrt{2C^2+2C-2-\frac{4}{C}}}{C+2} < \sqrt{\frac{2C}{C+2}}$. Therefore, $\rho_1 < \sqrt{\frac{2C}{C+2}} - \frac{C}{C+2}$. Thus, it holds that $\rho > \sqrt{\frac{2C}{C+2}} - \frac{C}{C+2} > \rho_1$.

When $C \to \infty$, it holds that $\frac{1}{C^k} \to 0$ for all $k \geq 1$. Then, the inequality $\rho^2[1+\frac{2}{C}-\frac{1}{C^2}-\frac{1}{C^3}]+\rho[2-\frac{3}{C}-\frac{1}{C^2}+\frac{2}{C^3}]+[-\frac{1}{C}+\frac{2}{C^2}-\frac{1}{C^3}] > 1$ becomes $\rho^2+2\rho-1 > 0$ which holds for $\rho > \sqrt{2}-1$. Note that if we have a sequence of equations $f_C(\rho)$ and there exists the limit $\lim_{C\to\infty} f_C(\rho) = f_\infty(\rho)$, then it holds fundamentally by the theory of function limits that if $\rho(C)$ is the root of $f_C(\rho)=0$, then $\lim_{C\to\infty}\rho(C)$ is the root of $f_\infty(\rho)$. Therefore, for $\rho > \sqrt{2}-1$ the system is unstable. Analyzing the inequality $\rho' > \sqrt{\frac{2C}{C+2}} - \frac{C}{C+2}$, we take $C > \frac{-[4(\rho')^2+4\rho'-4]+\sqrt{16[(\rho')^2+\rho'-1]^2-16(\rho')^2[(\rho')^2+2\rho'-1]}}{2[(\rho')^2+2\rho'-1]} > 1$. If we replace ρ' with $\sqrt{2}-1+\epsilon$ at this inequality, we estimate a $C > 1$ such that $s_{j+1} > s_j$. This argument can be repeated for an infinite number of phases showing that the number of packets in the system increases forever for $\rho > \sqrt{2}-1$. □

Then, we show an instability bound for the composition of LIS and NTS protocols on the network $\mathcal{G}_1$ (see Figure 2). The network $\mathcal{G}_1$ is also used for proving the instability of the composition of LIS and SIS protocols (Theorem 4.1). However in this case, the edges $f_2, f_2', f_4, f_4', f_5, f_5', f_6, f_6', f_7, f_7'$ of $\mathcal{G}_1$ use the NTS protocol instead of the SIS protocol, while the remaining edges of $\mathcal{G}_1$ use the LIS protocol.

Theorem 4.2 *Let $\rho' = \sqrt{2}-1+\epsilon$ with $0 < \epsilon \leq 3/2-\sqrt{2}$ and $C > 1$ where C is a particular function of ρ'. For the network $\mathcal{G}_1$ there is an adversary $\mathcal{A}_2$ of rate ρ that can change the link capacities of $\mathcal{G}_1$ between the two integer values 1 and C such that the system $\langle \mathcal{G}_1, \mathcal{A}_2, \mathsf{LIS}, \mathsf{NTS}\rangle$ is unstable for every $\rho > \rho'$. When $C \to \infty$ the system $\langle \mathcal{G}_1, \mathcal{A}_2, \mathsf{LIS}, \mathsf{NTS}\rangle$ is unstable for $\rho > \sqrt{2}-1$.*

PROOF. We break the construction of the adversary $\mathcal{A}_2$ into phases.

Inductive Hypothesis: At the beginning of phase j (suppose j is even), there are s_j packets that are queued in the queues f_1', f_4', f_5', f_6' (in total) requiring to traverse the edges e_0, f_2, f_3, f_4.

Induction Step: At the beginning of phase $j+1$, there will be $s_{j+1} > s_j$ packets that will be queued in the queues f_1, f_4, f_5, f_6 (in total) requiring to traverse the edges e_1, f_2', f_3', f_4'.

We will construct an adversary $\mathcal{A}_2$ such that the induction step will hold. Proving that the induction step holds, we ensure that the inductive hypothesis will hold at the beginning of phase $j+1$ for the symmetric edges with an increased value of s_j, $s_{j+1} > s_j$. By the symmetry of the network, repeating the phase construction an unbounded number of times, we will create an unbounded number of packets in the network.

From the inductive hypothesis, initially, there are s_j packets (that constitute the set of packets S) in the queues $f_1^{'}, f_4^{'}, f_5^{'}$, $f_6^{'}$ requiring to traverse the edges e_0, f_2, f_3, f_4. In order to prove the induction step, it is assumed that the set S has a large enough number of $|S| = s_j$ packets in the initial system configuration.

During phase j, the adversary plays three rounds of injections as follows:

- **Round 1:** It lasts $|T_1| = s_j/C$ time steps.

  Adversary's behavior. During this round the edge f_2 has unit capacity, while all the other edges have high capacity C. Also, the adversary injects a set X of $|X| = \rho C|T_1|$ packets in the queue e_0 wanting to traverse the edges $e_0, f_1,\ f_5, f_6, e_1, f_2^{'}, f_3^{'}, f_4^{'}$ and a set S_1 of $|S_1| = \rho|T_1|$ packets in the queue f_2 wanting to traverse the edge f_2. These injections satisfy the load condition because the edges $e_0, f_1,\ f_5, f_6, e_1, f_2^{'}, f_3^{'}, f_4^{'}$ have high capacity C and the edge f_2 has unit capacity during this round, and the injection paths of the different packet sets do not have overlapped edges.

  Evolution of the system configuration. The packets of the set S delay the packets of the set X in the queue e_0 that uses the LIS protocol because they are longer time in the system than the packets of the set X. At the same time, the packets of the set S are delayed in the queue f_2 that uses the NTS protocol due to the packets of the set S_1 which are nearest to their source (queue f_2) than the packets of the set S and the unit capacity of the edge f_2. At the end of this round, the remaining packets of the set S in f_2 are $|S_2| = |S| - (|T_1| - |S_1|)$. The packets of the set S that manage to traverse the edge f_2, traverse their remaining path and they are absorbed.

- **Round 2:** It lasts $|T_2| = |S_2|/C$ time steps.

  Adversary's behavior. During this round, the edge f_1 has unit capacity, while all the other edges have capacity C. Also, the adversary injects a set Y of $|Y| = \rho C|T_2|$ packets in the queue f_3 requiring to traverse the edges $f_3, f_4, f_6, e_1, f_2^{'},\ f_3^{'},\ f_4^{'}$. These packet injections satisfy the load condition because the assigned path consists of edges that have high capacity C during this round.

  Evolution of the system configuration. The packets of the set S_2 delay the packets of the set Y in the queue f_3 that uses the LIS protocol because they are longer time in the system than the packets of the set Y. The packets of the set S_2 traverse the edge f_3 and they are absorbed. At the same time, the packets of the set X are delayed in the queue f_1 due to the unit capacity of the edge f_1. Thus, at the end of this round the remaining packets of the set X in the queue f_1 is a set $|X^{'}|$ of $|X^{'}| = |X| - |T_2|$ packets.

- **Round 3:** It lasts $|T_3| = |X^{'}|/C$ time steps.

  Adversary's behavior. During this round the edges f_4, f_5, f_6 have unit capacity, while all the other edges have high capacity C. Also, the adversary injects a set Z of $|Z| = \rho C|T_3|$ packets in the queue f_1 requiring to traverse the edges $f_1, f_7, e_1, f_2^{'}, f_3^{'}, f_4^{'}$, a set S_3 of $|S_3| = \rho|T_3|$ packets in the queue f_4 wanting to traverse the edge f_4, a set S_4 of $|S_4| = \rho|T_3|$ packets in the queue f_5 wanting to traverse the edge f_5 and a set S_5 of $|S_5| = \rho|T_3|$ packets in the queue f_6 wanting to traverse the edge f_6. These injections satisfy the load condition because the edges $f_1, f_7, e_1, f_2^{'}, f_3^{'}, f_4^{'}$ have high capacity C and the edges f_4, f_5, f_6 have unit capacity during this round, and the injection paths of the different packet sets do not have overlapped edges.

  Evolution of the system configuration. The packets of the set $X^{'}$ delay the packets of the set Z in the queue f_1 that uses the LIS protocol because they are longer time in the system than the packets of the set Z. At the same time the packets of the set $X^{'}$ are delayed in the queue f_5 that uses the NTS protocol due to the unit capacity of the edge f_5 during this round and the packets of the set S_4 that are nearest to their source (queue f_5) than the packets of the set $X^{'}$ (queue e_0). Therefore, the remaining packets of the set $X^{'}$ in the queue f_5 is a set $|X^{''}|$ of $|X^{''}| = |X^{'}| - (|T_3| - |S_4|)$ packets. Moreover, the packets of the set Y are delayed in the queue f_4 that uses the NTS protocol due to the unit capacity of the edge f_4 during this round and the packets of the set S_3 that are nearest to their source (queue f_4) than the packets of the set Y (queue f_3). Therefore, the remaining packets of the set Y in the queue f_4 is a set $|Y^{'}|$ of $|Y^{'}| = |Y| - (|T_3| - S_3)$ packets.

