Facets of the Fully Mixed Nash Equilibrium Conjecture*

Rainer Feldmann† *Marios Mavronicolas*‡ *Andreas Pieris*§

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Abstract

In this work, we continue the study of the many facets of the *Fully Mixed Nash Equilibrium Conjecture*, henceforth abbreviated as the FMNE *Conjecture*, in selfish routing for the special case of n identical *users* over two (identical) parallel *links*. We introduce a new measure of *Social Cost*, defined to be the expectation of the square of the maximum *congestion* on a link; we call it *Quadratic Maximum Social Cost*. A *Nash equilibrium* is a stable state where no user can improve her (expected) latency by switching her mixed strategy; a *worst-case* Nash equilibrium is one that maximizes Quadratic Maximum Social Cost. In the *fully mixed* Nash equilibrium, all *mixed strategies* achieve full support.

Formulated within this framework is yet another facet of the FMNE *Conjecture*, which states that the fully mixed Nash equilibrium is the worst-case Nash equilibrium. We present an extensive proof of the FMNE *Conjecture*; the proof employs a mixture of combinatorial arguments and analytical estimations. Some of these analytical estimations are derived through some new bounds on *generalized medians* of the binomial distribution [22] we obtain, which are of independent interest.

1 Introduction

Motivation and Framework. In this work, we continue the study of the (multi-faceted) *Fully Mixed Nash Equilibrium Conjecture* [7], henceforth abbreviated as the FMNE *Conjecture*, in selfish routing. Specifically, we look at a special case of the KP model for selfish routing due to Koutsoupias and Papadimitriou [15]; here, a collection of n (*unweighed*) *users* wish to each transmit one unit of traffic from *source* to *destination*, which are joined through *two* (identical) parallel *links*. The *congestion* on a link is the total number of users choosing it; each user makes her choice using a *mixed strategy*, which is a probability distribution over links. In the special case case of the KP model we look at, the *latency* on a link is identified with the congestion on it.

In a *Nash equilibrium* [20, 21], no user can improve the expected congestion on the link she chooses by switching to a different (mixed) strategy. Originally considered by Kaplansky back in 1945 [14], *fully mixed Nash equilibria* have all their involved probabilities strictly positive; they were recently coined into the context of selfish routing by Mavronicolas and Spirakis [19]. Clearly, the fully mixed Nash equilibrium maximizes the randomization used in the mixed strategies of the players; so, it is a natural candidate to become a vehicle for the study of the effects of randomization on the quality of Nash equilibria.

We introduce a new measure of *Social Cost* [15] for the evaluation of Nash equilibria. The new measure is taken to be the expectation of the square of the maximum congestion on a link; call it *Quadratic Maximum Social Cost*. (The expectatiom is taken over all random choices of the users.) Note that the Quadratic Maximum Social Cost simultaneously generalizes the *Maximum Social Cost* (expectation of maximum latency) proposed in the seminal work of Koutsoupias and Papadimitriou [15], and the *Quadratic Social Cost* (expectation of the sum of the squares of the latencies) proposed in [16].

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†Faculty of Computer Science, Electrical Engineering and Mathematics, University of Paderborn, 33102 Paderborn, Germany. Email: obelix@uni-paderborn.de

‡Department of Computer Science, University of Cyprus, Nicosia CY-1678, Cyprus. Currently visiting Faculty of Computer Science, Electrical Engineering and Mathematics, University of Paderborn, 33102 Paderborn, Germany. Email: mavronic@cs.ucy.ac.cy

§Computing Laboratory, University of Oxford, Oxford OX1 3QD, United Kingdom. Email: andreas.pieris@keble.ox.ac.uk

The motivation to consider the square of the latency comes from the real application of scheduling transmissions among nodes positioned on the Euclidian plane. The received power at a receiver is proportional to the power $-\delta$ of the (generalized) Euclidian distance from the sender to the receiver; δ is the *path-loss exponent*, for which it has been empirically assumed that $\delta \geq 2$ (cf. [13]). In many natural cases, the latency is proportional to the (generalized) Euclidian distance, and the proportionality constant may have to do with external conditions of the medium and the transmission power; in those cases, the received power is proportional to the power $-\delta$ of the latency. So, investigating the expected maximum latency to the power δ for the initial case $\delta = 2$ is expected to give insights about the optimization of received power in selfish transmissions.

For any particular definition of Social Cost, the FMNE *Conjecture* states that the fully mixed Nash equilibrium maximizes the Social Cost among all Nash equilibria. The validity of the FMNE *Conjecture* implies that computing the worst-case Nash equilibrium (with respect to the fixed Social Cost) for a given instance is trivial; it may also allow an approximation to the *Price of Anarchy* [15] in case where there is a FPRAS for approximating the Social Cost of the fully mixed Nash equilibrium (cf. [6]).

Contribution. In this proposed framework, we formulate a corresponding facet of the FMNE *Conjecture*:

Conjecture 1.1 *The fully mixed Nash equilibrium maximizes the Quadratic Maximum Social Cost.*

We present an extensive proof of this FMNE *Conjecture* using a wealth of combinatorial and analytical tools. The proof amounts to a very sharp comparison of the Quadratic Maximum Social Cost of an *arbitrary* Nash equilibrium to that of the fully mixed Nash equilibrium.

The proof has required some very *sharp* analytical estimates of various combinatorial functions that entered the analysis; this provides some evidence that the proved inequality among the two compared Quadratic Maximum Social Costs is very *tight*. The employed analytical estimates may be applicable elsewhere; so, they are interesting on their own right. In more detail, we have provided some new estimations for some generalizations of the *median* of the binomial distribution [11, 22], which may be of independent interest.

Related Work. The FMNE *Conjecture* was first stated in [7]; it was motivated there by some initial observations in [6]. The FMNE *Conjecture* has been proved for the Maximum Social Cost for the cases of *(i)* two (*unweighted*) users and non-identical but *related links*, and *(ii)* an arbitrary number of (unweighted) users and two (identical) links in [17]. In fact, our estimation techniques significantly extend those for the case *(ii)* above in [17]; due to the increased complexity of the Quadratic Maximum Social Cost function (over Maximum Social Cost), far more involved estimations have been required in the present proof. Counterexamples to the FMNE *Conjecture* appeared *(i)* for the case of unrelated links in [17], and *(ii)* for the case of weighted users in [5]. In the context of selfish routing, the fully mixed Nash equilibrium and the FMNE *Conjecture* have attracted a lot of interest and attention; they both have been studied extensively in the last few years for a wide variety of theoretical models of selfish routing and Social Cost measures - see, e.g., [2, 4, 9, 10, 12, 16, 18].

The status of the studied facets of the FMNE *Conjecture* is summarized in Table 1. In the case of related links, latency is a linear function of congestion on a link; in the (special) case of identical links, the linear function is identity, while in the (more general) case of *player-specific links*, the linear function is specific to each player. In the (even more general) case of *unrelated links*, there is an additive contribution to latency on a link, which is both player-specific and *link-specific*. The *Quadratic Social Cost* [16], denoted as QSC, is the (expectation of the) sum of the squares of the latencies; more generally, the *Polynomial Social Cost*, denoted as PSC, is the (expectation of the) sum of polynomial functions of the latencies. The *Player-Average Social Cost* (considered in [9, 12] and denoted as $\Sigma_{\mathsf{IC}}\mathsf{SC}$) is the sum of Individual Costs of the players; the *Player-Maximum Social Cost* (considered in [9, 10] and denoted as $\mathsf{M}_{\mathsf{IC}}\mathsf{SC}$) is the maximum Individual Cost of a player.

2 Mathematical Tools

Notation. For any integer $n \geq 2$, denote $[n] = \{1, 2, \ldots, n\}$. For a random variable X following the distribution $\mathbb{P}$, denote as $\mathbb{E}_{\mathbb{P}}(X)$ the *expectation* of X; $X \sim \mathbb{P}$ denotes that X follows the distribution $\mathbb{P}$. For an integer n, the predicates $\mathsf{Even}(n)$ and $\mathsf{Odd}(n)$ will be 1 when n is even and odd, respectively, and 0 otherwise.

Two Combinatorial Facts. The first fact is an extension of *Stirling's* approximation $n! \approx \sqrt{2\pi} n^{n+\frac{1}{2}} \exp(-n)$ to $n!$, where $\exp(x)$ denotes e^x. The extension yields a double inequality for $n!$ (cf. [3, Chapter 2, Section 9]).

Lemma 2.1 *For all integers $n \geq 1$,* $\sqrt{2\pi} n^{n+\frac{1}{2}} \exp\left(-n + \frac{1}{12n+1}\right) \leq n! \leq \sqrt{2\pi} n^{n+\frac{1}{2}} \exp\left(-n + \frac{1}{12n}\right)$.

A double application of Lemma 2.1 in fractional expansions of binomial coefficients yields:

Model assumptions	Social Cost	FMNE *Conjecture*?	Reference
$n = 2$, weighted users & identical links	MSC	$\surd$	[6]
unweighted users & related links	MSC	49.02	[6]
weighted users & identical links	MSC	$2h(1+\varepsilon)$	[9]
$n = 2$, unweighted users & related links	MSC	$\surd$	[17]
$m = 2$, unweighted users & identical links	MSC	$\surd$	[17]
$m = 2, n = 2$ & unrelated links	MSC	$\surd$	[17]
$m = 2, n = 3$ & unrelated links	MSC	$\times$	[17]
unweighted users & identical links	QSC	$\surd$	[16]
unweighted users & links with (identical) non-constant and convex latency functions	Σ_{IC}SC	$\surd$	[9]
unweighted users & identical links	PSC	$\surd$	[10]
weighted users & player-specific links	Σ_{IC}SC	$\surd$	[12]
weighted users & player-specific links	M_{IC}SC	$\surd$	[12]
weighted users & identical links	MSC	$\times$	[5]
weighted users with types & identical links	Σ_{IC}SC	$\surd$	[10]
weighted users with types & identical links	M_{IC}SC	$\surd$	[10]

Table 1: The status of the studied facets of the FMNE *Conjecture*. A symbol $\surd$ (resp., $\times$) in the third column indicates that the FMNE *Conjecture* has been proven (resp., refuted) for the corresponding case. A number ρ in the third column indicates that an *approximate* version of the FMNE *Conjecture* has been shown: the Social Cost of an arbitrary Nash equilibrium is at most ρ times the one of the fully mixed. The symbol h denotes the factor by which the largest weight deviates from the average weight (in the case of weighted users).

Lemma 2.2 *For all integers $n \geq 1$,* $n\sqrt{\frac{n}{2\pi}}\exp\left(\frac{1}{12n+1}-\frac{1}{3n}\right) \leq \frac{n^2}{2^{n+1}}\binom{n}{\frac{n}{2}} \leq n\sqrt{\frac{n}{6}}$.

Lemma 2.3 *For all integers $n \geq 1$,* $\sqrt{\frac{n}{2\pi}}\exp\left(\frac{1}{12n+1}-\frac{1}{3n-3}\right) \leq \frac{n!}{2^n\left(\left(\frac{n-1}{2}\right)!\right)^2} \leq \left(\frac{n}{n-1}\right)^n\sqrt{\frac{n}{6}}$.

The second fact is a maximization property of the *Bernstein basis polynomial of order k and degree n* $\mathsf{b}_{k,n}(x) = \binom{n}{k}x^k(1-x)^{n-k}$, which forms a basis of the vector space of polynomials of degree n [1].

Lemma 2.4 *For all integers $0 \leq k \leq n$, $\max_{x\in[0,1]} \mathsf{b}_{k,n}(x) = \binom{n}{k}k^k n^{-n}(n-k)^{n-k}$, occurring at $x = \frac{k}{n}$.*

Generalized Medians of the Binomial Distribution. Consider a sequence of N *Bernoulli* trials, each succeeding with probability p. The number of successes out of these N trials follows the *binomial distribution*; that is, the probability of obtaining at most $k \leq N$ successes is $\Sigma_{\ell=0}^{k}\binom{N}{\ell}p^\ell(1-p)^{N-\ell}$.

Define the *binomial function* $\mathsf{B}_{N,k}(p) : [0,1] \rightarrow \mathbb{R}$ with $\mathsf{B}_{N,k}(p) = \Sigma_{\ell=0}^{k}\binom{N}{\ell}p^\ell(1-p)^{N-\ell}$. Clearly, $\mathsf{B}_{N,k}(p)$ is strictly decreasing in (and continuous with) p, with $\mathsf{B}_{N,k}(0) = 1$ and $\mathsf{B}_{N,k}(1) = 0$. By continuity, it follows that $\mathsf{B}_{N,k}$ attains all intermediate values between 0 and 1. For any $\alpha \in [0,1]$, define the ***α-median*** of the binomial distribution, denoted as $\mathsf{M}_{N,p}(\alpha)$ with $\mathsf{M}_{N,p}(\alpha) = \min\{k \in [0,N] \mid \mathsf{B}_{N,k}(p) \geq \alpha\}$; intuitively, the α-median of the binomial distribution is the *least* integer k such that the probability of obtaining at most k successes is at least α. Clearly, $\mathsf{B}_{N,k}(p) < \alpha$ for all indices $k < \mathsf{M}_{N,p}(\alpha)$. This definition of α-median generalizes the classical definition of median of the binomial distribution (which is the $\frac{1}{2}$-median). We will use one known fact about medians [11, Theorem 2.3]:

Lemma 2.5 $\mathsf{M}_{N,\frac{1}{2}}\left(\frac{1}{2}\right) = \lfloor\frac{N}{2}\rfloor$*; for $p < \frac{1}{2}$,* $\mathsf{M}_{N,p}\left(\frac{1}{2}\right) \geq (N+1)p - 1$.

Furthermore, we establish in this work some **new** bounds on generalized medians, which shall be employed in some later proofs:

Lemma 2.6 (Generalized Medians) *For any $\epsilon > 0$, the following bounds hold on generalized medians of the binomial distribution, where* $p = \frac{1}{2} - \frac{r}{2(n-r-1)}$:

(1) $\mathsf{M}_{n-r-2,p}\left(\frac{1}{2}+\epsilon\right) > \left\lceil \frac{n-3}{2} \right\rceil - r - 1$, *where* $1 \le r \le \left\lfloor \frac{n-3}{2} \right\rfloor - 4$.

(2) $\mathsf{M}_{n-r-2,p}\left(\frac{3}{7}+\epsilon\right) > \left\lceil \frac{n-3}{2} \right\rceil - r - 1$, *where* $n \ge 134$ *is even and* $r = \left\lfloor \frac{n-3}{2} \right\rfloor - 3$.

(3) $\mathsf{M}_{n-r-2,p}\left(\frac{2}{5}+\epsilon\right) > \left\lceil \frac{n-3}{2} \right\rceil - r - 1$, *where* $n \ge 134$ *is even and* $r = \left\lfloor \frac{n-3}{2} \right\rfloor - 2$.

(4) $\mathsf{M}_{n-r-2,p}\left(\frac{1}{3}+\epsilon\right) > \left\lceil \frac{n-3}{2} \right\rceil - r - 1$, *where* $n \ge 134$ *is even and* $r = \left\lfloor \frac{n-3}{2} \right\rfloor - 1$.

(5) $\mathsf{M}_{n-r-2,p}\left(\frac{1}{4}+\epsilon\right) > \left\lceil \frac{n-3}{2} \right\rceil - r - 1$, *where* $n \ge 134$ *is even and* $r = \left\lfloor \frac{n-3}{2} \right\rfloor$.

(6) $\mathsf{M}_{n-r-2,p}\left(\frac{3}{11}+\epsilon\right) > \left\lceil \frac{n-3}{2} \right\rceil - r - 1$, *where* $n \ge 135$ *is odd and* $r = \left\lfloor \frac{n-3}{2} \right\rfloor - 3$.

(7) $\mathsf{M}_{n-r-2,p}\left(\frac{2}{9}+\epsilon\right) > \left\lceil \frac{n-3}{2} \right\rceil - r - 1$, *where* $n \ge 135$ *is odd and* $r = \left\lfloor \frac{n-3}{2} \right\rfloor - 2$.

(8) $\mathsf{M}_{n-r-2,p}\left(\frac{1}{7}+\epsilon\right) > \left\lceil \frac{n-3}{2} \right\rceil - r - 1$, *where* $n \ge 135$ *is odd and* $r = \left\lfloor \frac{n-3}{2} \right\rfloor - 1$

(9) $\mathsf{M}_{n-r-2,p}(\epsilon) > \left\lceil \frac{n-3}{2} \right\rceil - r - 1$, *where* $n \ge 135$ *is odd and* $r = \left\lfloor \frac{n-3}{2} \right\rfloor$.

3 Framework and Preliminaries

Our definitions are based on (and depart from) the standard ones for the KP model; see, e.g., [17, Section 2].

General. We consider a ***network*** consisting of **two** parallel *links* $1, 2$ from a ***source*** to a ***destination*** node. Each of $n \ge 2$ *users* $1, 2, \ldots, n$ wishes to route one unit of traffic from source to destination.

A ***pure strategy*** s_i for user $i \in [n]$ is some specific link; a ***mixed strategy*** σ_i is a probability distribution over pure strategies— so, σ_i is a probability distribution over links. The ***support*** of user i in her mixed strategy σ_i, denoted as $\mathsf{support}(\sigma_i)$, is the set of pure strategies to which i assigns strictly positive probability. A ***pure profile*** is a vector $\mathbf{s} = \langle s_1, \ldots, s_n \rangle$ of pure strategies, one for each user; a ***mixed profile*** is a vector $\boldsymbol{\sigma} = \langle \sigma_1, \ldots, \sigma_n \rangle$ of mixed strategies, one for each user. The mixed profile $\boldsymbol{\sigma}$ is ***fully mixed*** if for each user $i \in [n]$ and link $j \in [2]$, $\sigma_i(j) > 0$. Note that a mixed profile $\boldsymbol{\sigma}$ induces a (product) probability measure $\mathbb{P}_{\boldsymbol{\sigma}}$ on the space of pure profiles. A user i is pure in the mixed profile $\boldsymbol{\sigma}$ if $|\mathsf{support}(\sigma_i)| = 1$; so, a pure profile is the degenerate of a mixed profile where all users are pure. A user i is fully mixed in the mixed profile $\boldsymbol{\sigma}$ if $|\mathsf{support}(\sigma_i)| = 2$; so, a fully mixed profile is the special case of a mixed profile where all users are fully mixed.

Cost measures and Nash equilibria. The ***congestion*** on the link ℓ in the pure profile $\mathbf{s}$, denoted as $\mathsf{c}(\ell, \mathbf{s})$, is the number of users choosing link ℓ in $\mathbf{s}$; so, $\mathsf{c}(\ell, \mathbf{s}) = |\{i \in [n] : s_i = \ell\}|$. The ***Individual Cost*** of user i in the profile $\mathbf{s}$, denoted as $\mathsf{IC}_i(\mathbf{s})$, is the congestion on her chosen link; so, $\mathsf{IC}_i(\mathbf{s}) = \mathsf{c}(s_i, \mathbf{s})$. The ***expected congestion*** on the link ℓ in the mixed profile $\boldsymbol{\sigma}$, denoted as $\mathsf{c}(\ell, \boldsymbol{\sigma})$, is the expectation (according to $\boldsymbol{\sigma}$) of the congestion on link ℓ; so, $\mathsf{c}(\ell, \boldsymbol{\sigma}) = \mathbb{E}_{\mathbf{s} \sim \mathbb{P}_{\boldsymbol{\sigma}}}(\mathsf{c}(\ell, \mathbf{s}))$. The ***Expected Individual Cost*** of user i in the mixed profile $\boldsymbol{\sigma}$, denoted as $\mathsf{IC}_i(\boldsymbol{\sigma})$, is the expectation (according to $\boldsymbol{\sigma}$) of her Individual Cost; so, $\mathsf{IC}_i(\boldsymbol{\sigma}) = \mathbb{E}_{\mathbf{s} \sim \mathbb{P}_{\boldsymbol{\sigma}}}(\mathsf{IC}_i(\mathbf{s}))$.

The ***Maximum Social Cost*** of the mixed profile $\boldsymbol{\sigma}$, denoted as $\mathsf{MSC}(\boldsymbol{\sigma})$, is the expectation of the maximum congestion; so, $\mathsf{MSC}(\boldsymbol{\sigma}) = \mathbb{E}_{\mathbf{s} \sim \mathbb{P}_{\boldsymbol{\sigma}}}\left(\max_{\ell \in [2]} \mathsf{c}(\ell, \mathbf{s})\right)$. The ***Quadratic Maximum Social Cost*** of the mixed profile $\boldsymbol{\sigma}$, denoted as $\mathsf{QMSC}(\boldsymbol{\sigma})$, is the expectation of the square of the maximum congestion; so,

$$\begin{aligned}
\mathsf{QMSC}(\boldsymbol{\sigma}) &= \mathbb{E}_{\mathbf{s} \sim \mathbb{P}_{\boldsymbol{\sigma}}}\left(\left(\max_{\ell \in [2]} \mathsf{c}(\ell, \mathbf{s})\right)^2\right) \\
&= \sum_{\mathbf{s} \in \mathcal{S}} \mathbb{P}_{\boldsymbol{\sigma}}(\mathbf{s}) . \left(\max_{\ell \in [2]} \mathsf{c}(\ell, \mathbf{s})\right)^2 \\
&= \sum_{\mathbf{s} \in \mathcal{S}} \left(\prod_{k \in [n]} \sigma_k(s_k)\right) . \left(\max_{\ell \in [2]} \mathsf{c}(\ell, \mathbf{s})\right)^2 .
\end{aligned}$$

The mixed profile $\boldsymbol{\sigma}$ is a ***(mixed) Nash equilibrium*** [20, 21] if for each user $i \in [n]$, for each mixed strategy σ'_i of player i, $\mathsf{IC}_i(\boldsymbol{\sigma}) \le \mathsf{IC}_i(\boldsymbol{\sigma}_{-i} \diamond \sigma'_i)$; so, player i has no incentive to unilaterally change her mixed strategy. (Note that $\boldsymbol{\sigma}_{-i} \diamond \sigma'_i$ is the mixed profile obtained by substituting the mixed strategy σ_i of player i in $\boldsymbol{\sigma}$ with the mixed strategy $\sigma_{i'}$.)

The fully mixed Nash equilibrium. We are especially interested in the fully mixed Nash equilibrium $\boldsymbol{\phi}$ which is known to exist uniquely in the setting we consider [19]; it is also known that for each pair of user $i \in [n]$ and a link $\ell \in [2]$, $\boldsymbol{\phi}_i(\ell) = \frac{1}{2}$, so that all 2^n pure profiles are equiprobable, each occurring with probability $\frac{1}{2^n}$ [19, Lemma 15]. The Maximum Social Cost of $\boldsymbol{\phi}$ is given by $\mathsf{MSC}(\boldsymbol{\phi}) = \frac{n}{2} + \frac{n}{2^n}\binom{n-1}{\lceil \frac{n}{2} \rceil - 1}$ [17]. We now calculate the Quadratic Maximum Social Cost of the fully mixed Nash equilibrium $\boldsymbol{\phi}$.

Lemma 3.1 $\mathsf{QMSC}(\boldsymbol{\phi}) = \frac{n}{4} + \frac{n^2}{4} + \frac{n^2}{2^n}\binom{n-1}{\lceil \frac{n}{2} \rceil - 1}$.

The arbitrary Nash equilibrium. Fix now an arbitrary Nash equilibrium $\boldsymbol{\sigma}$. It is known that $\mathsf{MSC}(\boldsymbol{\phi}) \geq \mathsf{MSC}(\boldsymbol{\sigma})$ [17] (for the particular case of unweighted users and two identical links). We consider three sets:

- The set $\mathcal{U}_1 = \{i : \mathsf{support}(\sigma_i) = \{1\}\}$ of (pure) users choosing link 1.
- The set $\mathcal{U}_2 = \{i : \mathsf{support}(\sigma_i) = \{2\}\}$ of (pure) users choosing link 2.
- The set $\mathcal{U}_{12} = \{i : \mathsf{support}(\sigma_i) = \{1, 2\}\}$ of (fully) mixed users choosing either link 1 or link 2.

Denote $u = \min\{|\mathcal{U}_1|, |\mathcal{U}_2|\}$. So, there are $2u$ (pure) users each of which chooses either link 1 or link 2 with probability 1. Denote $\widehat{\boldsymbol{\sigma}}$ the mixed profile derived from $\boldsymbol{\sigma}$ by eliminating those $2u$ users; note that $\widehat{\boldsymbol{\sigma}}$ is a (mixed) Nash equilibrium. Also, denote as $\widehat{\boldsymbol{\phi}}$ the fully mixed Nash equilibrium with $n-2u$ users. Note that $\widehat{\boldsymbol{\sigma}}$ has simpler form that $\boldsymbol{\sigma}$. Hence, it would be more convenient to compare $\mathsf{QMSC}(\widehat{\boldsymbol{\phi}})$ and $\mathsf{QMSC}(\widehat{\boldsymbol{\sigma}})$ (instead of comparing $\mathsf{QMSC}(\boldsymbol{\phi})$ and $\mathsf{QMSC}(\boldsymbol{\sigma})$). To do so, we need to prove a relation between $\mathsf{QMSC}(\widehat{\boldsymbol{\sigma}})$ and $\mathsf{QMSC}(\boldsymbol{\sigma})$, and another relation between $\mathsf{QMSC}(\widehat{\boldsymbol{\phi}})$ and $\mathsf{QMSC}(\boldsymbol{\phi})$.We first prove a relation between the Quadratic Maximum Social Costs of $\boldsymbol{\sigma}$ and $\widehat{\boldsymbol{\sigma}}$. Note that

$$\begin{aligned}
\mathsf{QMSC}(\widehat{\boldsymbol{\sigma}}) &= \mathbb{E}_{\mathbb{P}_{\boldsymbol{\sigma}}}\left((\max\{\mathsf{c}(1,\boldsymbol{\sigma}), \mathsf{c}(2,\boldsymbol{\sigma})\} - u)^2\right) \\
&= \mathbb{E}_{\mathbb{P}_{\boldsymbol{\sigma}}}\left((\max\{\mathsf{c}(1,\boldsymbol{\sigma}), \mathsf{c}(2,\boldsymbol{\sigma})\})^2 - 2u\max\{\mathsf{c}(1,\boldsymbol{\sigma}), \mathsf{c}(2,\boldsymbol{\sigma})\} + u^2\right) \\
&= \mathbb{E}_{\mathbb{P}_{\boldsymbol{\sigma}}}\left((\max\{\mathsf{c}(1,\boldsymbol{\sigma}), \mathsf{c}(2,\boldsymbol{\sigma})\})^2\right) - 2u\mathbb{E}_{\mathbb{P}_{\boldsymbol{\sigma}}}\left(\max\{\mathsf{c}(1,\boldsymbol{\sigma}), \mathsf{c}(2,\boldsymbol{\sigma})\}\right) + u^2 \\
&= \mathsf{QMSC}(\boldsymbol{\sigma}) - 2u\mathsf{MSC}(\boldsymbol{\sigma}) + u^2.
\end{aligned}$$

Hence it follows:

Lemma 3.2 $\mathsf{QMSC}(\widehat{\boldsymbol{\sigma}}) = \mathsf{QMSC}(\boldsymbol{\sigma}) - 2u\mathsf{MSC}(\boldsymbol{\sigma}) + u^2$.