  Note that during this round $|K| = 2|T_3| - |S_3| - |S_4|$ packets arrive in the queue f_6 from the queues f_4, f_5. However, the edge f_6 has unit capacity and uses the NTS protocol that gives priority to the packets of the set S_5. Furthermore, the duration of this round is $|T_3|$ time steps. Consequently, at the end of this round the number of packets that remain in the queue f_6 requiring to traverse the edges $f_6, e_1, f_2^{'}, f_3^{'}, f_4^{'}$ is $|L| = |K| + |S_5| - |T_3|$. Thus, the number of packets in the queues f_1, f_4, f_5, f_6 requiring to traverse the edges $e_1, f_2^{'}, f_3^{'}, f_4^{'}$ at the end of this round is $s_{j+1} = |X^{''}| + |Y^{'}| + |Z| + |L|$. Substituting the quantities $|X^{''}|, |Y^{'}|, |Z|$ and $|L|$, we take $s_{j+1} = \rho s_j + \frac{\rho^2-\rho-1}{C} s_j + 2\frac{1-\rho}{C^2} s_j + \frac{-\rho^2+2\rho-1}{C^3} s_j + \rho s_j + \frac{2\rho^2-2\rho}{C} s_j + \frac{1-\rho}{C^2} s_j + \frac{-\rho^2+2\rho-1}{C^3} s_j + \rho^2 s_j - \frac{\rho}{C} s_j + \frac{\rho-\rho^2}{C^2} s_j + \frac{\rho-\rho^2}{C} s_j + 2\frac{\rho-1}{C^2} s_j + \frac{\rho^2-2\rho+1}{C^3} s_j$.

  In order to have instability, we must have $s_{j+1} > s_j$, that is $\rho^2[1 + \frac{2}{C} - \frac{1}{C^2} - \frac{1}{C^3}] + \rho[2 - \frac{3}{C} - \frac{1}{C^2} + \frac{2}{C^3}] + [-\frac{1}{C} + \frac{2}{C^2} - \frac{1}{C^3}] > 1$. Initially, note that the following inequalities hold: (i) $1 + \frac{2}{C} - \frac{1}{C^2} - \frac{1}{C^3} < 1 + \frac{2}{C}$, (ii) $2 - \frac{3}{C} - \frac{1}{C^2} + \frac{2}{C^3} < 2$, (iii) $1 + \frac{1}{C} - \frac{2}{C^2} + \frac{1}{C^3} > 1 - \frac{2}{C^2}$. Therefore, the inequality becomes $\rho^2[1 + \frac{2}{C}] + 2\rho > 1 - \frac{2}{C^2}$. Thus, it suffices to be shown that $\rho^2[C^2 + 2C] +$

$2\rho C^2 > C^2 - 2$. This inequality holds for ρ larger than the largest root ρ_1 of the polynomial $\rho^2[C^2 + 2C] + 2\rho C^2 - (C^2 - 2)$. The largest root of the polynomial is $\rho_1 = \frac{-2C^2+\sqrt{4C^4+4(C^2-2)(C^2+2C)}}{2C(C+2)} = \frac{\sqrt{2C^2+2C-2-\frac{4}{C}}}{C+2} - \frac{C}{C+2}$. But, $\frac{\sqrt{2C^2+2C-2-\frac{4}{C}}}{C+2} < \sqrt{\frac{2C}{C+2}}$. Therefore, $\rho_1 < \sqrt{\frac{2C}{C+2}} - \frac{C}{C+2}$. Thus, it holds that $\rho > \sqrt{\frac{2C}{C+2}} - \frac{C}{C+2} > \rho_1$.

When $C \to \infty$, it holds that $\frac{1}{C^k} \to 0$ for all $k \geq 1$. Then, the inequality $\rho^2[1 + \frac{2}{C} - \frac{1}{C^2} - \frac{1}{C^3}] + \rho[2 - \frac{3}{C} - \frac{1}{C^2} + \frac{2}{C^3}] + [-\frac{1}{C} + \frac{2}{C^2} - \frac{1}{C^3}] > 1$ becomes $\rho^2 + 2\rho - 1 > 0$ which holds for $\rho > \sqrt{2} - 1$. Note that if we have a sequence of equations $f_C(\rho)$ and there exists the limit $\lim_{C\to\infty} f_C(\rho) = f_\infty(\rho)$, then it holds fundamentally by the theory of function limits that if $\rho(C)$ is the root of $f_C(\rho) = 0$, then $\lim_{C\to\infty} \rho(C)$ is the root of $f_\infty(\rho)$. Therefore, for $\rho > \sqrt{2} - 1$ the system is unstable. Analyzing the inequality $\rho' > \sqrt{\frac{2C}{C+2}} - \frac{C}{C+2}$, we take $C > \frac{-[4(\rho')^2+4\rho'-4]+\sqrt{16[(\rho')^2+\rho'-1]^2-16(\rho')^2[(\rho')^2+2\rho'-1]}}{2[(\rho')^2+2\rho'-1]} > 1$. If we replace ρ' with $\sqrt{2}-1+\epsilon$ at this inequality, we estimate a $C > 1$ such that $s_{j+1} > s_j$. This argument can be repeated for an infinite number of phases showing that the number of packets in the system increases forever for $\rho > \sqrt{2} - 1$. □

Then, we show an instability bound for the composition of LIS and FTG protocols on the network $\mathcal{G}_3$ (see Figure 3). The edges $e_0, e_1, f_1, f_1', f_3, f_3'$ of $\mathcal{G}_3$ use the LIS protocol, while the remaining edges use the FTG protocol.

Theorem 4.3 *Let $\rho' = \sqrt{2} - 1 + \epsilon$ with $0 < \epsilon \leq 3/2 - \sqrt{2}$ and $C > 1$ where C is a particular function of ρ'. For the network $\mathcal{G}_3$ there is an adversary $\mathcal{A}_3$ of rate ρ that can change the link capacities of $\mathcal{G}_3$ between the two integer values 1 and C such that the system $\langle \mathcal{G}_3, \mathcal{A}_3, \mathsf{LIS}, \mathsf{FTG}\rangle$ is unstable for every $\rho > \rho'$. When $C \to \infty$ the system $\langle \mathcal{G}_3, \mathcal{A}_3, \mathsf{LIS}, \mathsf{FTG}\rangle$ is unstable for $\rho > \sqrt{2} - 1$.*

PROOF. We break the construction of the adversary $\mathcal{A}_3$ into phases.

Inductive Hypothesis: At the beginning of phase j (suppose j is even), there are s_j packets that are queued in the queues f_1', f_4', f_5', f_6' (in total) requiring to traverse the edges e_0, f_2, f_3, f_4.

Induction Step: At the beginning of phase $j+1$, there will be $s_{j+1} > s_j$ packets that will be queued in the queues f_1, f_4, f_5, f_6 (in total) requiring to traverse the edges e_1, f_2', f_3', f_4'.

We will construct an adversary $\mathcal{A}_3$ such that the induction step will hold. Proving that the induction step holds, we ensure that the inductive hypothesis will hold at the beginning of phase $j + 1$ for the symmetric edges with an increased value of s_j, $s_{j+1} > s_j$. By the symmetry of the network, repeating

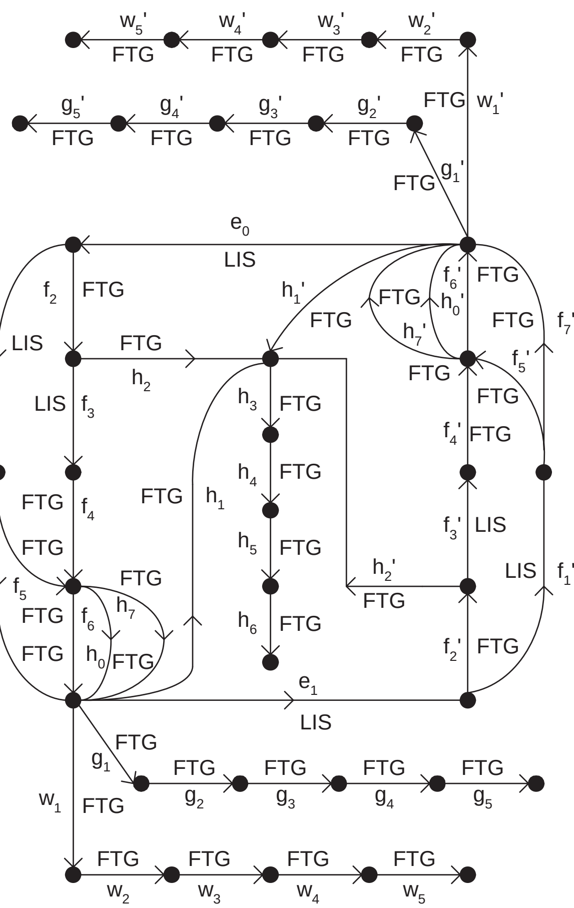

Fig. 3. The network $\mathcal{G}_3$.

the phase construction an unbounded number of times, we will create an unbounded number of packets in the network.

From the inductive hypothesis, initially, there are s_j packets (that constitute the set of packets S) in the queues $f_1', f_4', f_5',\ f_6'$ requiring to traverse the edges e_0, f_2, f_3, f_4. In order to prove the induction step, it is assumed that the set S has a large enough number of $|S| = s_j$ packets in the initial system configuration.

During phase j, the adversary plays three rounds of injections as follows:

- **Round 1:** It lasts $|T_1| = s_j/C$ time steps.

  Adversary's behavior. During this round the edge f_2 has unit capacity, while all the other edges have high capacity C. Also, the adversary injects a set X of $|X| = \rho C|T_1|$ packets in the queue e_0 wanting to traverse the edges $e_0, f_1,\ f_5, f_6, e_1, f_2', f_3', f_4'$ and a set S_1 of $|S_1| = \rho|T_1|$ packets in the queue f_2 wanting to traverse the edges f_2, h_2, h_3, h_4. These injections satisfy the load condition because the edges $e_0, f_1,\ f_5, f_6, e_1, f_2', f_3', f_4'$ have high capacity C and the edge f_2 has unit capacity during this round, and the injection paths of the different packet sets do not have overlapped edges.

Evolution of the system configuration. The packets of the set S delay the packets of the set X in the queue e_0 that uses the LIS protocol because they are longer time in the system than the packets of the set X. At the same time, the packets of the set S are delayed in the queue f_2 that uses the FTG protocol due to the packets of the set S_1 which have furthest to go (queue h_4) than the packets of the set S (queue f_4) and the unit capacity of the edge f_2. At the end of this round, the remaining packets of the set S in f_2 are $|S_2| = |S| - (|T_1| - |S_1|)$. The packets of the set S that manage to traverse the edge f_2 traverse their remaining path and they are absorbed.

- **Round 2:** It lasts $|T_2| = |S_2|/C$ steps.

  Adversary's behavior. During this round the edge f_1 has unit capacity, while all the other edges have high capacity C. Also, the adversary injects a set Y of $|Y| = \rho C|T_2|$ packets in the queue f_3 requiring to traverse the edges $f_3, f_4, f_6, e_1, f_2^{'}, f_3^{'}, f_4^{'}$. These packet injections satisfy the load condition because the assigned path consists of edges that have high capacity C during this round.

  Evolution of the system configuration. The packets of the set S_2 delay the packets of the set Y in the queue f_3 that uses the LIS protocol because they are longer time in the system than the packets of the set Y. The packets of the set S_2 traverse the edge f_3 and they are absorbed. At the same time, the packets of the set X are delayed in the queue f_1 due to the unit capacity of the edge f_1. Therefore, the remaining packets of the set X in the queue f_1 at the end of this round is a set $|X^{'}|$ of $|X^{'}| = |X| - |T_2|$ packets.

- **Round 3:** It lasts $|T_3| = |X^{'}|/C$ time steps.

  Adversary's behavior. During this round the edges f_4, f_5, f_6 have unit capacity, while all the other edges have capacity C. Also, the adversary injects a set Z of $|Z| = \rho C|T_3|$ packets in the queue f_1 requiring to traverse the edges $f_1, f_7, e_1, f_2^{'}, f_3^{'}, f_4^{'}$, a set S_3 of $|S_3| = \rho|T_3|$ packets in the queue f_4 wanting to traverse the edges $f_4, h_0, h_1, h_3, h_4, h_5, h_6$, a set S_4 of $|S_4| = \rho|T_3|$ packets in the queue f_5 wanting to traverse the edges $f_5, h_7, g_1, g_2, g_3, g_4, g_5$ and a set S_5 of $|S_5| = \rho|T_3|$ packets in the queue f_6 wanting to traverse the edges $f_6, w_1, w_2, w_3, w_4, w_5$. These injections satisfy the load condition because the edges $f_1, f_7, e_1, f_2^{'}, f_3^{'}, f_4^{'}$ have high capacity C and the edges f_4, f_5, f_6 have unit capacity during this round, and the injection paths of the different packet sets do not have overlapped edges.

  Evolution of the system configuration. The packets of the set $X^{'}$ delay the packets of the set Z in the queue f_1 that uses the LIS protocol because they are longer time in the system than the packets of the set Z. At the same time the packets of the set $X^{'}$ are delayed in the queue f_5 that uses the FTG protocol due to the unit capacity of the edge f_5 during this round and the packets of the set S_4 that have furthest to go (queue g_5) than the packets of the set X' (queue f_4'). Therefore, the remaining packets of the set X' in the queue f_5 is a set $|X''|$ of $|X''| = |X'| - (|T_3| - |S_4|)$ packets. Moreover, the packets of the set Y are delayed in the queue f_4 that uses the FTG protocol due to the unit capacity of the edge f_4 during this round and the packets of the set S_3 that have furthest to go (queue h_6) than the packets of the set Y (queue f_4'). Thus, the remaining packets of the set Y in the queue f_4 at the end of this round is a set $|Y'|$ of $|Y'| = |Y| - (|T_3| - |S_3|)$ packets.

Note that during this round $|K| = 2|T_3| - |S_3| - |S_4|$ packets arrive in the queue f_6 from the queues f_4, f_5. However, the edge f_6 has unit capacity and uses the FTG protocol that gives priority to the packets of the set S_5. Furthermore, the duration of this round is $|T_3|$ time steps. Consequently, at the end of this round the number of packets that remain in the queue f_6 requiring to traverse the edges $f_6, e_1, f_2', f_3', f_4'$ is $|L| = |K| + |S_5| - |T_3|$. Thus, the number of packets in the queues f_1, f_4, f_5, f_6 requiring to traverse the edges e_1, f_2', f_3', f_4' at the end of this round is $s_{j+1} = |X''| + |Y'| + |Z| + |L|$. Substituting the quantities $|X''|, |Y'|, |Z|$ and $|L|$, we take $s_{j+1} = \rho s_j + \frac{\rho^2-\rho-1}{C} s_j + 2\frac{1-\rho}{C^2} s_j + \frac{-\rho^2+2\rho-1}{C^3} s_j + \rho s_j + \frac{2\rho^2-2\rho}{C} s_j + \frac{1-\rho}{C^2} s_j + \frac{-\rho^2+2\rho-1}{C^3} s_j + \rho^2 s_j - \frac{\rho}{C} s_j + \frac{\rho-\rho^2}{C^2} s_j + \frac{\rho-\rho^2}{C} s_j + 2\frac{\rho-1}{C^2} s_j + \frac{\rho^2-2\rho+1}{C^3} s_j$.

In order to have instability, we must have $s_{j+1} > s_j$, that is $\rho^2[1 + \frac{2}{C} - \frac{1}{C^2} - \frac{1}{C^3}] + \rho[2 - \frac{3}{C} - \frac{1}{C^2} + \frac{2}{C^3}] + [-\frac{1}{C} + \frac{2}{C^2} - \frac{1}{C^3}] > 1$. Initially, note that the following inequalities hold: (i) $1 + \frac{2}{C} - \frac{1}{C^2} - \frac{1}{C^3} < 1 + \frac{2}{C}$, (ii) $2 - \frac{3}{C} - \frac{1}{C^2} + \frac{2}{C^3} < 2$, (iii) $1 + \frac{1}{C} - \frac{2}{C^2} + \frac{1}{C^3} > 1 - \frac{2}{C^2}$. Therefore, the inequality becomes $\rho^2[1 + \frac{2}{C}] + 2\rho > 1 - \frac{2}{C^2}$. Thus, it suffices to be shown that $\rho^2[C^2 + 2C] + 2\rho C^2 > C^2 - 2$. This inequality holds for ρ larger than the largest root ρ_1 of the polynomial $\rho^2[C^2 + 2C] + 2\rho C^2 - (C^2 - 2)$. The largest root of the polynomial is $\rho_1 = \frac{-2C^2+\sqrt{4C^4+4(C^2-2)(C^2+2C)}}{2C(C+2)} = \frac{\sqrt{2C^2+2C-2-\frac{4}{C}}}{C+2} - \frac{C}{C+2}$. But, $\frac{\sqrt{2C^2+2C-2-\frac{4}{C}}}{C+2} < \sqrt{\frac{2C}{C+2}}$. Therefore, $\rho_1 < \sqrt{\frac{2C}{C+2}} - \frac{C}{C+2}$. Thus, it holds that $\rho > \sqrt{\frac{2C}{C+2}} - \frac{C}{C+2} > \rho_1$.