We continue to compare the Quadratic Social Costs of $\boldsymbol{\phi}$ and $\widehat{\boldsymbol{\phi}}$. Lemma 3.1 implies that

$$\begin{aligned}
&\mathsf{QMSC}(\boldsymbol{\phi}) - \mathsf{QMSC}(\widehat{\boldsymbol{\phi}}) \\
= \;& \frac{n}{4} + \frac{n^2}{4} + \frac{n^2}{2^n}\binom{n-1}{\left\lceil \frac{n}{2} \right\rceil - 1} - \frac{n-2u}{4} - \frac{(n-2u)^2}{4} - \frac{(n-2u)^2}{2^{n-2u}}\binom{n-2u-1}{\left\lceil \frac{n-2u}{2} \right\rceil - 1} \\
= \;& -u\left(u - n - \frac{1}{2}\right) + \frac{n^2}{2^n}\binom{n-1}{\left\lceil \frac{n}{2} \right\rceil - 1} - \frac{(n-2u)^2}{2^{n-2u}}\binom{n-2u-1}{\left\lceil \frac{n-2u}{2} \right\rceil - 1} \\
= \;& -\mathsf{QMSC}(\widehat{\boldsymbol{\sigma}}) + \mathsf{QMSC}(\boldsymbol{\sigma}) - 2u\mathsf{MSC}(\boldsymbol{\sigma}) + u\left(n + \frac{1}{2}\right) + \frac{n^2}{2^n}\binom{n-1}{\left\lceil \frac{n}{2} \right\rceil - 1} - \frac{(n-2u)^2}{2^{n-2u}}\binom{n-2u-1}{\left\lceil \frac{n-2u}{2} \right\rceil - 1}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\mathsf{QMSC}(\boldsymbol{\phi}) - \mathsf{QMSC}(\boldsymbol{\sigma}) - (\mathsf{QMSC}(\widehat{\boldsymbol{\phi}}) - \mathsf{QMSC}(\widehat{\boldsymbol{\sigma}})) \\
= \;& -2u\mathsf{MSC}(\boldsymbol{\sigma}) + u\left(n + \frac{1}{2}\right) + \frac{n^2}{2^n}\binom{n-1}{\left\lceil \frac{n}{2} \right\rceil - 1} - \frac{(n-2u)^2}{2^{n-2u}}\binom{n-2u-1}{\left\lceil \frac{n-2u}{2} \right\rceil - 1} \\
\geq \;& -2u\mathsf{MSC}(\boldsymbol{\phi}) + u\left(n + \frac{1}{2}\right) + \frac{n^2}{2^n}\binom{n-1}{\left\lceil \frac{n}{2} \right\rceil - 1} - \frac{(n-2u)^2}{2^{n-2u}}\binom{n-2u-1}{\left\lceil \frac{n-2u}{2} \right\rceil - 1} \\
= \;& -2u\left(\frac{n}{2} + \frac{n}{2^n}\binom{n-1}{\left\lceil \frac{n}{2} \right\rceil - 1}\right) + u\left(n + \frac{1}{2}\right) + \frac{n^2}{2^n}\binom{n-1}{\left\lceil \frac{n}{2} \right\rceil - 1} - \frac{(n-2u)^2}{2^{n-2u}}\binom{n-2u-1}{\left\lceil \frac{n-2u}{2} \right\rceil - 1} \\
= \;& \frac{u}{2} - 2u\frac{n}{2^n}\binom{n-1}{\left\lceil \frac{n}{2} \right\rceil - 1} + \frac{n^2}{2^n}\binom{n-1}{\left\lceil \frac{n}{2} \right\rceil - 1} - \frac{(n-2u)^2}{2^{n-2u}}\binom{n-2u-1}{\left\lceil \frac{n-2u}{2} \right\rceil - 1}.
\end{aligned}$$

We now prove a technical claim:

Lemma 3.3 *For all pairs of integers n and u such that $n \geq 2u$,*

$$-2u\frac{n}{2^n}\binom{n-1}{\left\lceil\frac{n}{2}\right\rceil-1}+\frac{n^2}{2^n}\binom{n-1}{\left\lceil\frac{n}{2}\right\rceil-1}-\frac{(n-2u)^2}{2^{n-2u}}\binom{n-2u-1}{\left\lceil\frac{n-2u}{2}\right\rceil-1} \quad \geq \quad 0.$$

Lemma 3.3 implies that (to prove that $\mathsf{QMSC}(\boldsymbol{\phi}) \geq \mathsf{QMSC}(\boldsymbol{\sigma})$) it suffices to prove that $\mathsf{QMSC}(\widehat{\boldsymbol{\phi}}) \geq \mathsf{QMSC}(\widehat{\boldsymbol{\sigma}})$. The rest of the paper is devoted to proving this inequality. For notational convenience, rename now the variables so that both $\widehat{\boldsymbol{\sigma}}$ and $\widehat{\boldsymbol{\phi}}$ henceforth refer to an instance with n users. All n users are fully mixed in $\widehat{\boldsymbol{\phi}}$; assume that in $\widehat{\boldsymbol{\sigma}}$, $r \geq 1$ (pure) users choose link 1 with probability 1 and $n-r$ (mixed) users choose both links with positive probability. Lücking *et al.* [17] proved:

Lemma 3.4 *For the Nash equilibrium $\widehat{\boldsymbol{\sigma}}$, for each mixed user $i \in [n]$, $\sigma_i(1) = \frac{1}{2} - \frac{r}{2(n-r-1)}$. Furthermore, $r \leq \left\lfloor \frac{n-3}{2} \right\rfloor$. (Henceforth, we shall denote, for each user $i \in [n]$, $p = \sigma_i(1)$ and $q = \sigma_i(2)$, where $p + q = 1$.)*

We now calculate $\mathsf{QMSC}(\widehat{\boldsymbol{\sigma}})$:

Lemma 3.5 $\mathsf{QMSC}(\widehat{\boldsymbol{\sigma}}) = \mathsf{Even}(n) \cdot \frac{n^2}{4}\binom{n-r}{\frac{n}{2}-r}p^{\frac{n}{2}-r}q^{\frac{n}{2}} + \sum_{i=\lfloor\frac{n}{2}\rfloor+1}^{n} i^2\binom{n-r}{i-r}p^{i-r}q^{n-i} + \sum_{i=\lfloor\frac{n}{2}\rfloor+1}^{n-r} i^2\binom{n-r}{i}p^{n-r-i}q^{i}$.

The next technical claim expresses $\mathsf{QMSC}(\widehat{\boldsymbol{\sigma}})$ in a different way by adding and subtracting terms.

Lemma 3.6 $\mathsf{QMSC}(\widehat{\boldsymbol{\sigma}}) = A + B - C + \mathsf{Even}(n) \cdot \frac{n^2}{4}\binom{n-r}{\frac{n}{2}-r}p^{\frac{n}{2}-r}q^{\frac{n}{2}}$, where :

$$\begin{aligned}
A &= \sum_{i=\lceil\frac{n+1}{2}\rceil}^{n}(i-r)\binom{n-r}{i-r}p^{i-r-1}q^{n+1-i} + \sum_{i=\lceil\frac{n+1}{2}\rceil}^{n-r}(n-r)\binom{n-r-1}{i-1}p^{n-r-i}q^{i}\\
B &= \sum_{i=\lceil\frac{n+1}{2}\rceil}^{n}(n-r)(i-r-1)\binom{n-r-1}{i-r-1}p^{i-r-2}q^{n+2-i} + \sum_{i=\lceil\frac{n+1}{2}\rceil}^{n-r}(n-r)(n-r-1)\binom{n-r-2}{i-2}p^{n-r-i}q^{i}\\
C &= \sum_{i=\lceil\frac{n+1}{2}\rceil}^{n}\binom{n-r}{i-r}\Big((i-r)p^{i-r-1}q^{n+1-i} + (i-r)^2p^{i-r-2}q^{n+2-i} - (i-r)p^{i-r-2}q^{n+2-i} - i^2p^{i-r}q^{n-i}\Big).
\end{aligned}$$

We calculate that

$$A = q(n-r) + \mathsf{Odd}(n)\cdot q(n-r)\binom{n-r-1}{\frac{n-1}{2}-r}p^{\frac{n-1}{2}-r}q^{\frac{n-1}{2}},$$

$$B \quad = \quad q^2(n-r)(n-r-1)\left(1+\binom{n-r-2}{\left\lceil\frac{n-2}{2}\right\rceil-r}p^{\lceil\frac{n-2}{2}\rceil-r}q^{\lfloor\frac{n-2}{2}\rfloor} + \mathsf{Odd}(n)\cdot\binom{n-r-2}{\frac{n-3}{2}-r}p^{\frac{n-3}{2}-r}q^{\frac{n-1}{2}}\right)$$

and

$$\begin{aligned}
C &= (n-r)\Bigg(\big((pq-p^2)+q(q^2-p^2)\big)\sum_{i=\lceil\frac{n+3}{2}\rceil-r}^{n-r}\binom{n-r-2}{i-2}p^{i-2}q^{n-r-i}+\\
&\qquad (q^2-p^2)(n-r-1)\sum_{i=\lceil\frac{n+1}{2}\rceil-r}^{n-r}\binom{n-r-2}{i-2}p^{i-2}q^{n-r-i} + (pq-p^2)\binom{n-r-2}{\left\lceil\frac{n-3}{2}\right\rceil-r}p^{\lceil\frac{n-3}{2}\rceil-r}q^{\lfloor\frac{n-1}{2}\rfloor}\Bigg)\\
&> (n-r)\Bigg((q^2-p^2)(n-r-1)\sum_{i=\lceil\frac{n+1}{2}\rceil-r}^{n-r}\binom{n-r-2}{i-2}p^{i-2}q^{n-r-i} + (pq-p^2)\binom{n-r-2}{\lceil\frac{n-3}{2}\rceil-r}p^{\lceil\frac{n-3}{2}\rceil-r}q^{\lfloor\frac{n-1}{2}\rfloor}\Bigg).
\end{aligned}$$

4 The FMNE Conjecture is Valid

The proof will use some estimations and technical claims which have been deferred to Sections 5 and 6, respectively. We establish:

Theorem 4.1 *For the fully mixed Nash equilibrium* $\widehat{\boldsymbol{\phi}}$ *and the Nash equilibrium* $\widehat{\boldsymbol{\sigma}}$*,* $\mathsf{QMSC}(\widehat{\boldsymbol{\phi}}) \geq \mathsf{QMSC}(\widehat{\boldsymbol{\sigma}})$*.*

Proof: Assume that $n \geq 134$. (For smaller n, the claim is verified directly.) Lemmas 3.1 and 3.6 imply that

$$\begin{aligned}
&\mathsf{QMSC}(\widehat{\boldsymbol{\phi}}) - \mathsf{QMSC}(\widehat{\boldsymbol{\sigma}}) \\
= \ & \frac{n}{4} + \frac{n^2}{4} + \frac{n^2}{2^n}\binom{n-1}{\left\lceil \frac{n}{2} \right\rceil - 1} - q(n-r) - q^2(n-r)(n-r-1) + (q^2-p^2)(n-r)(n-r-1)\mathsf{Q} + \mathsf{D},
\end{aligned}$$

where

$$\begin{aligned}
\mathsf{D} \ = \ & -q^2(n-r)(n-r-1)\binom{n-r-2}{\left\lceil \frac{n-2}{2} \right\rceil - r} p^{\lceil \frac{n-2}{2} \rceil - r} q^{\lfloor \frac{n-2}{2} \rfloor} + (pq-p^2)(n-r)\binom{n-r-2}{\left\lceil \frac{n-3}{2} \right\rceil - r} p^{\lceil \frac{n-3}{2} \rceil - r} q^{\lfloor \frac{n-1}{2} \rfloor} \\
& -q(n-r)\mathsf{Odd}(n) \cdot \left(\binom{n-r-1}{\frac{n-1}{2} - r} p^{\frac{n-1}{2} - r} q^{\frac{n-1}{2}} + q(n-r-1)\binom{n-r-2}{\frac{n-3}{2} - r} p^{\frac{n-3}{2} - r} q^{\frac{n-1}{2}} \right) - \\
& \mathsf{Even}(n) \cdot \frac{n^2}{4}\binom{n-r}{\frac{n}{2} - r} p^{\frac{n}{2} - r} q^{\frac{n}{2}} \\
\mathsf{Q} \ = \ & \sum_{i=\lceil \frac{n+1}{2} \rceil - r}^{n-r} \binom{n-r-2}{i-2} p^{i-2} q^{n-r-i} \ = \ \sum_{i=\lceil \frac{n-3}{2} \rceil - r}^{n-r} \binom{n-r-2}{i} p^{i} q^{n-r-2-i} \ = \ 1 - \mathsf{B}_{n-r-2, \lceil \frac{n-3}{2} \rceil - r - 1}(p).
\end{aligned}$$

Note that Lemma 2.6 implies *lower* bounds on Q for various values of r, which will be used in some later proofs. We proceed by case analysis. We consider separately the two cases where n is even or odd.

Case 1: n is even By substituting p and q from Lemma 3.4, we get that

$$\begin{aligned}
\mathsf{D} \ = \ & -\frac{(n-1)^2(n-r)}{4(n-r-1)}\binom{n-r-2}{\frac{n-2}{2} - r} p^{\frac{n-2}{2} - r} q^{\frac{n-2}{2}} + \\
& \frac{r(n-r)(n-2r-1)}{2(n-r-1)^2}\binom{n-r-2}{\frac{n-2}{2} - r} p^{\frac{n-2}{2} - r} q^{\frac{n-2}{2}} - \frac{n^2}{4}\binom{n-r}{\frac{n}{2} - r} p^{\frac{n}{2} - r} q^{\frac{n}{2}} \\
= \ & \binom{n-r-2}{\frac{n-2}{2} - r} p^{\frac{n-2}{2} - r} q^{\frac{n-2}{2}} \left(-\frac{(n-1)^2(n-r)}{4(n-r-1)} + \frac{r(n-r)(n-2r-1)}{2(n-r-1)^2} - pq\frac{n(n-r)(n-r-1)}{n-2r} \right) \\
\geq \ & \binom{n-r-2}{\frac{n-2}{2} - r} p^{\frac{n-2}{2} - r} q^{\frac{n-2}{2}} \left(-\frac{(n-1)^2(n-r)}{4(n-r-1)} - \frac{n(n-1)(n-2r-1)(n-r)}{4(n-r-1)(n-2r)} \right)
\end{aligned}$$

It follows that

$$\begin{aligned}
& \mathsf{QMSC}(\widehat{\boldsymbol{\phi}}) - \mathsf{QMSC}(\widehat{\boldsymbol{\sigma}}) \\
\geq \ & \frac{n}{4} + \frac{n^2}{4} + \frac{n^2}{2^{n+1}}\binom{n}{\frac{n}{2}} - \frac{(n-1)(n-r)}{2(n-r-1)} - \frac{(n-1)^2(n-r)}{4(n-r-1)} + r(n-r)\mathsf{Q} \\
& -\binom{n-r-2}{\frac{n-2}{2} - r} p^{\frac{n-2}{2} - r} q^{\frac{n-2}{2}} \left(\frac{(n-1)^2(n-r)}{4(n-r-1)} + \frac{n(n-1)(n-2r-1)(n-r)}{4(n-r-1)(n-2r)} \right) \\
= \ & \underbrace{\frac{n^2}{2^{n+1}}\binom{n}{\frac{n}{2}} + r(n-r)\mathsf{Q} - \binom{n-r-2}{\frac{n-2}{2} - r} p^{\frac{n-2}{2} - r} q^{\frac{n-2}{2}} \left(\frac{(n-1)^2(n-r)}{4(n-r-1)} + \frac{n(n-1)(n-2r-1)(n-r)}{4(n-r-1)(n-2r)} \right)}_{\mathsf{G}} \\
& -\frac{r(n+1)}{4(n-r-1)} \\
> \ & \frac{n^2}{2^{n+1}}\binom{n}{\frac{n}{2}} - \binom{n-r-2}{\frac{n-2}{2} - r} p^{\frac{n-2}{2} - r} q^{\frac{n-2}{2}} \left(\frac{(n-1)^2(n-r)}{4(n-r-1)} + \frac{n(n-1)(n-r)}{4(n-r-1)} \right) + r(n-r)\mathsf{Q} \\
& -\frac{r(n+1)}{4(n-r-1)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{n^2}{2^{n+1}}\binom{n}{\frac{n}{2}} - \binom{n-r-2}{\frac{n-2}{2}-r} p^{\frac{n-2}{2}-r} q^{\frac{n-2}{2}} \left(\frac{(n-1)(2n-1)(n-r)}{4(n-r-1)}\right) + r(n-r)\mathsf{Q} - \frac{r(n+1)}{4(n-r-1)} \\
&> \frac{n^2}{2^{n+1}}\binom{n}{\frac{n}{2}} - \binom{n-r-2}{\frac{n-2}{2}-r} p^{\frac{n-2}{2}-r} q^{\frac{n-2}{2}} \left(\frac{n(n-1)(n-r)}{2(n-r-1)}\right) + r(n-r)\mathsf{Q} - \frac{r(n+1)}{4(n-r-1)} \\
&\geq \frac{n^2}{2^{n+1}}\binom{n}{\frac{n}{2}} - \binom{n-r-2}{\frac{n-2}{2}-r} p^{\frac{n-2}{2}-r} q^{\frac{n-2}{2}} \left(\frac{n^2(n-r)}{2(n-r-2)}\right) + r(n-r)\mathsf{Q} - \frac{r(n+1)}{4(n-r-1)}.
\end{aligned}
$$

We proceed by case analysis on the range of values of r. For each range, we shall use the corresponding case(s) of Lemma 2.6 to infer a lower bound on Q.

$\boxed{1. \quad 1 \leq r \leq \left\lfloor \frac{n-3}{2} \right\rfloor - 4:}$ Note that in this case, Lemma 2.6 (Case (1)) implies that $\mathsf{Q} \geq \frac{1}{2}$. Hence, substituting p and q from Lemma 3.4, we get that

$$
\begin{aligned}
\mathsf{G} &\geq \frac{n^2}{2^{n+1}}\binom{n}{\frac{n}{2}} - \frac{n^2(n-r)}{2(n-r-2)}\binom{n-r-2}{\frac{n-2}{2}-r}\left(\frac{n-2r-1}{2(n-r-1)}\right)^{\frac{n-2r-2}{2}}\left(\frac{n-1}{2(n-r-1)}\right)^{\frac{n-2}{2}} + \frac{r(n-r)}{2} \\
&= \frac{n^2}{2^{n+1}}\binom{n}{\frac{n}{2}}\left(1 - \frac{n}{n-r-2}\frac{\prod_{i=0}^{r}(n-2r+2i)}{\prod_{i=1}^{r}(n-r+i)}\frac{(n-1)(n-2r-1)^{1-r}}{(n-r-1)^{3-r}}\right. \\
&\qquad \left.\left(\frac{n-2r-1}{n-r-1}\right)^{\frac{n-4}{2}}\left(\frac{n-1}{n-r-1}\right)^{\frac{n-4}{2}}\right) + \frac{r(n-r)}{2}.
\end{aligned}
$$

We consider two different subcases:

$\boxed{1.1. \quad 1 \leq r \leq 4:}$ We use Lemma 2.2 and the estimation in Lemma 5.1. Clearly,

$$
\begin{aligned}
\mathsf{G} &\overset{(2.2),(5.1)}{\geq} n\sqrt{\frac{n}{6}}\left(1 - \frac{n}{n-r-2}\frac{\prod_{i=0}^{r}(n-2r+2i)}{\prod_{i=1}^{r}(n-r+i)}\frac{(n-1)(n-2r-1)^{1-r}}{(n-r-1)^{3-r}}\left(\frac{n-2r-1}{n-r-1}\right)^{\frac{n-4}{2}}\right. \\
&\qquad \left.\left(\frac{n-1}{n-r-1}\right)^{\frac{n-4}{2}}\right) + \frac{r(n-r)}{2} \\
&\overset{(5.1)}{\geq} \frac{r(n+1)}{4(n-r-1)},
\end{aligned}
$$

and the claim follows.

$\boxed{1.2. \quad 5 \leq r \leq \left\lfloor \frac{n-3}{2} \right\rfloor - 4:}$ We use the estimation in Lemma 5.2 and Lemmas 6.1 and 6.2. Clearly,

$$
\begin{aligned}
\mathsf{G} &\overset{(2.2),(5.2)}{\geq} n\sqrt{\frac{n}{2\pi}}\exp\left(\frac{1}{12n+1} - \frac{1}{3n}\right) - \frac{n^2}{2}\frac{1}{\sqrt{2\pi}}\frac{n-r}{\sqrt{(n-r-2)(\frac{n}{2}-1)(\frac{n}{2}-r-1)}} \\
&\qquad \exp\left(\frac{1}{12(n-r-2)} - \frac{1}{6n-11} - \frac{1}{6n-12r-11}\right) + \frac{r(n-r)}{2} \\
&= n\sqrt{\frac{n}{2\pi}}\exp\left(\frac{1}{12n+1} - \frac{1}{3n}\right)\left(1 - \sqrt{\frac{n(n-r)^2}{(n-2)(n-r-2)(n-2r-2)}}\right. \\
&\qquad \left.\exp\left(\frac{1}{12(n-r-2)} - \frac{1}{6n-11} - \frac{1}{6n-12r-11} - \frac{1}{12n+1} + \frac{1}{3n}\right)\right) + \frac{r(n-r)}{2} \\
&\overset{(6.1)}{\geq} n\sqrt{\frac{n}{2\pi}}\exp\left(\frac{1}{12n+1} - \frac{1}{3n}\right)\left(1 - \sqrt{\frac{n(n-r)^2}{(n-2)(n-r-2)(n-2r-2)}}\right) + \frac{r(n-r)}{2} \\
&\geq n\sqrt{\frac{n}{6}}\left(1 - \sqrt{\frac{n(n-r)^2}{(n-2)(n-r-2)(n-2r-2)}}\right) + \frac{r(n-r)}{2} \\
&\overset{r \leq \lfloor \frac{n-3}{2} \rfloor - 4}{\geq} n\sqrt{\frac{n}{6}}\left(1 - \sqrt{\frac{n(n-r)^2}{(n-2)(n-r-2)(n-2r-2)}}\right) + \frac{r(n+1)}{4} \\
&\overset{(6.2)}{\geq} \frac{r(n+1)}{4(n-r-1)}
\end{aligned}
$$

and the claim follows.

2. $\left\lfloor \frac{n-3}{2} \right\rfloor - 3 \le r \le \left\lfloor \frac{n-3}{2} \right\rfloor$: We shall use the estimation in Lemma 5.3 (Case (1)). (Note that this way, we are implicitly using corresponding cases of Lemma 2.6 to get lower bounds on Q since the proof of Lemma 5.3 uses such lower bounds from Lemma 2.6.) By substituting p and q from Lemma 3.4, we get that

$$\begin{aligned}
&\mathsf{QMSC}(\widehat{\boldsymbol{\phi}}) - \mathsf{QMSC}(\widehat{\boldsymbol{\sigma}}) \\
\ge\ & \frac{n}{4} + \frac{n^2}{4} - \frac{(n-1)(n-r)}{2(n-r-1)} - \frac{(n-1)^2(n-r)}{4(n-r-1)} + r(n-r)\mathsf{Q} - \binom{n-r-2}{\frac{n-2}{2}-r} \cdot \\
& \left(\frac{n-2r-1}{2(n-r-1)}\right)^{\frac{n-2}{2}-r} \left(\frac{n-1}{2(n-r-1)}\right)^{\frac{n-2}{2}} \left(\frac{(n-1)^2(n-r)}{4(n-r-1)} + \frac{n(n-1)(n-2r-1)(n-r)}{4(n-r-1)(n-2r)}\right) \\
=\ & \frac{n}{4} + \frac{n^2}{4} - (n-r)\left(\frac{(n-1)(n+1)}{4(n-r-1)} - r\mathsf{Q}\right. \\
& \left. + \frac{\prod_{i=0}^{\frac{n-2r-4}{2}}\left(\frac{n}{2}+i\right)}{\left(\frac{n-2}{2}-r\right)!} \left(\frac{n-2r-1}{2(n-r-1)}\right)^{\frac{n-2}{2}-r} \left(\frac{n-1}{2(n-r-1)}\right)^{\frac{n}{2}} \left(\frac{n^2-n-2nr+r}{n-2r}\right)\right) \\
\ge\ & \frac{n}{4} + \frac{n^2}{4} - \frac{n-r}{2}\left(\frac{(n-1)(n+1)}{2(n-r-1)} - 2r\mathsf{Q} + \frac{n(n-2r-1)^{\frac{n-2}{2}-r}(n^2-n-2nr+r)}{2^{\frac{n-2r-4}{2}}\left(\frac{n-2}{2}-r\right)!2(n-r-1)(n-2r)} \left(\frac{n-1}{2(n-r-1)}\right)^{\frac{n}{2}}\right) \\
\ge\ & -\frac{n-r}{2}\left(\frac{n^2}{2(n-r-1)} - \frac{n^2}{2(n-r)} - 2r\mathsf{Q} + \frac{n(n-2r-1)^{\frac{n-2}{2}-r}(n^2-n-2nr+r)}{2^{\frac{n-2r-4}{2}}\left(\frac{n-2}{2}-r\right)!(n-2r)} \left(\frac{n-1}{2(n-r-1)}\right)^{\frac{n}{2}}\right) \\
\overset{(5.3)}{\ge}\ & 0,
\end{aligned}$$

and the claim follows. This completes the proof for even n.