When $C \to \infty$, it holds that $\frac{1}{C^k} \to 0$ for all $k \geq 1$. Then, the inequality $\rho^2[1 + \frac{2}{C} - \frac{1}{C^2} - \frac{1}{C^3}] + \rho[2 - \frac{3}{C} - \frac{1}{C^2} + \frac{2}{C^3}] + [-\frac{1}{C} + \frac{2}{C^2} - \frac{1}{C^3}] > 1$ becomes $\rho^2 + 2\rho - 1 > 0$ which holds for $\rho > \sqrt{2} - 1$. Note that if we have a sequence of equations $f_C(\rho)$ and there exists the limit $\lim_{C\to\infty} f_C(\rho) = f_\infty(\rho)$, then it holds fundamentally by the theory of function limits that if $\rho(C)$ is the root of $f_C(\rho) = 0$, then $\lim_{C\to\infty} \rho(C)$ is the root of $f_\infty(\rho)$. Therefore, for $\rho > \sqrt{2} - 1$ the system is unstable. Analyzing the inequality $\rho' > \sqrt{\frac{2C}{C+2}} - \frac{C}{C+2}$, we take $C > \frac{-[4(\rho')^2+4\rho'-4]+\sqrt{16[(\rho')^2+\rho'-1]^2-16(\rho')^2[(\rho')^2+2\rho'-1]}}{2[(\rho')^2+2\rho'-1]} > 1$. If we replace ρ' with $\sqrt{2}-1+\epsilon$ at this inequality, we estimate a $C > 1$ such that $s_{j+1} > s_j$. This argument can be repeated for an infinite number of phases showing that the number of packets in the system increases forever for $\rho > \sqrt{2} - 1$. $\square$

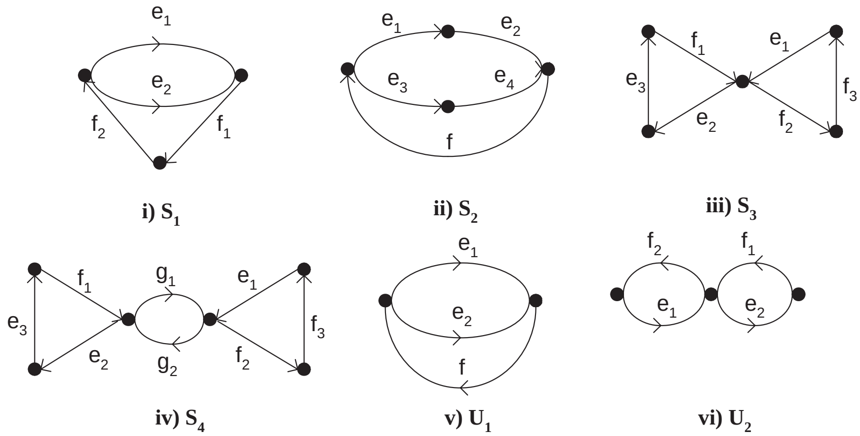

Fig. 4. The pure simple-path networks $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4$, and the not pure simple-path networks $\mathcal{U}_1, \mathcal{U}_2$.

5 Instability Bounds for Forbidden Subgraphs

In this section, we present lower bounds on the injection rate that guarantee instability for forbidden subgraphs. First, we consider the pure simple-path networks $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4$ (see Figure 4) that use the NTG-U-LIS protocol. We show:

Theorem 5.1 *Let $\rho \geq 0.82$. For the network $\mathcal{S}_1$ there is an adversary $\mathcal{A}_1$ of rate ρ that can change the link capacities of $\mathcal{S}_1$ between the two integer values 1 and $C > 1000$ such that the system $\langle \mathcal{S}_1, \mathcal{A}_1, \mathsf{NTG-U-LIS} \rangle$ is unstable. When $C \rightarrow \infty$ the system $\langle \mathcal{S}_1, \mathcal{A}_1, \mathsf{NTG-U-LIS} \rangle$ is unstable for $\rho > 0.8191$.*

PROOF. We break the construction of the adversary $\mathcal{A}_1$ into phases.

Inductive Hypothesis: At the beginning of phase j, there are s_j packets that are queued in the queues e_1, e_2 requiring to traverse the edge f_1.

Induction Step: At the beginning of phase $j+1$, there will be $s_{j+1} > s_j$ packets that will be queued in the queues e_1, e_2 requiring to traverse the edge f_1.

We will construct an adversary $\mathcal{A}_1$ such that the induction step will hold. Proving that the induction step holds, we ensure that the inductive hypothesis will hold at the beginning of phase $j+1$ with an increased value of s_j, $s_{j+1} > s_j$. From the inductive hypothesis, initially, there are s_j packets (that constitute the set of packets S) in the queues e_1, e_2 requiring to traverse the edge f_1. In order to prove that the induction step works, we consider that there is a large enough number of packets s_j in the initial system configuration.

During phase j, the adversary plays four rounds of injections as follows:

- **Round 1:** It lasts $|T_1| = s_j/C$ time steps.

  Adversary's behavior. During this round all the network edges have high capacity C and the adversary injects in the queue f_1 a set X of $|X| = \rho C|T_1|$ packets wanting to traverse the edges f_1, f_2. These packet injections satisfy the load condition because the assigned path consists of edges that have high capacity C during this round.

  Evolution of the system configuration. The packets of the set S delay the packets of the set X in the queue f_1 because they are nearest to their destination (queue f_1) than the packets of the set X (queue f_2). The packets of the set S traverse the edge f_1 and they are absorbed.

- **Round 2:** It lasts $|T_2| = |X|/C$ time steps.

  Adversary's behavior. During this round all the network edges have high capacity C and the adversary injects a set Y of $|Y| = \rho C|T_2|$ packets in the queue f_2 requiring to traverse the edges f_2, e_1. These packet injections satisfy the load condition because the assigned path consists of edges that have high capacity C during this round.

  Evolution of the system configuration. The packets of the set X delay the packets of the set Y in the queue f_2 because they have nearest to go (queue f_2) than the packets of the set Y (queue e_1). The packets of the set X traverse the edge f_2 and they are absorbed.

- **Round 3:** It lasts $|T_3| = |Y|/C$ time steps.

  Adversary's behavior. During this round all the network edges have high capacity C and the adversary injects a set Z of $|Z| = \rho C|T_3|$ packets in the queue f_2 requiring to traverse the edges f_2, e_2. Also, it injects a set Z_1 of $|Z_1| = \rho C|T_3|$ packets in the queue e_1 requiring to traverse the edges e_1, f_1. These injections satisfy the load condition because all the network edges have high capacity C, and the injection paths of the different packet sets do not have overlapped edges.

  Evolution of the system configuration. The packets of the set Y delay the packets of the set Z in the queue f_2 because they are longer time in the system than the packets of the set Z (the packets of the sets Y and Z have to traverse the same distance to reach their destination). Moreover, the packets of the set Y delay the packets of the set Z_1 in the queue e_1 because they have nearest to go (queue e_1) than the packets of the set Z_1 (queue f_1). The packets of the set Y traverse the edge e_1 and they are absorbed.

- **Round 4:** It lasts $|T_4| = |Z|/C$ time steps.

  Adversary's behavior. During this round the edge e_1 has unit capacity, while all the other edges have high capacity C. The adversary injects a set Z_2 of $|Z_2| = \rho C|T_4|$ packets in the queue e_2 requiring to traverse the edges e_2, f_1. These packet injections satisfy the load condition because the assigned path consists of edges that have high capacity C during this round.

Evolution of the system configuration. The packets of the set Z delay the packets of the set Z_2 in the queue e_2 because they have nearest to go (queue e_2) than the packets of the set Z_2 (queue f_1). The packets of the set Z traverse the edge e_2 and they are absorbed. Moreover, the packets of the set Z_1 are delayed in the queue e_1 due to the unit capacity of the edge e_1 during this round. Therefore, the remaining packets of the set Z_1 in the queue e_1 at the end of this round is a set $|Z_1^{'}|$ of $|Z_1^{'}| = |Z_1| - |T_4|$ packets, while the remaining packets of the set Z_1 traverse their remaining path and they are absorbed. Thus, the number of packets in the queues e_1, e_2 requiring to traverse the edge f_1 at the end of this round is $s_{j+1} = |Z_2| + |Z_1^{'}| = \rho^4 s_j + \rho^3 s_j - \frac{\rho^3}{C} s_j$.

In order to have instability, we must have $s_{j+1} > s_j$, that is $\rho^4 s_j + \rho^3 s_j - \frac{\rho^3}{C} s_j > s_j$. Therefore, $\rho^4 C + \rho^3 (C-1) > C$. Dividing by C the inequality, we take $\rho^4 + \rho^3(1 - \frac{1}{C}) > 1$. If we let $C = 1000$ and $\rho = 0.82$, the inequality holds. Thus, for $C > 1000$ and $\rho = 0.82$ the inequality holds, too.

When $C \to \infty$, it holds that $\frac{1}{C^k} \to 0$ for all $k \geq 1$. Then, our inequality becomes $\rho^4 + \rho^3 - 1 > 0$ which holds for $\rho > 0.8191$. Note that if we have a sequence of equations $f_C(\rho)$ and there exists the limit $\lim_{C \to \infty} f_C(\rho) = f_\infty(\rho)$, then it holds fundamentally by the theory of function limits that if $\rho(C)$ is the root of $f_C(\rho) = 0$, then $\lim_{C \to \infty} \rho(C)$ is the root of $f_\infty(\rho)$. Therefore, for $\rho > 0.8191$ the system is unstable. This argument can be repeated for an infinite number of phases showing that the number of packets in the system increases forever for $\rho > 0.8191$. $\square$

Theorem 5.2 *Let $\rho \geq 0.82$. For the network $\mathcal{S}_2$ there is an adversary $\mathcal{A}_2$ of rate ρ that can change the link capacities of $\mathcal{S}_2$ between the two integer values 1 and $C > 1000$ such that the system $\langle \mathcal{S}_2, \mathcal{A}_2, \mathsf{NTG-U-LIS} \rangle$ is unstable. When $C \to \infty$ the system $\langle \mathcal{S}_2, \mathcal{A}_2, \mathsf{NTG-U-LIS} \rangle$ is unstable for $\rho > 0.8191$.*

PROOF. We break the construction of the adversary $\mathcal{A}_2$ into phases.