Case 2: n is odd By substituting p and q from Lemma 3.4, we get that

$$\begin{aligned}
\mathsf{D} =\ & -(n-r)(n-r-1)\binom{n-r-2}{\frac{n-1}{2}-r}p^{\frac{n-1}{2}-r}q^{\frac{n+1}{2}} + \frac{r(n-r)(n-2r-1)}{2(n-r-1)^2}\binom{n-r-2}{\frac{n-3}{2}-r}p^{\frac{n-3}{2}-r}q^{\frac{n-1}{2}} \\
& -(n-r)\binom{n-r-1}{\frac{n-1}{2}-r}p^{\frac{n-1}{2}-r}q^{\frac{n+1}{2}} - \frac{(n-1)^2(n-r)}{4(n-r-1)}\binom{n-r-2}{\frac{n-3}{2}-r}p^{\frac{n-3}{2}-r}q^{\frac{n-1}{2}} \\
=\ & \left(-(n-r)(n-r-1) - (n-r)\frac{n-r-1}{n-r-1-(\frac{n-1}{2}-r)}\right)\binom{n-r-2}{\frac{n-1}{2}-r}p^{\frac{n-1}{2}-r}q^{\frac{n+1}{2}} \\
& + \left(\frac{r(n-r)(n-2r-1)}{2(n-r-1)^2} - \frac{(n-1)^2(n-r)}{4(n-r-1)}\right)\binom{n-r-2}{\frac{n-3}{2}-r}p^{\frac{n-3}{2}-r}q^{\frac{n-1}{2}} \\
=\ & -\frac{(n-r)(n-r-1)(n+1)}{n-1}\binom{n-r-2}{\frac{n-1}{2}-r}p^{\frac{n-1}{2}-r}q^{\frac{n+1}{2}} \\
& + \left(\frac{2r(n-r)}{n-1} - \frac{(n-1)(n-r)(n-r-1)}{n-2r-1}\right)\binom{n-r-2}{\frac{n-3}{2}-r}p^{\frac{n-1}{2}-r}q^{\frac{n+1}{2}} \\
\ge\ & -(n-r)(n-r-1)(n+1)p^{\frac{n-1}{2}-r}q^{\frac{n+1}{2}}\left(\frac{1}{n-1}\binom{n-r-2}{\frac{n-1}{2}-r} + \frac{1}{n-2r-1}\binom{n-r-2}{\frac{n-3}{2}-r}\right) \\
=\ & -(n-r)(n+1)p^{\frac{n-1}{2}-r}q^{\frac{n+1}{2}}\left(\binom{n-r-2}{\frac{n-1}{2}-r} + \binom{n-r-2}{\frac{n-3}{2}-r}\right) \\
=\ & -(n-r)(n+1)\binom{n-r-1}{\frac{n-1}{2}-r}p^{\frac{n-1}{2}-r}q^{\frac{n+1}{2}}.
\end{aligned}$$

It follows that

$$
\begin{aligned}
&\mathsf{QMSC}(\widehat{\boldsymbol{\phi}}) - \mathsf{QMSC}(\widehat{\boldsymbol{\sigma}}) \\
\geq\ & \frac{n}{4} + \frac{n^2}{4} + \frac{n(n+1)}{2^{n+1}}\binom{n}{\frac{n+1}{2}} - \frac{(n-1)(n-r)}{2(n-r-1)} - \frac{(n-1)^2(n-r)}{4(n-r-1)} + r(n-r)\mathsf{Q} \\
& -(n-r)(n+1)\binom{n-r-1}{\frac{n-1}{2}-r} p^{\frac{n-1}{2}-r} q^{\frac{n+1}{2}} \\
=\ & \underbrace{\frac{n(n+1)}{2^{n+1}}\binom{n}{\frac{n+1}{2}} - (n-r)(n+1)\binom{n-r-1}{\frac{n-1}{2}-r} p^{\frac{n-1}{2}-r} q^{\frac{n+1}{2}} + r(n-r)\mathsf{Q}}_{\mathsf{H}} - \frac{r(n+1)}{4(n-r-1)}.
\end{aligned}
$$

We proceed by case analysis on the range of values of r. For each range, we shall use the corresponding case(s) of Lemma 2.6 to infer a lower bound on Q.

1. $1 \leq r \leq \left\lfloor \frac{n-3}{2} \right\rfloor - 4$: Note that Lemma 2.6 (Case (1)) implies that $\mathsf{Q} \geq \frac{1}{2}$. There are two subcases:

1.1. $1 \leq r \leq 3$: We use Lemma 2.3 and the estimation in Lemma 5.4. By substituting p and q from Lemma 3.4, we get that

$$
\begin{aligned}
\mathsf{H} \quad \geq\ & \frac{n(n+1)}{2^{n+1}}\binom{n}{\frac{n+1}{2}} - (n-r)(n+1)\binom{n-r-1}{\frac{n-1}{2}-r}\left(\frac{n-2r-1}{2(n-r-1)}\right)^{\frac{n-2r-1}{2}}\left(\frac{n-1}{2(n-r-1)}\right)^{\frac{n+1}{2}} + \\
& \frac{r(n-r)}{2} \\
\geq\ & \frac{(n-1)(n+1)}{2^{n+1}}\binom{n}{\frac{n+1}{2}} - (n-1)(n+1)\frac{\prod_{i=0}^{r}(\frac{n-2r+1+2i}{2})}{\prod_{i=0}^{r}(n-r+i)}\binom{n}{\frac{n+1}{2}}\left(\frac{n-2r-1}{2(n-r-1)}\right)^{\frac{n-2r-1}{2}} \\
& \left(\frac{n-1}{2(n-r-1)}\right)^{\frac{n+1}{2}} + \frac{r(n-r)}{2} \\
=\ & \frac{n-1}{2^n}\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2}\left(1 - \frac{\prod_{i=0}^{r}(n-2r+1+2i)}{\prod_{i=0}^{r}(n-r+i)}\frac{(n-1)(n-2r-1)^{-r}}{(n-r-1)^{1-r}}\left(\frac{n-2r-1}{n-r-1}\right)^{\frac{n-1}{2}}\right. \\
& \left.\left(\frac{n-1}{n-r-1}\right)^{\frac{n-1}{2}}\right) + \frac{r(n-r)}{2} \\
\overset{(2.3),(5.4)}{\geq}\ & (n-1)\left(\frac{n}{n-1}\right)^n\sqrt{\frac{n}{6}}\left(1 - \frac{\prod_{i=0}^{r}(n-2r+1+2i)}{\prod_{i=0}^{r}(n-r+i)}\frac{(n-1)(n-r-1)^{r-1}}{(n-2r-1)^r}\right. \\
& \left.\left(\frac{n-2r-1}{n-r-1}\right)^{\frac{n-1}{2}}\left(\frac{n-1}{n-r-1}\right)^{\frac{n-1}{2}}\right) + \frac{n-1}{2} \\
\overset{(5.4)}{\geq}\ & \frac{r(n+1)}{4(n-r-1)},
\end{aligned}
$$

and the claim follows.

1.2. $4 \leq r \leq \left\lfloor \frac{n-3}{2} \right\rfloor - 4$: We use the estimation in Lemma 5.5 and Lemmas 6.3 and 6.4. We get that

$$
\begin{aligned}
\mathsf{H} \overset{(2.3),(5.5)}{\geq}\ & n\sqrt{\frac{n}{2\pi}}\exp\left(\frac{1}{12n+1} - \frac{1}{3n-3}\right) - (n-r)(n+1)\frac{1}{\sqrt{2\pi}}\frac{n-1}{\sqrt{(n-1)(n-r-1)(n-2r-1)}} \\
& \exp\left(\frac{1}{12(n-r-1)} - \frac{1}{6n-5} - \frac{1}{6n-12r-5}\right) + \frac{r(n-r)}{2} \\
\geq\ & n\sqrt{\frac{n}{2\pi}}\exp\left(\frac{1}{12n+1} - \frac{1}{3n-3}\right)\cdot\left(1 - \sqrt{\frac{(n+1)^2(n-r)^2}{n(n-1)(n-r-1)(n-2r-1)}}\right. \\
& \left.\exp\left(\frac{1}{12(n-r-1)} - \frac{1}{6n-5} - \frac{1}{6n-12r-5} - \frac{1}{12n+1} + \frac{1}{3n-3}\right)\right) + \frac{r(n-r)}{2}
\end{aligned}
$$

$$\stackrel{(6.3)}{\geq} \quad n\sqrt{\frac{n}{2\pi}}\exp\left(\frac{1}{12n+1}-\frac{1}{3n-3}\right)\cdot\left(1-\sqrt{\frac{(n+1)^2(n-r)^2}{n(n-1)(n-r-1)(n-2r-1)}}\right)+\frac{r(n-r)}{2}$$

$$\geq \quad n\sqrt{\frac{n}{6}}\left(1-\sqrt{\frac{(n+1)^2(n-r)^2}{n(n-1)(n-r-1)(n-2r-1)}}\right)+\frac{r(n-r)}{2}$$

$$\stackrel{r\leq\lfloor\frac{n-3}{2}\rfloor-4}{\geq} \quad n\sqrt{\frac{n}{6}}\left(1-\sqrt{\frac{(n+1)^2(n-r)^2}{n(n-1)(n-r-1)(n-2r-1)}}\right)+\frac{r(n+1)}{4}$$

$$\stackrel{(6.4)}{\geq} \quad \frac{r(n+1)}{4(n-r-1)},$$

and the claim follows.

2. $\left\lfloor\frac{n-3}{2}\right\rfloor-3\leq r\leq\left\lfloor\frac{n-3}{2}\right\rfloor$: We shall use the estimation in Lemma 5.3 (Case (2)). (Note that this way, we are implicitly using corresponding cases of Lemma 2.6 to get lower bounds on Q, since the proof of Lemma 5.3 uses such lower bounds from Lemma 2.6.) By substituting p and q from Lemma 3.4, we get that

$$\begin{aligned}
&\mathsf{QMSC}(\widehat{\boldsymbol{\phi}})-\mathsf{QMSC}(\widehat{\boldsymbol{\sigma}})\\
\geq\quad & \frac{n(n+1)}{2^{n+1}}\binom{n}{\frac{n+1}{2}}-\frac{(n-1)(n-r)}{2(n-r-1)}-\frac{(n-1)^2(n-r)}{4(n-r-1)}+r(n-r)\mathsf{Q}-(n-r)(n+1)\binom{n-r-1}{\frac{n-1}{2}-r}p^{\frac{n-1}{2}-r}q^{\frac{n+1}{2}}\\
\geq\quad & -(n-r)\left(\frac{(n-1)(n+1)}{4(n-r-1)}-r\mathsf{Q}+(n+1)\binom{n-r-1}{\frac{n-1}{2}-r}\left(\frac{n-2r-1}{2(n-r-1)}\right)^{\frac{n-1}{2}-r}\left(\frac{n-1}{2(n-r-1)}\right)^{\frac{n+1}{2}}\right)\\
=\quad & -\frac{n-r}{2}\left(\frac{(n-1)(n+1)}{2(n-r-1)}-2r\mathsf{Q}+2(n+1)\binom{n-r-1}{\frac{n-1}{2}-r}\left(\frac{n-2r-1}{2(n-r-1)}\right)^{\frac{n-1}{2}-r}\left(\frac{n-1}{2(n-r-1)}\right)^{\frac{n+1}{2}}\right)\\
\stackrel{(5.3)}{\geq}\quad & 0.
\end{aligned}$$

This completes the proof for odd n. ■

5 Estimations

In this section, we collect together all estimations which were used in the proof of Theorem 4.1. Some of these estimations refer to the probabilities p and q introduced in Lemma 3.4. Some other estimations refer to the quantity $\mathsf{Q}=\sum_{i=\lceil\frac{n+1}{2}\rceil-r}^{n-r}\binom{n-r-2}{i-2}p^{i-2}q^{n-r-i}$ introduced in the proof of Theorem 4.1.

Lemma 5.1 *For all integers n and r such that $1\leq r\leq 4$,*

(1) $$\frac{n}{n-r-2}\,\frac{\prod_{i=0}^{r}(n-2r+2i)}{\prod_{i=1}^{r}(n-r+i)}\,\frac{(n-2r-1)^{1-r}}{(n-r-1)^{2-r}}\left(\frac{n-2r-1}{n-r-1}\right)^{\frac{n-4}{2}}\left(\frac{n-1}{n-r-1}\right)^{\frac{n-2}{2}}\ \geq\ 1.$$

(2) $$n\sqrt{\frac{n}{6}}\left(1-\frac{n}{n-r-2}\,\frac{\prod_{i=0}^{r}(n-2r+2i)}{\prod_{i=1}^{r}(n-r+i)}\,\frac{(n-1)(n-2r-1)^{1-r}}{(n-r-1)^{3-r}}\left(\frac{n-2r-1}{n-r-1}\right)^{\frac{n-4}{2}}\left(\frac{n-1}{n-r-1}\right)^{\frac{n-4}{2}}\right)\geq \frac{r(n+1)}{4(n-r-1)}-\frac{r(n-r)}{2}.$$

Lemma 5.2 *For all integers n and r such that $5\leq r\leq\left\lfloor\frac{n-3}{2}\right\rfloor-4$,*

$$\binom{n-r-2}{\frac{n-2}{2}-r}p^{\frac{n-2}{2}-r}q^{\frac{n-2}{2}}\leq\sqrt{\frac{n-r-2}{2\pi(\frac{n}{2}-1)(\frac{n}{2}-r-1)}}\exp\left(\frac{1}{12(n-r-2)}-\frac{1}{6n-11}-\frac{1}{6n-12r-11}\right).$$

Lemma 5.3 *For all integers r such that $\left\lfloor\frac{n-3}{2}\right\rfloor-3\leq r\leq\left\lfloor\frac{n-3}{2}\right\rfloor$:*

(1) *For all even integers $n\geq 134$,*

$$\frac{n^2}{2(n-r-1)}-\frac{n^2}{2(n-r)}-2r\mathsf{Q}+\frac{n(n-2r-1)^{\frac{n-2}{2}-r}(n^2-n-2nr+r)}{2^{\frac{n-2r-4}{2}}\left(\frac{n-2}{2}-r\right)!(n-2r)}\left(\frac{n-1}{2(n-r-1)}\right)^{\frac{n}{2}}\ \leq\ 0.$$

(2) *For all odd integers* $n \geq 135$,
$$\frac{n}{4} + \frac{n^2}{4} - \frac{n-r}{2}\left(\frac{(n-1)(n+1)}{2(n-r-1)} - 2r\mathsf{Q} + 2(n+1)\binom{n-r-1}{\frac{n-1}{2}-r}\left(\frac{n-2r-1}{2(n-r-1)}\right)^{\frac{n-1}{2}-r}\left(\frac{n-1}{2(n-r-1)}\right)^{\frac{n+1}{2}}\right) \geq 0.$$

Lemma 5.4 *For all integers* n *and* r *such that* $1 \leq r \leq 4$,

(1) $$\frac{\prod_{i=0}^{r}(n-2r+1+2i)}{\prod_{i=0}^{r}(n-r+i)}\frac{(n-2r-1)^{-r}}{(n-r-1)^{-r}}\left(\frac{n-2r-1}{n-r-1}\right)^{\frac{n-1}{2}}\left(\frac{n-1}{n-r-1}\right)^{\frac{n+1}{2}} \geq 1.$$

(2) $$(n-1)\left(\frac{n}{n-1}\right)^{n}\sqrt{\frac{n}{6}}\left(1 - \frac{\prod_{i=0}^{r}(n-2r+1+2i)}{\prod_{i=0}^{r}(n-r+i)}\frac{(n-1)(n-r-1)^{r-1}}{(n-2r-1)^{r}}\left(\frac{n-2r-1}{n-r-1}\right)^{\frac{n-1}{2}}\left(\frac{n-1}{n-r-1}\right)^{\frac{n-1}{2}}\right) + \frac{n-1}{2} \geq \frac{r(n+1)}{4(n-r-1)}.$$

Lemma 5.5 *For all integers* n *and* r,

$$\binom{n-r-1}{\frac{n-1}{2}-r}p^{\frac{n-1}{2}-r}q^{\frac{n+1}{2}} \leq \frac{n-1}{\sqrt{2\pi(n-1)(n-r-1)(n-2r-1)}}\exp\left(\frac{1}{12(n-r-1)} - \frac{1}{6n-5} - \frac{1}{6n-12r-5}\right).$$

6 Technical Claims

In this section, we collect together some simple technical claims which were used in the proof of Theorem 4.1.

Lemma 6.1 *For all* $n \geq 1$ *and* $r > 0$, $\frac{1}{12(n-r-2)} - \frac{1}{6n-11} - \frac{1}{6n-12r-11} - \frac{1}{12n+1} + \frac{1}{3n} < 0.$

Lemma 6.2 *For all even integers* $n \geq 134$ *and integers* r *such that* $5 \leq r \leq \left\lfloor \frac{n-3}{2} \right\rfloor - 4$,

$$n\sqrt{\frac{n}{6}}\left(1 - \sqrt{\frac{n(n-r)^2}{(n-2)(n-r-2)(n-2r-2)}}\right) + \frac{r(n+1)}{4} \geq \frac{r(n+1)}{4(n-r-1)}.$$

Lemma 6.3 *For all* $n \geq 1$ *and* $r > 3$, $\frac{1}{12(n-r-1)} - \frac{1}{6n-5} - \frac{1}{6n-12r-5} - \frac{1}{12n+1} + \frac{1}{3n-3} < 0.$

Lemma 6.4 *For all odd integers* $n \geq 135$ *and integers* r *such that* $4 \leq r \leq \left\lfloor \frac{n-3}{2} \right\rfloor - 4$,

$$n\sqrt{\frac{n}{6}}\left(1 - \sqrt{\frac{(n+1)^2(n-r)^2}{n(n-1)(n-r-1)(n-2r-1)}}\right) + \frac{r(n+1)}{4} \geq \frac{r(n+1)}{4(n-r-1)}.$$

7 Conclusions

We have presented an extensive proof for the validity of the FMNE *Conjecture* for a special case of the selfish routing model of Koutsoupias and Papadimitriou [15] where users are unweighted and there are only two identical (related) links. We adopted a new, well-motivated kind of Social Cost, called Quadratic Maximum Social Cost. To carry out the proof, we developed some new estimations of (generalized) medians of the binomial distribution, which are of independent interest and value. In turn, those estimations were used as tools, together with a variety of combinatorial arguments and other analytical estimations, in the main proof.

We believe that our work contributes significantly, both conceptually and technically, to enriching our knowledge about the many facets of the FMNE *Conjecture*. Based on this improved understanding, we extend the FMNE *Conjecture* formulated and proven in this work to an *Extended* FMNE *Conjecture* for the more general case with an arbitrary number of unweighted users, an arbitrary number of identical (related) links and Social Cost as the expectation of a polynomial with non-negative coefficients of the maximum congestion on a link. Settling this Extended FMNE *Conjecture* remains a major challenge.

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References

[1] S. Bernstein, "Démonstration du Théoreme de Weierstrass Fondée sur le Calcul des Probabilities," *Commun. Soc. Math. Kharkow.*, Vol. 2, No. 13, pp. 1–2, 1912/13.

[2] R. Elsässer, M. Gairing, T. Lücking, M. Mavronicolas and B. Monien, "A Simple Graph-Theoretic Model for Selfish Restricted Scheduling," *Proceedings of the 1st International Workshop on Internet and Network Economics,* pp. 195–209, Vol. 3828, LNCS, Springer-Verlag, 2005.

[3] W. Feller, *An Introduction to Probability Theory and its Applications*, Third Edition, Wiley, 1968.

[4] A. Ferrante and M. Parente, "Existence of Nash Equilibria in Selfish Routing Problems," *Proceedings of the 11th International Colloquium on Structural Information and Communication Complexity,* pp. 149–160, LNCS, Springer-Verlag, 2004.

[5] S. Fischer and B. Vöcking, "On the Structure and Complexity of Worst-Case Equilibria," *Theoretical Computer Science,* Vol. 378, No. 2, pp. 165–174, 2007.

[6] D. Fotakis, S. Kontogiannis, E. Koutsoupias, M. Mavronicolas and P. Spirakis, "The Structure and Complexity of Nash Equilibria for a Selfish Routing Game," *Proceedings of the 29th International Colloqium on Automata, Languages and Programming,* pp. 123–134, Vol. 2380, LNCS, Springer-Verlag, 2002.

[7] M. Gairing, T. Lücking, M. Mavronicolas, B. Monien and P. Spirakis, "Structure and Complexity of Extreme Nash Equilibria," *Theoretical Computer Science,* Vol. 343, Nos. 1–2, pp. 133–157, 2005.

[8] M. Gairing, T. Lücking, M. Mavronicolas and B. Monien, "The Price of Anarchy for Polynomial Social Cost," *Theoretical Computer Science,* Vol. 369, Nos. 1–3, pp. 116–135, 2006.

[9] M. Gairing, T. Lücking, M. Mavronicolas, B. Monien and M. Rode, "Nash Equilibria in Discrete Routing Games with Convex Latency Functions," *Proceedings of the 31st International Colloqium on Automata, Languages and Programming,* pp. 645–657, Vol. 3142, LNCS, Springer-Verlag, 2004.

[10] M. Gairing, B. Monien and K. Tiemann, "Selfish Routing with Incomplete Information," *Proceedings of the 17th Annual ACM Symposium on Parallelism in Algorithms and Architectures,* pp. 203–212, 2005.

[11] R. Göb, "Bounds for Median and 50 Percentage Point of Binomial and Negative Binomial Distribution," *Metrika*, Vol. 41, No. 1, pp. 43–54, 1994.

[12] Ch. Georgiou, Th. Pavlides and A. Philippou, "Uncertainty in Selfish Routing," *CD-ROM Proceedings of the 20th IEEE International Parallel and Distributed Processing Symposium,* 2006.

[13] O. Goussevskala, Y. A. Oswald and R. Wattenhofer, "Complexity in Geometric SINR," *Proceedings of the 8th ACM International Symposium on Mobile Ad Hoc Networking and Computing,* pp. 100–109, 2007.

[14] I. Kaplansky, "A Contribution to von-Neumann's Theory of Games," *Annals of Mathematics,* Vol. 46, No. 3, pp. 474–479, 1945.

[15] E. Koutsoupias and C. H. Papadimitriou, "Worst-Case Equilibria," *Proceedings of the 16th Annual Symposium on Theoretical Aspects of Computer Science,* pp. 404–413, Vol. 1563, LNCS, Springer-Verlag, 1999.

[16] T. Lücking, M. Mavronicolas, B. Monien and M. Rode, "A New Model for Selfish Routing," *Proceedings of the 41st International Symposium on Theoretical Aspects of Computer Science,* pp. 547–548, Vol. 2996, LNCS, Springer-Verlag, 2004.

[17] T. Lücking, M. Mavronicolas, B. Monien, M. Rode, P. Spirakis and I. Vrto, "Which is the Worst-Case Nash Equilibrium?," *Proceedings of the 26th International Symposium on Mathematical Foundations of Computer Science,* pp. 551–561, Vol. 2747, LNCS, Springer-Verlag, 2003.

[18] M. Mavronicolas, P. Panagopoulou and P. Spirakis, "A Cost Mechanism for Fair Pricing of Resource Usage," *Proceedings of the 1st International Workshop on Internet and Network Economics,* pp. 210–224, Vol. 3838, LNCS, Springer-Verlag, 2005. Extended vesrion accepted to *Algorithmica.*

[19] M. Mavronicolas and P. Spirakis, "The Price of Selfish Routing," *Algorithmica*, Vol. 48, pp. 91–126, 2007.

[20] J. F. Nash, "Equilibrium Points in N-Person Games," *Proceedings of the National Academy of Sciences of the United States of America,* Vol. 36, pp. 48–49, 1950.

[21] J. F. Nash, "Non-Cooperative Games," *Annals of Mathematics,* Vol. 54, No. 2, pp. 286–295, 1951.

[22] W. Uhlmann, "Vergleich der Hypergeometrischen mit der Binomial-Verteilung," *Metrika*, Vol. 10, pp. 145–158, 1966.