Inductive Hypothesis: At the beginning of phase j, there are s_j packets that are queued in the queues e_2, e_4 requiring to traverse the edge f.

Induction Step: At the beginning of phase $j+1$, there will be $s_{j+1} > s_j$ packets that will be queued in the queues e_2, e_4 requiring to traverse the edge f.

We will construct an adversary $\mathcal{A}_2$ such that the induction step will hold. Proving that the induction step holds, we ensure that the inductive hypothesis will hold at the beginning of phase $j+1$ with an increased value of s_j, $s_{j+1} > s_j$. From the inductive hypothesis, initially, there are s_j packets (that constitute the set of packets S) in the queues e_2, e_4 requiring to traverse the edge f. In order to prove that the induction step works, we consider that there is a large enough number of packets s_j in the initial system configuration.

During phase j, the adversary plays four rounds of injections as follows:

- **Round 1:** It lasts $|T_1| = s_j/C$ time steps.

  Adversary's behavior. During this round all the network edges have high capacity C and the adversary injects in the queue f a set X of $|X| = \rho C|T_1|$ packets wanting to traverse the edges f, e_3. These packet injections satisfy the load condition because the assigned path consists of edges that have high capacity C during this round.

  Evolution of the system configuration. The packets of the set S delay the packets of the set X in the queue f because they have nearest to go (queue f) than the packets of the set X (queue e_3). The packets of the set S traverse the edge f and they are absorbed.

- **Round 2:** It lasts $|T_2| = |X|/C$ time steps.

  Adversary's behavior. During this round all the network edges have high capacity C and the adversary injects a set Y of $|Y| = \rho C|T_2|$ packets in the queue e_3 requiring to traverse the edges e_3, e_4 and a set Z of $|Z| = \rho C|T_2|$ packets in the queue f requiring to traverse the edges f, e_1. These injections satisfy the load condition because all the network edges have high capacity C, and the injection paths of the different packet sets do not have overlapped edges.

  Evolution of the system configuration. The packets of the set X delay the packets of the set Z in the queue f because they are longer time in the system than the packets of the set Z (the packets of the sets X and Z have to traverse the same distance to reach their destination). Furthermore, the packets of the set X delay the packets of the set Y in the queue e_3 because they have nearest to go (queue e_3) than the packets of the set Y (queue e_4). The packets of the set X traverse the edge e_3 and they are absorbed.

- **Round 3:** It lasts $|T_3| = |Y|/C$ time steps.

  Adversary's behavior. During this round all the network edges have high capacity C. Also, the adversary injects a set Z_1 of $|Z_1| = \rho C|T_3|$ packets in the queue e_4 requiring to traverse the edges e_4, f and a set Z_2 of $|Z_2| = \rho C|T_3|$ packets in e_1 requiring to traverse the edges e_1, e_2. These injections satisfy the load condition because all the network edges have high capacity C, and the injection paths of the different packet sets do not have overlapped edges.

  Evolution of the system configuration. The packets of the set Y delay the packets of the set Z_1 in e_4 because they have nearest to go (queue e_4) than the packets of the set Z_1 (queue f). Furthermore, for the same reason the packets of the set Z delay the packets of the set Z_2 in the queue e_1. The packets of the sets Y and Z traverse the edges e_4 and e_1 correspondingly and they are absorbed.

- **Round 4:** It lasts $|T_4| = |Z_1|/C$ time steps.

Adversary's behavior. During this round the edge e_4 has unit capacity, while all the other edges have high capacity C. Also, the adversary injects a set Z_3 of $|Z_3| = \rho C|T_4|$ packets in the queue e_2 requiring to traverse the edges e_2, f. These packet injections satisfy the load condition because the assigned path consists of edges that have high capacity C during this round.

Evolution of the system configuration. The packets of the set Z_2 delay the packets of the set Z_3 in the queue e_2 because they have nearest to go (queue e_2) than the packets of the set Z_3 (queue f). Moreover, the packets of the set Z_1 are delayed in e_4 due to the unit capacity of the edge e_4 during this round. Therefore, the remaining packets of the set Z_1 in the queue e_4 at the end of this round is a set Z_4 of $|Z_4| = |Z_1| - |T_4|$ packets, while the remaining packets of the set Z_1 traverse their path and they are absorbed. Thus, the number of packets in the queues e_2, e_4 requiring to traverse the edge f at the end of this round is $s_{j+1} = |Z_3| + |Z_4| = \rho^4 s_j + \rho^3 s_j - \frac{\rho^3}{C} s_j$.

In order to have instability, we must have $s_{j+1} > s_j$, that is $\rho^4 s_j + \rho^3 s_j - \frac{\rho^3}{C} s_j > s_j$. Therefore, $\rho^4 C + \rho^3 (C-1) > C$. Dividing by C the inequality, we take $\rho^4 + \rho^3(1 - \frac{1}{C}) > 1$. If we let $C = 1000$ and $\rho = 0.82$, the inequality holds. Thus, for $C > 1000$ and $\rho = 0.82$ the inequality holds, too.

When $C \to \infty$, it holds that $\frac{1}{C^k} \to 0$ for all $k \geq 1$. Then, our inequality becomes $\rho^4 + \rho^3 - 1 > 0$ which holds for $\rho > 0.8191$. Note that if we have a sequence of equations $f_C(\rho)$ and there exists the limit $\lim_{C\to\infty} f_C(\rho) = f_\infty(\rho)$, then it holds fundamentally by the theory of function limits that if $\rho(C)$ is the root of $f_C(\rho) = 0$, then $\lim_{C\to\infty} \rho(C)$ is the root of $f_\infty(\rho)$. Therefore, for $\rho > 0.8191$ the system is unstable. This argument can be repeated for an infinite number of phases showing that the number of packets in the system increases forever for $\rho > 0.8191$. $\square$

Theorem 5.3 *Let $\rho \geq 0.82$. For the network $\mathcal{S}_3$ there is an adversary $\mathcal{A}_3$ of rate ρ that can change the link capacities of $\mathcal{S}_3$ between the two integer values 1 and $C > 1000$ such that the system $\langle \mathcal{S}_3, \mathcal{A}_3, \mathsf{NTG-U-LIS} \rangle$ is unstable. When $C \to \infty$ the system $\langle \mathcal{S}_3, \mathcal{A}_3, \mathsf{NTG-U-LIS} \rangle$ is unstable for $\rho > 0.8191$.*

PROOF. We break the construction of the adversary $\mathcal{A}_3$ into phases.

Inductive Hypothesis: At the beginning of phase j, there are s_j packets that are queued in the queues f_1, f_3 requiring to traverse the edges f_1, e_2 and f_3, e_1, e_2 correspondingly.

Induction Step: At the beginning of phase $j+1$, there will be $s_{j+1} > s_j$ packets that will be queued in the queues f_1, f_3 requiring to traverse the edges f_1, e_2 and f_3, e_1, e_2 correspondingly.

We will construct an adversary $\mathcal{A}_3$ such that the induction step will hold. Proving that the induction step holds, we ensure that the inductive hypothesis will hold at the beginning of phase $j+1$ with an increased value of s_j, $s_{j+1} > s_j$. From the inductive hypothesis, initially, there are s_j packets (that constitute the set of packets S) in the queues f_1, f_3 requiring to traverse the edges f_1, e_2 and f_3, e_1, e_2 correspondingly. In order to prove that the induction step works, we consider that there is a large enough number of packets s_j in the initial system configuration.

During phase j, the adversary plays four rounds of injections as follows:

- **Round 1:** It lasts $|T_1| = s_j/C$ time steps.

  Adversary's behavior. During this round all the network edges have high capacity C and the adversary injects in the queue e_2 a set X of $|X| = \rho C|T_1|$ packets wanting to traverse the edges e_2, e_3. These packet injections satisfy the load condition because the assigned path consists of edges that have high capacity C during this round.

  Evolution of the system configuration. The packets of the set S delay the packets of the set X in the queue e_2 because they have nearest to go (queue e_2) than the packets of the set X (queue e_3). The packets of the set S traverse the edge e_2 and they are absorbed.

- **Round 2:** It lasts $|T_2| = |X|/C$ time steps.

  Adversary's behavior. During this round all the network edges have high capacity C and the adversary injects a set Y of $|Y| = \rho C|T_2|$ packets in the queue e_3 requiring to traverse the edges e_3, f_1, f_2. These packet injections satisfy the load condition because the assigned path consists of edges that have high capacity C during this round.

  Evolution of the system configuration. The packets of the set X delay the packets of the set Y in the queue e_3 because they have nearest to go (queue e_3) than the packets of the set Y (queue f_2). The packets of the set X packets traverse the edge e_3 and they are absorbed.

- **Round 3:** It lasts $|T_3| = |Y|/C$ time steps.

  Adversary's behavior. During this round all the network edges have high capacity C. Also, the adversary injects a set Z of $|Z| = \rho C|T_3|$ packets in the queue f_1 requiring to traverse the edges f_1, e_2 and a set Z_1 of $|Z_1| = \rho C|T_3|$ packets in f_2 requiring to traverse the edges f_2, f_3. These injections satisfy the load condition because all the network edges have high capacity C, and the injection paths of the different packet sets do not have overlapped edges.

  Evolution of the system configuration. The packets of the set Y delay the packets of the set Z in f_1 because they are longer time in the system than the packets of the set Z (the packets of the sets Y and Z have to traverse the same distance to reach their destination). Furthermore, the packets of the set Y delay the packets of the set Z_1 in the queue f_2 because they have nearest to go (queue f_2) than the packets of the set Z_1 (queue f_3). The packets of the set Y traverse the edge f_2 and they are absorbed.

- **Round 4:** It lasts $|T_4| = |Z|/C$ time steps.

  Adversary's behavior. During this round the edge f_1 has unit capacity, while all the other edges have high capacity C. Also, the adversary injects a set Z_2 of $|Z_2| = \rho C|T_4|$ packets in the queue f_3 requiring to traverse the edges f_3, e_1, e_2. These packet injections satisfy the load condition because the assigned path consists of edges that have high capacity C during this round.

  Evolution of the system configuration. The packets of the set Z_1 delay the packets of the set Z_2 in the queue e_1 because they have nearest to go (queue f_3) than the packets of the set Z_2 (queue e_2). The packets of the set Z_1 traverse the edge f_3 and they are absorbed. Moreover, the packets of the set Z are delayed in f_1 due to the unit capacity of the edge f_1 during this round. Therefore, the remaining packets of the set Z in the queue f_1 at the end of this round is a set Z_3 of $|Z_3| = |Z| - |T_4|$ packets, while the remaining packets of the set Z traverse their path and they are absorbed. Thus, the number of packets in the queues f_1, f_3 requiring to traverse the edges f_1, e_2 and f_3, e_1, e_2 correspondingly at the end of this round is $s_{j+1} = |Z_2| + |Z_3| = \rho^4 s_j + \rho^3 s_j - \frac{\rho^3}{C} s_j$.

  In order to have instability, we must have $s_{j+1} > s_j$, that is $\rho^4 s_j + \rho^3 s_j - \frac{\rho^3}{C} s_j > s_j$. Therefore, $\rho^4 C + \rho^3(C-1) > C$. Dividing by C the inequality, we take $\rho^4 + \rho^3(1 - \frac{1}{C}) > 1$. If we let $C = 1000$ and $\rho = 0.82$, the inequality holds. Thus, for $C > 1000$ and $\rho = 0.82$ the inequality holds, too.

  When $C \to \infty$, it holds that $\frac{1}{C^k} \to 0$ for all $k \geq 1$. Then, our inequality becomes $\rho^4 + \rho^3 - 1 > 0$ which holds for $\rho > 0.8191$. Note that if we have a sequence of equations $f_C(\rho)$ and there exists the limit $\lim_{C\to\infty} f_C(\rho) = f_\infty(\rho)$, then it holds fundamentally by the theory of function limits that if $\rho(C)$ is the root of $f_C(\rho) = 0$, then $\lim_{C\to\infty} \rho(C)$ is the root of $f_\infty(\rho)$. Therefore, for $\rho > 0.8191$ the system is unstable. This argument can be repeated for an infinite number of phases showing that the number of packets in the system increases forever for $\rho > 0.8191$. $\square$

Theorem 5.4 *Let $\rho \geq 0.82$. For the network $\mathcal{S}_4$ there is an adversary $\mathcal{A}_4$ of rate ρ that can change the link capacities of $\mathcal{S}_4$ between the two integer values 1 and $C > 1000$ such that the system $\langle \mathcal{S}_4, \mathcal{A}_4, \mathsf{NTG-U-LIS} \rangle$ is unstable. When $C \to \infty$ the system $\langle \mathcal{S}_4, \mathcal{A}_4, \mathsf{NTG-U-LIS} \rangle$ is unstable for $\rho > 0.8191$.*

PROOF. We break the construction of the adversary $\mathcal{A}_4$ into phases.

Inductive Hypothesis: At the beginning of phase j, there are s_j packets that are queued in the queues f_1, f_3 requiring to traverse the edges f_1, e_2 and f_3, e_1, g_2, e_2 correspondingly.

Induction Step: At the beginning of phase $j+1$, there will be $s_{j+1} > s_j$ packets that will be queued in the queues f_1, f_3 requiring to traverse the edges f_1, e_2 and f_3, e_1, g_2, e_2 correspondingly.

We will construct an adversary $\mathcal{A}_4$ such that the induction step will hold. Proving that the induction step holds, we ensure that the inductive hypothesis will hold at the beginning of phase $j+1$ with an increased value of s_j, $s_{j+1} > s_j$. From the inductive hypothesis, initially, there are s_j packets (that constitute the set of packets S) in the queues f_1, f_3 requiring to traverse the edges f_1, e_2 and f_3, e_1, g_2, e_2 correspondingly. In order to prove that the induction step works, we consider that there is a large enough number of packets s_j in the initial system configuration.

During phase j, the adversary plays four rounds of injections as follows:

- **Round 1:** It lasts $|T_1| = s_j/C$ time steps.

  Adversary's behavior. During this round all the network edges have high capacity C and the adversary injects in the queue e_2 a set X of $|X| = \rho C|T_1|$ packets wanting to traverse the edges e_2, e_3. These packet injections satisfy the load condition because the assigned path consists of edges that have high capacity C during this round.

  Evolution of the system configuration. The packets of the set S delay the packets of the set X in the queue e_2 because they have nearest to go (queue e_2) than the packets of the set X (queue e_3). The packets of the set S traverse the edge e_2 and they are absorbed.

- **Round 2:** It lasts $|T_2| = |X|/C$ time steps.

  Adversary's behavior. During this round all the network edges have high capacity C and the adversary injects a set Y of $|Y| = \rho C|T_2|$ packets in the queue e_3 requiring to traverse the edges e_3, f_1, g_1. These packet injections satisfy the load condition because the assigned path consists of edges that have high capacity C during this round.

  Evolution of the system configuration. The packets of the set X delay the packets of the set Y in the queue e_3 because they have nearest to go (queue e_3) than the packets of the set Y (queue g_1). The packets of the set X traverse the edge e_3 and they are absorbed.

- **Round 3:** It lasts $|T_3| = |Y|/C$ time steps.

  Adversary's behavior. During this round all the network edges have high capacity C. Also, the adversary injects a set Z_1 of $|Z_1| = \rho C|T_3|$ packets in the queue f_1 requiring to traverse the edges f_1, e_2 and a set Z_2 of $|Z_2| = \rho C|T_3|$ packets in g_1 requiring to traverse the edges g_1, f_2, f_3. These injections satisfy the load condition because all the network edges have high capacity C, and the injection paths of the different packet sets do not have overlapped edges.

Evolution of the system configuration. The packets of the set Y delay the packets of the set Z_1 in f_1 because they are longer time in the system than the packets of the set Z_1 (the packets of the sets Y and Z_1 have to traverse the same distance to reach their destination). Furthermore, the packets of the set Y delay the packets of the set Z_2 in the queue g_1 because they have nearest to go (queue g_1) than the packets of the set Z_2 (queue f_3). The packets of the set Y packets traverse the edge g_1 and they are absorbed.

- **Round 4:** It lasts $|T_4| = |Z_1|/C$ time steps.

  Adversary's behavior. During this round the edge f_1 has unit capacity, while all the other edges have high capacity C. Also, the adversary injects a set Z_3 of $|Z_3| = \rho C|T_4|$ packets in the queue f_3 requiring to traverse the edges f_3, e_1, g_2, e_2. These packet injections satisfy the load condition because the assigned path consists of edges that have high capacity C during this round.

  Evolution of the system configuration. The packets of the set Z_2 delay the packets of the set Z_3 in the queue f_3 because they have nearest to go (queue f_3) than the packets of the set Z_3 (queue e_2). Moreover, the packets of the set Z_1 are delayed in f_1 due to the unit capacity of the edge f_1 during this round. Therefore, the remaining packets of the set Z_1 in the queue f_1 at the end of this round is a set Z_4 of $|Z_4| = |Z_1| - |T_4|$ packets, while the remaining packets of the set Z_1 traverse their remaining path and they are absorbed. Thus, the number of packets in the queues f_1, f_3 requiring to traverse the edges f_1, e_2 and f_3, e_1, g_2, e_2 correspondingly at the end of this round is $s_{j+1} = |Z_3| + |Z_4| = \rho^4 s_j + \rho^3 s_j - \frac{\rho^3 s_j}{C}$.

  In order to have instability, we must have $s_{j+1} > s_j$, that is $\rho^4 s_j + \rho^3 s_j - \frac{\rho^3}{C} s_j > s_j$. Therefore, $\rho^4 C + \rho^3(C-1) > C$. Dividing by C the inequality, we take $\rho^4 + \rho^3(1 - \frac{1}{C}) > 1$. If we let $C = 1000$ and $\rho = 0.82$, the inequality holds. Thus, for $C > 1000$ and $\rho = 0.82$ the inequality holds, too.