A Proofs from Section 2

A.1 Lemma 2.2

We apply Lemma 2.1 on both the numerator and the denominator of the fractional expansion of $\binom{n}{\frac{n}{2}}$ to obtain that

$$\begin{aligned}\frac{n^2}{2^{n+1}}\binom{n}{\frac{n}{2}} &\leq \frac{n^2}{2^{n+1}}\frac{\sqrt{2\pi}n^{n+\frac{1}{2}}\exp(-n+\frac{1}{12n})}{\left(\sqrt{2\pi}\left(\frac{n}{2}\right)^{\frac{n+1}{2}}\exp\left(-\frac{n}{2}+\frac{1}{6n+1}\right)\right)^2}\\ &= n\sqrt{\frac{n}{2\pi}}\exp\left(\frac{1}{12n}-\frac{2}{6n+1}\right)\\ &\leq n\sqrt{\frac{n}{6}}\end{aligned}$$

and that

$$\begin{aligned}\frac{n^2}{2^{n+1}}\binom{n}{\frac{n}{2}} &\geq \frac{n^2}{2^{n+1}}\frac{\sqrt{2\pi}n^{n+\frac{1}{2}}\exp(-n+\frac{1}{12n+1})}{\left(\sqrt{2\pi}\left(\frac{n}{2}\right)^{\frac{n+1}{2}}\exp\left(-\frac{n}{2}+\frac{1}{6n}\right)\right)^2}\\ &= n\sqrt{\frac{n}{2\pi}}\exp\left(\frac{1}{12n+1}-\frac{1}{3n}\right).\end{aligned}$$

A.2 Lemma 2.3

By applying Lemma 2.1 to both the numerator and the denominator of the fraction, we obtain that

$$\begin{aligned}\frac{n!}{2^n\left(\left(\frac{n-1}{2}\right)!\right)^2} &\leq \frac{\sqrt{2\pi}n^{n+\frac{1}{2}}\exp(-n+\frac{1}{12n})}{2^n\left(\sqrt{2\pi}\left(\frac{n-1}{2}\right)^{\frac{n}{2}}\exp\left(-\frac{n-1}{2}+\frac{1}{6n-5}\right)\right)^2}\\ &= \left(\frac{n}{n-1}\right)^n\sqrt{\frac{n}{2\pi}}\exp\left(\frac{1}{12n}-\frac{2}{6n-5}-1\right)\\ &\leq \left(\frac{n}{n-1}\right)^n\sqrt{\frac{n}{6}}.\end{aligned}$$

and that

$$\begin{aligned}\frac{n!}{2^n\left(\left(\frac{n-1}{2}\right)!\right)^2} &\geq \frac{\sqrt{2\pi}n^{n+\frac{1}{2}}\exp(-n+\frac{1}{12n+1})}{2^n\left(\sqrt{2\pi}\left(\frac{n-1}{2}\right)^{\frac{n}{2}}\exp\left(-\frac{n-1}{2}+\frac{1}{6n-6}\right)\right)^2}\\ &= \left(\frac{n}{n-1}\right)^n\sqrt{\frac{n}{2\pi}}\exp\left(\frac{1}{12n+1}-\frac{1}{3n-3}-1\right)\\ &\geq \sqrt{\frac{n}{2\pi}}\exp\left(\frac{1}{12n+1}-\frac{1}{3n-3}\right).\end{aligned}$$

A.3 Lemma 2.6

1. Proof of (1): By the definition of generalized medians, the claim is equivalent to $\mathsf{B}_{n-r-2,\lceil\frac{n-3}{2}\rceil-r-1}(p)\leq\frac{1}{2}$. From Lemma 2.5, we have that $\mathsf{B}_{n-r-2,(n-r-2+1)p-1}(p)\leq\frac{1}{2}$, where $p=\frac{1}{2}-\frac{r}{2(n-r-1)}$. Hence, by the definition of the binomial function, it suffices to prove that $\left\lceil\frac{n-3}{2}\right\rceil-r-1\leq(n-r-2+1)p-1$ which is equivalent to $\left\lceil\frac{n-3}{2}\right\rceil-r\leq\frac{n-2r-1}{2}$. If n is even, $\left\lceil\frac{n-3}{2}\right\rceil-r=\frac{n-2r-2}{2}<\frac{n-2r-1}{2}$. If n is odd $\left\lceil\frac{n-3}{2}\right\rceil-r=\frac{n-2r-2}{2}<\frac{n-2r-1}{2}$. So, the claim follows in all cases.

2. Proof of (2), (3), (4) and (5): Define $b(x) = \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}$ for $x = 0, 1, 2, 3$, respectively. By the definition of generalized medians, the claim is equivalent to $\mathsf{B}_{n-r-2, \lceil \frac{n-3}{2} \rceil - (\lfloor \frac{n-3}{2} \rfloor - x) - 1}(p) \leq b(x)$ for $x \in \{0, 1, 2, 3\}$. Since n is even, $\left\lceil \frac{n-3}{2} \right\rceil - \left\lfloor \frac{n-3}{2} \right\rfloor = 1$ and the claim is equivalent to $\mathsf{B}_{n-r-2,x}(p) \leq b(x)$ for $x \in \{0, 1, 2, 3\}$. Note that

$$\begin{aligned}
\mathsf{B}_{n-r-2,x}(p) &= \sum_{i=0}^{x} \binom{n-r-2}{i} p^i (1-p)^{n-r-2-i} \\
&= \sum_{i=0}^{x} \binom{n - \lfloor \frac{n-3}{2} \rfloor + x - 2}{i} p^i (1-p)^{n-r-2-i} \\
&= \sum_{i=0}^{x} \binom{\frac{n}{2} + x}{i} p^i (1-p)^{\frac{n}{2}+x-i} \\
&= (1-p)^{\frac{n}{2}} \sum_{i=0}^{x} \binom{\frac{n}{2} + x}{i} p^i (1-p)^{x-i} \\
&= \underbrace{\left(\frac{n-1}{n+2x+2}\right)^{\frac{n}{2}}}_{g_x(n)} \cdot \underbrace{\sum_{i=0}^{x} \binom{\frac{n}{2} + x}{i} \left(\frac{2x+3}{n+2x+2}\right)^i \left(\frac{n-1}{n+2x+2}\right)^{x-i}}_{f_x(n)}.
\end{aligned}$$

We observe that: *(i)* The function $f_x(n)$ is increasing in n for each $x \in \{0, 1, 2, 3\}$ with $f_x(n) = 1, \frac{7n+8}{2n+8}, \frac{85n^2+362n+288}{8n^2+96n+288}, \frac{1497n^3+13842n^2+36648n+24576}{48n^3+1152n^2+9216n+24576}$ for $x = \{0, 1, 2, 3\}$, respectively, with $\lim_{n\to\infty} f_x(n) = 1, \frac{7}{2}, \frac{85}{8}, \frac{1479}{48}$, respectively; *(ii)* The function $\frac{b(x)}{g_x(n)} = b(x) \cdot \left(\frac{n+2x+2}{n-1}\right)^{\frac{n}{2}}$ is increasing in n for each $x \in \{0, 1, 2, 3\}$ with $\frac{b(x)}{g_x(n)} \geq 1, \frac{7}{2}, \frac{85}{8}, \frac{1479}{48}$ for $n = 134$. The claim follows from *(i)* and *(ii)*.

3. Proof of (6), (7), (8) and (9): Define $b(x) = 0, \frac{1}{7}, \frac{2}{9}, \frac{3}{11}$ for $x = 0, 1, 2, 3$, respectively. By the definition of generalized medians, the claim is equivalent to $\mathsf{B}_{n-r-2, \lceil \frac{n-3}{2} \rceil - (\lfloor \frac{n-3}{2} \rfloor - x) - 1}(p) \leq b(x)$ for $x \in \{0, 1, 2, 3\}$. Since n is odd, $\left\lceil \frac{n-3}{2} \right\rceil - \left\lfloor \frac{n-3}{2} \right\rfloor = 0$ and the claim is equivalent to $\mathsf{B}_{n-r-2,x-1}(p) \leq b(x)$ for $x \in \{0, 1, 2, 3\}$. Note that

$$\begin{aligned}
\mathsf{B}_{n-r-2,x-1}(p) &= \sum_{i=0}^{x-1} \binom{n-r-2}{i} p^i (1-p)^{n-r-2-i} \\
&= \sum_{i=0}^{x-1} \binom{n - \lfloor \frac{n-3}{2} \rfloor + x - 2}{i} p^i (1-p)^{n-r-2-i} \\
&= \sum_{i=0}^{x-1} \binom{\frac{n-1}{2} + x}{i} p^i (1-p)^{\frac{n-1}{2}+x-i} \\
&= (1-p)^{\frac{n-1}{2}} \sum_{i=0}^{x-1} \binom{\frac{n-1}{2} + x}{i} p^i (1-p)^{x-i} \\
&= \underbrace{\left(\frac{n-1}{n+2x+2}\right)^{\frac{n-1}{2}}}_{g_x(n)} \cdot \underbrace{\sum_{i=0}^{x-1} \binom{\frac{n-1}{2} + x}{i} \left(\frac{2x+3}{n+2x+2}\right)^i \left(\frac{n-1}{n+2x+2}\right)^{x-i}}_{f_x(n)}.
\end{aligned}$$

We observe that: *(i)* The function $f_x(n)$ is increasing in n for each $x \in \{0, 1, 2, 3\}$ with $f_x(n) = 0, 1, \frac{4n+8}{n+5}, \frac{13n^2+78n+101}{n^2+14n+49}$ for $x = \{0, 1, 2, 3\}$, respectively, with $\lim_{n\to\infty} f_x(n) = 0, 1, 4, 13$, respectively; *(ii)* The function $\frac{b(x)}{g_x(n)} = b(x) \cdot \left(\frac{n+2x+2}{n-1}\right)^{\frac{n}{2}}$ is increasing in n for each $x \in \{0, 1, 2, 3\}$ with $\frac{b(x)}{g_x(n)} \geq 0, 1, 4, 13$ for $n = 135$. The claim follows from *(i)* and *(ii)*.

B Proofs and Calculations from Section 3

B.1 Lemma 3.1

Note that the maximum congestion attains the following values:

- $\frac{n^2}{4}$, attained when $\frac{n}{2}$ users are assigned to both links 1 and 2; this occurs in $\binom{n}{\frac{n}{2}}$ ways if n is even and cannot occur for odd n.
- i^2, where $\frac{n}{2} < i \leq n$, attained when i users are assigned to one link and the remaining $n-i < i$ users are assigned to the other link; this occurs in $2\binom{n}{i}$ ways (where, the factor 2 takes care of exchanging the links where the maximum latency is attained).

By the equiprobability of all 2^n pure profiles, it follows that

$$\begin{aligned}
\mathsf{QMSC}(\phi) &= \frac{1}{2^n}\left(\begin{cases} \frac{n^2}{4}\binom{n}{\frac{n}{2}}, & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases} + 2\sum_{i=\lfloor\frac{n}{2}\rfloor+1}^{n} i^2\binom{n}{i}\right) \\
&= \frac{n}{2^n}\left(\begin{cases} \frac{n}{2}\binom{n-1}{\lceil\frac{n}{2}\rceil-1}, & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases} + 2\sum_{i=\lfloor\frac{n}{2}\rfloor}^{n-1} (i+1)\binom{n-1}{i}\right) \\
&= \frac{n}{2^n}\begin{cases} \frac{n}{2}\binom{n-1}{\lceil\frac{n}{2}\rceil-1} + 2^{n-1} + (n-1)\left(2^{n-2} + \binom{n-2}{\lceil\frac{n}{2}\rceil-1}\right), & n \text{ is even} \\ \binom{n-1}{\lceil\frac{n}{2}\rceil-1} + 2^{n-1} + (n-1)\left(2^{n-2} + 2\binom{n-2}{\lceil\frac{n}{2}\rceil-2}\right), & n \text{ is odd} \end{cases} \\
&= \frac{n}{4} + \frac{n^2}{4} + \frac{n}{2^n}\binom{n-1}{\lceil\frac{n}{2}\rceil-1} + \frac{n(n-1)}{2^n}\left(\binom{n-2}{\lceil\frac{n}{2}\rceil-2} + \binom{n-2}{\lfloor\frac{n}{2}\rfloor-1}\right) \\
&= \frac{n}{4} + \frac{n^2}{4} + \frac{n^2}{2^n}\binom{n-1}{\lceil\frac{n}{2}\rceil-1},
\end{aligned}$$

as needed.

B.2 Lemma 3.3

Clearly,

$$\begin{aligned}
& -2u\frac{n}{2^n}\binom{n-1}{\lceil\frac{n}{2}\rceil-1} + \frac{n^2}{2^n}\binom{n-1}{\lceil\frac{n}{2}\rceil-1} - \frac{(n-2u)^2}{2^{n-2u}}\binom{n-2u-1}{\lceil\frac{n-2u}{2}\rceil-1} \\
=\ & (n-2u)\frac{n}{2^n}\binom{n-1}{\lceil\frac{n}{2}\rceil-1} - \frac{(n-2u)^2}{2^{n-2u}}\binom{n-2u-1}{\lceil\frac{n-2u}{2}\rceil-1} \\
=\ & \frac{n-2u}{2^{n-2u}}\left(\frac{n}{2^{2u}}\binom{n-1}{\lceil\frac{n}{2}\rceil-1} - (n-2u)\binom{n-2u-1}{\lceil\frac{n-2u}{2}\rceil-1}\right).
\end{aligned}$$

Hence, it suffices to show that

$$\binom{n-1}{\lceil\frac{n}{2}\rceil-1} \geq \frac{2^{2u}(n-2u)}{n}\binom{n-2u-1}{\lceil\frac{n-2u}{2}\rceil-1}.$$

We consider separately the two cases where n is even or odd.