  When $C \to \infty$, it holds that $\frac{1}{C^k} \to 0$ for all $k \geq 1$. Then, our inequality becomes $\rho^4 + \rho^3 - 1 > 0$ which holds for $\rho > 0.8191$. Note that if we have a sequence of equations $f_C(\rho)$ and there exists the limit $\lim_{C\to\infty} f_C(\rho) = f_\infty(\rho)$, then it holds fundamentally by the theory of function limits that if $\rho(C)$ is the root of $f_C(\rho) = 0$, then $\lim_{C\to\infty} \rho(C)$ is the root of $f_\infty(\rho)$. Therefore, for $\rho > 0.8191$ the system is unstable. This argument can be repeated for an infinite number of phases showing that the number of packets in the system increases forever for $\rho > 0.8191$. □

Now, we consider the not pure simple-path networks $\mathcal{U}_1, \mathcal{U}_2$ (see Figure 4) that use the NTG-U-LIS protocol.

Theorem 5.5 *Let $\rho = 0.8$. For the network $\mathcal{U}_1$ there is an adversary $\mathcal{A}$ of rate ρ that can change the link capacities of $\mathcal{U}_1$ between the two integer values 1 and $C > 1000$ such that the system $\langle \mathcal{U}_1, \mathcal{A}, \mathsf{NTG-U-LIS} \rangle$ is unstable. When $C \to \infty$ the system $\langle \mathcal{U}_1, \mathcal{A}, \mathsf{NTG-U-LIS} \rangle$ is unstable for $\rho > \sqrt[3]{0.5}$.*

PROOF. We break the construction of the adversary $\mathcal{A}$ into phases.

Inductive Hypothesis: At the beginning of phase j, there are s_j packets that are queued in the queues e_1, e_2 requiring to traverse the edge f.

Induction Step: At the beginning of phase $j+1$, there will be $s_{j+1} > s_j$ packets that will be queued in the queues e_1, e_2 requiring to traverse the edge f.

We will construct an adversary $\mathcal{A}$ such that the induction step will hold. Proving that the induction step holds, we ensure that the inductive hypothesis will hold at the beginning of phase $j+1$ with an increased value of s_j, $s_{j+1} > s_j$. From the inductive hypothesis, initially, there are s_j packets (that constitute the set of packets S) in the queues e_1, e_2 requiring to traverse the edge f. In order to prove that the induction step works, we consider that there is a large enough number of packets s_j in the initial system configuration.

During phase j the adversary plays four rounds of injections as follows:

- **Round 1:** It lasts $|T_1| = s_j/C$ time steps.

  Adversary's behavior. During this round all the network edges have high capacity C and the adversary injects in the queue f a set X of $|X| = \rho C|T_1|$ packets wanting to traverse the edges f, e_1. These packet injections satisfy the load condition because the assigned path consists of edges that have high capacity C during this round.

  Evolution of the system configuration. The packets of the set S delay the packets of the set X in the queue f because they have nearest to go (queue f) than the packets of the set X (queue e_1). The packets of the set S traverse the edge f and they are absorbed.

- **Round 2:** It lasts $|T_2| = |X|/C$ time steps.

  Adversary's behavior. During this round all the network edges have high capacity C. Also, the adversary injects a set Y of $|Y| = \rho C|T_2|$ packets in the queue f requiring to traverse the edges f, e_2 and a set Z of $|Z| = \rho C|T_2|$ packets in the queue e_1 requiring to traverse the edge e_1. These injections satisfy the load condition because all the network edges have high capacity C, and the injection paths of the different packet sets do not have overlapped edges.

  Evolution of the system configuration. The packets of the set X delay the packets of the set Y in the queue f because they are longer time in the system than the packets of the set Y (the packets of the sets X and Y have to traverse the same distance to reach their destination). Furthermore, for the same reason the packets of the set X delay the packets of the set Z in the queue e_1. The packets of the set X traverse the edge e_1 and they are absorbed.

- **Round 3:** It lasts $|T_3| = |Y|/C$ time steps.

  Adversary's behavior. During this round the edge e_2 has unit capacity, while all the network edges have high capacity C. Also, the adversary injects a set Z_1 of $|Z_1| = \rho C|T_3|$ packets in the queue e_1 requiring to traverse the edges e_1, f. These packet injections satisfy the load condition because the assigned path consists of edges that have high capacity C during this round.

  Evolution of the system configuration. The packets of the set Z delay the packets of the set Z_1 in the queue e_1 because they have nearest to go (queue e_1) than the packets of the set Z_1 (queue f). Furthermore, the packets of the set Y are delayed in e_2 due to the unit capacity of the edge e_2 during this round. Thus, the remaining packets of the set Y in the queue e_2 at the end of this round is a set Y' of $|Y'| = |Y| - |T_3|$ packets, while the remaining packets of the set Y traverse their remaining path and they are absorbed.

- **Round 4:** It lasts $|T_4| = |Y'|/C$ time steps.

  Adversary's behavior. During this round the edge e_1 has unit capacity, while all the other edges have high capacity C. Also, the adversary injects a set Z_2 of $|Z_2| = \rho C|T_4|$ packets in the queue e_2 requiring to traverse the edges e_2, f. These packet injections satisfy the load condition because the assigned path consists of edges that have high capacity C during this round.

  Evolution of the system configuration. The packets of the set Y' delay the packets of the set Z_2 in the queue e_2 because they have nearest to go (queue e_2) than Z_2 packets (queue f). The packets of the set Y' packets traverse the edge e_2 and they are absorbed. Moreover, the packets of the set Z_1 are delayed in the queue e_1 due to the unit capacity of the edge e_1 during this round. Therefore, the packets of the set Z_1 in the queue e_1 at the end of this round is a set Z_1' of $|Z_1'| = |Z_1| - |T_4|$ packets, while the remaining packets of the set Z_1 traverse their path and they are absorbed. Thus, the number of packets in the queues e_1, e_2 requiring to traverse the edges f at the end of this round is $s_{j+1} = |Z_2| + |Z_1'| = \rho^3 s_j - \frac{\rho^3}{C} s_j + \rho^3 s_j - \frac{\rho^2}{C} s_j + \frac{\rho^2}{C^2} s_j$.

In order to have instability, we must have $s_{j+1} > s_j$, that is $2\rho^3 s_j - \frac{\rho^3}{C} s_j - \frac{\rho^2}{C} s_j + \frac{\rho^2}{C^2} s_j > s_j$. Therefore, $2\rho^3 C^2 - \rho^3 C - \rho^2 C + \rho^2 > C^2$. Dividing by C^2 the inequality, we take $\rho^3(2 - \frac{1}{C}) - \rho^2(\frac{1}{C} - \frac{1}{C^2}) > 1$. If we let $C = 1000$ and $\rho = 0.8$, the inequality holds. Thus, for $C > 1000$ and $\rho = 0.8$ the inequality holds, too.

When $C \to \infty$, it holds that $\frac{1}{C^k} \to 0$ for all $k \geq 1$. Then, our inequality becomes $2\rho^3 - 1 > 0$ which holds for $\rho > \sqrt[3]{0.5}$. Note that if we have a sequence of equations $f_C(\rho)$ and there exists the limit $\lim_{C\to\infty} f_C(\rho) = f_\infty(\rho)$, then it holds fundamentally by the theory of function limits that if $\rho(C)$ is the root of $f_C(\rho) = 0$, then $\lim_{C\to\infty} \rho(C)$ is the root of $f_\infty(\rho)$. Therefore, for $\rho > \sqrt[3]{0.5}$ the system is unstable. This argument can be repeated for an infinite number of phases showing that the number of packets in the system increases forever for $\rho > \sqrt[3]{0.5}$. □

Theorem 5.6 *Let $\rho = 0.76$. For the network $\mathcal{U}_2$ there is an adversary $\mathcal{A}'$ of rate ρ that can change the link capacities of $\mathcal{U}_2$ between the two integer values 1 and $C > 1000$ such that the system $\langle \mathcal{U}_2, \mathcal{A}', \mathsf{NTG-U-LIS} \rangle$ is unstable. When $C \to \infty$ the system $\langle \mathcal{U}_2, \mathcal{A}', \mathsf{NTG-U-LIS} \rangle$ is unstable for $\rho > 0.754$.*

PROOF. We break the construction of the adversary $\mathcal{A}'$ into phases.

Inductive Hypothesis: At the beginning of phase j, there are s_j packets that are queued in the queues e_1, f_1 requiring to traverse the edge f_2.

Induction Step: At the beginning of phase $j+1$ there will be $s_{j+1} > s_j$ packets which will be queued in the queues e_1, f_1 requiring to traverse the edge f_2.

We will construct an adversary $\mathcal{A}'$ such that the induction step will hold. Proving that the induction step holds, we ensure that the inductive hypothesis will hold at the beginning of phase $j+1$ with an increased value of s_j, $s_{j+1} > s_j$. From the inductive hypothesis, initially, there are s_j packets (that constitute the set of packets S) in the queues e_1, f_1 requiring to traverse the edge f_2. In order to prove that the induction step works, we consider that there is a large enough number of packets s_j in the initial system configuration.

During phase j the adversary plays three rounds of injections as follows:

- **Round 1:** It lasts $|T_1| = s_j/C$ time steps.

  Adversary's behavior. During this round all the network edges have high capacity C and the adversary injects in the queue f_2 a set X of $|X| = \rho C|T_1|$ packets wanting to traverse the edges f_2, e_1, e_2. These packet injections satisfy the load condition because the assigned path consists of edges that have high capacity C during this round.

  Evolution of the system configuration. The packets of the set S delay the packets of the set X in the queue f_2 because they have nearest to go (queue f_2) than the packets of the set X (queue e_2). The packets of the set S traverse the edge f_2 and they are absorbed.

- **Round 2:** It lasts $|T_2| = |X|/C$ time steps.

  Adversary's behavior. During this round all the network edges have high capacity C. Also, the adversary injects a set Y of $|Y| = \rho C|T_2|$ packets in the queue e_1 requiring to traverse the edges e_1, f_2 and a set Z of $|Z| = \rho C|T_2|$ packets in the queue e_2 requiring to traverse the edges e_2, f_1. These injections satisfy the load condition because all the network edges have high capacity C, and the injection paths of the different packet sets do not have overlapped edges.

  Evolution of the system configuration. The packets of the set X delay the packets of the set Y in the queue e_1 because they are longer time in the system than the packets of the set Y (the packets of the sets X and Y have to traverse the same distance to reach their destination). Furthermore, the packets of the set X delay the packets of the set Z in the queue e_2 because they have nearest to go (queue e_2) than the packets of the set Z (queue f_1). The packets of the set X traverse the edge e_2 and they are absorbed.

- **Round 3:** It lasts $|T_3| = |Y|/C$ time steps.

  Adversary's behavior. During this round the edge e_1 has unit capacity, while all the other edges have high capacity C. Also, the adversary injects a set Z_1 of $|Z_1| = \rho C|T_3|$ packets in the queue f_1 requiring to traverse the edges f_1, f_2. These packet injections satisfy the load condition because the assigned path consists of edges that have high capacity C during this round.

  Evolution of the system configuration. The packets of the set Z delay the packets of the set Z_1 in the queue f_1 because they have nearest to go (queue f_1) than the packets of the set Z_1 (queue f_2). Moreover, the packets of the set Y are delayed in e_1 due to the unit capacity of the edge e_1 during this round. Therefore, the packets of the set Y in the queue e_1 at the end of this round is a set $|Y^{'}|$ of $|Y^{'}| = |Y| - |T_3|$ packets, while the remaining packets of the set Y traverse their path and they are absorbed. Thus, the number of packets in the queues e_1, f_1 requiring to traverse the edge f_2 at the end of this round is $s_{j+1} = |Z_1| + |Y^{'}| = \rho^3 s_j + \rho^2 s_j - \frac{\rho^2 s_j}{C}$.

  In order to have instability, we must have $s_{j+1} > s_j$, that is $\rho^3 s_j + \rho^2 s_j - \frac{\rho^2 s_j}{C} > s_j$. Therefore, $\rho^3 C + \rho^2(C-1) > C$. Dividing by C the inequality, we take $\rho^3 + \rho^2(1 - \frac{1}{C}) > 1$. If we let $C = 1000$ and $\rho = 0.76$, the inequality holds. Thus, for $C > 1000$ and $\rho = 0.76$ the inequality holds, too.

  When $C \to \infty$, it holds that $\frac{1}{C^k} \to 0$ for all $k \geq 1$. Then, our inequality becomes $\rho^3 + \rho^2 - 1 > 0$, which holds for $\rho > 0.754$. Note that if we have a sequence of equations $f_C(\rho)$ and there exists the limit $\lim_{C\to\infty} f_C(\rho) = f_\infty(\rho)$, then it holds fundamentally by the theory of function limits that if $\rho(C)$ is the root of $f_C(\rho) = 0$, then $\lim_{C\to\infty} \rho(C)$ is the root of $f_\infty(\rho)$. Therefore, for $\rho > 0.754$ the system is unstable. This argument can be repeated for an infinite number of phases showing that the number of packets in the system increases forever for $\rho > 0.754$. □

6 Discussion and Directions for Further Research

In this work, we studied how the dynamic changing of the network link capacities affects the instability properties of greedy contention-resolution protocols and networks using an extension of the adversarial model that was first initiated by Borodin *et al.* in [5], the Adversarial, Quasi-Static Queueing Theory Model. Thus, we studied the instability properties of LIS showing an instability bound on the injection rate that represents the current state-of-the-art. Also, we proved that the lower bound on the injection rate that guarantees instability for specific compositions of universally stable protocols can be dropped lower than 1/2 when the link capacities change by an adversary. This result improves the corresponding one in the classical adversarial model [10]. Finally, we studied the instability behavior which is induced by all the known forbidden subgraphs on networks running the NTG-U-LIS protocol when the link capacities change dynamically showing lower instability bounds than their counterparts in the classical adversarial model [2].

However, a lot of problems remain open. Our results suggest that, for every unstable network, its instability bound in the model of quasi-static capacities may be lower than for the classical adversarial queueing model. Proving (or disproving) this remains an open problem. Another avenue for further research is whether there are upper bounds on the injection rate that guarantee stability for forbidden subgraphs when the link capacities can change dynamically. Studying the impact of dynamically changing link capacities on other greedy protocols and networks or whether the lower bound on the injection rate that guarantees instability for compositions of protocols can be dropped further is another interesting problem. Finally, it worths to receive attention the study of the stability behavior of networks and protocols in environments where the adversary controls the movement of the network nodes.

References

[1] M. Andrews, B. Awerbuch, A. Fernandez, J. Kleinberg, T. Leighton, Z. Liu, Universal Stability Results for Greedy Contention-Resolution Protocols, Journal of the ACM, 48 (1) (2001) 39–69.

[2] C. Alvarez, M. Blesa, M. Serna, A Characterization of Universal Stability in the Adversarial Queuing model, SIAM Journal on Computing, 34 (1) (2004) 41–66.

[3] W. Aiello, E. Kushilevitz, R. Ostrovsky, and A. Rosén, Adaptive Packet Routing for Bursty Adversarial Traffic, in: Proceedings of the 30th Annual ACM Symposium on Theory of Computing, Dallas, Texas, USA, 1998, pp. 359–368.

[4] A. Borodin, J. Kleinberg, P. Raghavan, M. Sudan, D. Williamson, Adversarial Queueing Theory, Journal of the ACM, 48 (1) (2001) 13–38.

[5] A. Borodin, R. Ostrovsky, Y. Rabani, Stability Preserving Transformations: Packet Routing Networks with Edge Capacities and Speeds, in: Proceedings of the 12th Annual ACM-SIAM Symposium on Discrete Algorithms, Washington, DC, USA, 2001, pp. 601–610.

[6] H. Chen, D. D. Yao, Fundamentals of Queueing Networks, Springer-Verlag, 2000.

[7] J. Diaz, D. Koukopoulos, S. Nikoletseas, M. Serna, P. Spirakis, D. Thilikos, Stability and Non-Stability of the FIFO Protocol, in: Proceedings of the 13th Annual ACM Symposium on Parallel Algorithms and Architectures, Crete, Greece, 2001, pp. 48–52.

[8] A. Goel, Stability of Networks and Protocols in the Adversarial Queueing Model for Packet Routing, Networks, 37 (4) (2001) 219–224.

[9] D. Koukopoulos, M. Mavronicolas, S. Nikoletseas, P. Spirakis, On the Stability of Compositions of Universally Stable, Greedy, Contention-Resolution Protocols, in: Proceedings of the 16th International Symposium on DIStributed Computing, Lecture Notes in Computer Science, Toulouse, France, Springer-Verlag, 2002, Vol. 2508, pp. 88–102.

[10] D. Koukopoulos, M. Mavronicolas, S. Nikoletseas, P. Spirakis, The Impact of Network Structure on the Stability of Greedy Protocols, in: Proceedings of the 5th Italian Conference on Algorithms and Complexity, Lecture Notes in Computer Science, Rome, Italy, Springer-Verlag, 2003, Vol. 2653, pp. 251–263. Also, accepted in the Journal Theory of Computing Systems.

[11] D. Koukopoulos, S. Nikoletseas, P. Spirakis, Stability Issues in Heterogeneous and FIFO Networks under the Adversarial Queueing Model, in: Proceedings of the 8th International Conference on High Performance Computing, Lecture Notes in Computer Science, Hyderabad, India, Springer-Verlag, 2001, Vol. 2228, pp. 3–14. Invited Keynote Address.

[12] Z. Lotker, B. Patt-Shamir, A. Rosén, New Stability Results for Adversarial Queuing, SIAM Journal on Computing, 33 (2) (2004) 286–303.

[13] P. Tsaparas, Stability in Adversarial Queueing Theory, M.Sc. Thesis, Computer Science Department, University of Toronto, 1997.