- n is even: Then,

$$\begin{aligned}
\binom{n-1}{\frac{n}{2}-1} &= \frac{(n-2u)(n-2u+1)\dots(n-2)(n-1)}{\left(\frac{n}{2}-u+1\right)\dots\frac{n}{2}\cdot\left(\frac{n}{2}-u\right)\dots\left(\frac{n}{2}-1\right)}\binom{n-2u-1}{\frac{n-2u}{2}-1} \\
&= \frac{(n-2u+1)(n-2u+3)\dots(n-1)2^{2u}}{(n-2u+2)(n-2u+4)\dots n}\binom{n-2u-1}{\frac{n-2u}{2}-1} \\
&\geq \frac{2^{2u}(n-1)}{n}\binom{n-2u-1}{\frac{n-2u}{2}-1} \\
&\geq \frac{2^{2u}(n-2u)}{n}\binom{n-2u-1}{\frac{n-2u}{2}-1}.
\end{aligned}$$

- n is odd: Then,

$$\begin{aligned}
\binom{n-1}{\frac{n+1}{2}-1} &= \frac{(n-2u)(n-2u+1)\dots(n-2)(n-1)}{\left(\frac{n-1}{2}-u+1\right)\dots\left(\frac{n-1}{2}\right)\cdot\left(\frac{n+1}{2}-u\right)\dots\left(\frac{n+1}{2}-1\right)}\binom{n-2u-1}{\frac{n-2u+1}{2}-1} \\
&= \frac{(n-2u)(n-2u+2)\dots(n-2)2^{2u}}{(n-2u+1)(n-2u+3)\dots(n-1)}\binom{n-2u-1}{\frac{n-2u+1}{2}-1} \\
&\geq \frac{(n-2u)2^{2u}}{n-1}\binom{n-2u-1}{\frac{n-2u+1}{2}-1} \\
&\geq \frac{(n-2u)2^{2u}}{n}\binom{n-2u-1}{\frac{n-2u+1}{2}-1}.
\end{aligned}$$

B.3 Lemma 3.5

Note that the maximum congestion attains the following values:

- $\frac{n^2}{4}$, attained when $\frac{n}{2}$ users are assigned to both links 1 and 2. There are $\binom{n-r}{\frac{n}{2}-r}$ such profiles when n is even, and each one occurs with probability $p^{\frac{n}{2}-r}q^{\frac{n}{2}}$. (There are no such profiles when n is odd.)
- i^2, when i users are assigned to link 1, where $\left\lfloor\frac{n}{2}\right\rfloor+1 \leq \kappa \leq n$. There are $\binom{n-r}{i-r}$ such profiles and each one occurs with probability $p^{i-r}q^{n-i}$.
- i^2, when i users are assigned to link 2, where $\left\lfloor\frac{n}{2}\right\rfloor+1 \leq i \leq n-r$. There are $\binom{n-r}{i}$ profiles and each one occurs with probability $p^{n-r-i}q^i$.

Thus, it follows:

$$\mathsf{QMSC}(\widehat{\boldsymbol{\sigma}}) \;=\; \mathsf{Even}(n)\cdot\frac{n^2}{4}\binom{n-r}{\frac{n}{2}-r}p^{\frac{n}{2}-r}q^{\frac{n}{2}} + \sum_{i=\left\lfloor\frac{n}{2}\right\rfloor+1}^{n} i^2\binom{n-r}{i-r}p^{i-r}q^{n-i} + \sum_{i=\left\lfloor\frac{n}{2}\right\rfloor+1}^{n-r} i^2\binom{n-r}{i}p^{n-r-i}q^i,$$

as needed.

B.4 Lemma 3.6

Clearly,

$$
\begin{aligned}
& A+B-C \\
= & \sum_{i=\lceil \frac{n+1}{2} \rceil}^{n} (i-r)\binom{n-r}{i-r} p^{i-r-1} q^{n+1-i} + \overbrace{\sum_{i=\lceil \frac{n+1}{2} \rceil}^{n-r} (n-r)\binom{n-r-1}{i-1} p^{n-r-i} q^{i}} \\
& \underbrace{\sum_{i=\lceil \frac{n+1}{2} \rceil}^{n} (n-r)(i-r-1)\binom{n-r-1}{i-r-1} p^{i-r-2} q^{n+2-i}} + \overbrace{\sum_{i=\lceil \frac{n+1}{2} \rceil}^{n-r} (n-r)(n-r-1)\binom{n-r-2}{i-2} p^{n-r-i} q^{i}} \\
& - \sum_{i=\lceil \frac{n+1}{2} \rceil}^{n} (i-r)\binom{n-r}{i-r} p^{i-r-1} q^{n+1-i} - \underbrace{\sum_{i=\lceil \frac{n+1}{2} \rceil}^{n} (i-r)^2\binom{n-r}{i-r} p^{i-r-2} q^{n+2-i}} \\
& \underbrace{+ \sum_{i=\lceil \frac{n+1}{2} \rceil}^{n} (i-r)\binom{n-r}{i-r} p^{i-r-2} q^{n+2-i}} + \overbrace{\sum_{i=\lceil \frac{n+1}{2} \rceil}^{n} i^2\binom{n-r}{i-r} p^{i-r} q^{n-i}}
\end{aligned}
$$

Note that the three underbraced terms cancel out because for each i,

$$
\begin{aligned}
& (n-r)(i-r-1)\binom{n-r-1}{i-r-1} - (i-r)^2\binom{n-r}{i-r} + (i-r)\binom{n-r}{i-r} \\
= & (n-r)(i-r-1)\binom{n-r-1}{i-r-1} - (i-r)(i-r-1)\binom{n-r}{i-r} \\
= & (n-r)(i-r-1)\binom{n-r-1}{i-r-1} - (n-r)(i-r-1)\binom{n-r-1}{i-r-1} \\
= & 0.
\end{aligned}
$$

Note also that for the three overbraced terms, for each i,

$$
\begin{aligned}
& (n-r)\binom{n-r-1}{i-1} + (n-r)(n-r-1)\binom{n-r-1}{i-2} + i^2\binom{n-r}{i-r} \\
= & (n-r)\binom{n-r-1}{i-1} + (n-r)(i-1)\binom{n-r-1}{i-1} + i^2\binom{n-r}{i-r} \\
= & (n-r)i\binom{n-r-1}{i-1} + i^2\binom{n-r}{i-r} \\
= & i^2\binom{n-r}{i} + i^2\binom{n-r}{i-r}.
\end{aligned}
$$

Note finally the the two remaining terms cancel out. The claim now follows.

B.5 Calculations for A, B and C

Clearly,

$$\begin{aligned}
A &= q(n-r)\left(\sum_{i=\lceil\frac{n+1}{2}\rceil}^{n}\binom{n-r-1}{i-r-1}p^{i-r-1}q^{n-i}+\sum_{i=\lceil\frac{n+1}{2}\rceil}^{n-r}\binom{n-r-1}{n-r-i}p^{n-r-i}q^{i-1}\right)\\
&= q(n-r)\left(\sum_{i=\lceil\frac{n+1}{2}\rceil}^{n}\binom{n-r-1}{i-r-1}p^{i-r-1}q^{n-i}+\sum_{i=r+1}^{n+1-\lceil\frac{n+1}{2}\rceil}\binom{n-r-1}{i-r-1}p^{i-r-1}q^{n-i}\right)\\
&= q(n-r)\left(\sum_{i=\lceil\frac{n+1}{2}\rceil}^{n}\binom{n-r-1}{i-r-1}p^{i-r-1}q^{n-i}+\sum_{i=r+1}^{\lfloor\frac{n+1}{2}\rfloor}\binom{n-r-1}{i-r-1}p^{i-r-1}q^{n-i}\right)\\
&= q(n-r)\left(\sum_{i=\lceil\frac{n+1}{2}\rceil-r-1}^{n-r-1}\binom{n-r-1}{i}p^{i}q^{n-i-r-1}+\sum_{i=0}^{\lfloor\frac{n+1}{2}\rfloor-r-1}\binom{n-r-1}{i}p^{i}q^{n-i-r-1}\right)\\
&= q(n-r)\sum_{i=0}^{n-r-1}\binom{n-r-1}{i}p^{i}q^{n-r-i-1}+\mathsf{Odd}(n)\cdot q(n-r)\binom{n-r-1}{\frac{n-1}{2}-r}p^{\frac{n-1}{2}-r}q^{\frac{n-1}{2}}\\
&= q(n-r)+\mathsf{Odd}(n)\cdot q(n-r)\binom{n-r-1}{\frac{n-1}{2}-r}p^{\frac{n-1}{2}-r}q^{\frac{n-1}{2}},
\end{aligned}$$

and

$$\begin{aligned}
B &= q^2(n-r)(n-r-1)\left(\sum_{i=\lceil\frac{n+1}{2}\rceil}^{n}\binom{n-r-2}{i-r-2}p^{i-r-2}q^{n-i}+\sum_{i=\lceil\frac{n+1}{2}\rceil}^{n}\binom{n-r-2}{n-r-i}p^{n-r-i}q^{i-2}\right)\\
&= q^2(n-r)(n-r-1)\left(\sum_{i=\lceil\frac{n+1}{2}\rceil}^{n}\binom{n-r-2}{i-r-2}p^{i-r-2}q^{n-i}+\sum_{i=r+2}^{\lfloor\frac{n+1}{2}\rfloor+1}\binom{n-r-2}{i-r-2}p^{i-r-2}q^{n-i}\right)\\
&= q^2(n-r)(n-r-1)\left(\sum_{i-r-2=\lceil\frac{n+1}{2}\rceil-r-2}^{n-r-2}\binom{n-r-2}{i-r-2}p^{i-r-2}q^{n-(i-r-2)-r-2}+\right.\\
&\qquad\left.\sum_{i-r-2=0}^{\lfloor\frac{n+1}{2}\rfloor+1-r-2}\binom{n-r-2}{i-r-2}p^{i-r-2}q^{n-(i-r-2)-r-2}\right)\\
&= q^2(n-r)(n-r-1)\left(\sum_{i=0}^{n-r-2}\binom{n-r-2}{i}p^{i}q^{n-r-i-2}+\binom{n-r-2}{\lceil\frac{n-2}{2}\rceil-r}p^{\lceil\frac{n-2}{2}\rceil-r}q^{\lfloor\frac{n-2}{2}\rfloor}\right.\\
&\qquad\left.+\mathsf{Odd}(n)\cdot\binom{n-r-2}{\frac{n-3}{2}-r}p^{\frac{n-3}{2}-r}q^{\frac{n-1}{2}}\right)\\
&= q^2(n-r)(n-r-1)\left(1+\binom{n-r-2}{\lceil\frac{n-2}{2}\rceil-r}p^{\lceil\frac{n-2}{2}\rceil-r}q^{\lfloor\frac{n-2}{2}\rfloor}+\mathsf{Odd}(n)\cdot\binom{n-r-2}{\frac{n-3}{2}-r}p^{\frac{n-3}{2}-r}q^{\frac{n-1}{2}}\right)
\end{aligned}$$

and

$$
\begin{aligned}
C &= \sum_{i=\lceil \frac{n+1}{2} \rceil}^{n} \binom{n-r}{i-r} p^{i-r-2} q^{n-i} \left((i-r)pq + (i-r)^2 q^2 - (i-r) q^2 - i^2 p^2\right) \\
&= \sum_{i=\lceil \frac{n+1}{2} \rceil - r}^{n-r} \binom{n-r}{i} p^{i-2} q^{n-r-i} \left(i(pq - q^2) + i^2 (q^2 - p^2)\right) \\
&= (pq - q^2)(n-r) \sum_{i=\lceil \frac{n+1}{2} \rceil - r}^{n-r} \binom{n-r-1}{i-1} p^{i-2} q^{n-r-i} \\
&\quad + (q^2 - p^2)(n-r)(n-r-1) \sum_{i=\lceil \frac{n+1}{2} \rceil - r}^{n-r} \binom{n-r-2}{i-2} p^{i-2} q^{n-r-i} \\
&\quad + (q^2 - p^2)(n-r) \sum_{i=\lceil \frac{n+1}{2} \rceil - r}^{n-r} \binom{n-r-1}{i-1} p^{i-2} q^{n-r-i} \\
&= (pq - p^2)(n-r) \sum_{i=\lceil \frac{n+1}{2} \rceil - r}^{n-r} \binom{n-r-1}{i-1} p^{i-2} q^{n-r-i} \\
&\quad + (q^2 - p^2)(n-r)(n-r-1) \sum_{i=\lceil \frac{n+1}{2} \rceil - r}^{n-r} \binom{n-r-2}{i-2} p^{i-2} q^{n-r-i} \\
&= \left((pq - p^2)(n-r) + (q^2 - p^2)(n-r)(n-r-1)\right) \sum_{i=\lceil \frac{n+1}{2} \rceil - r}^{n-r} \binom{n-r-2}{i-2} p^{i-2} q^{n-r-i} \\
&\quad + (q-p)(n-r) \sum_{i=\lceil \frac{n+1}{2} \rceil - r}^{n-r} \binom{n-r-2}{i-1} p^{i-1} q^{n-r-i} \\
&= \left((pq - p^2)(n-r) + (q^2 - p^2)(n-r)(n-r-1)\right. \\
&\quad \left. + q(q-p)(n-r)\right) \sum_{i=\lceil \frac{n+3}{2} \rceil - r}^{n-r} \binom{n-r-2}{i-2} p^{i-2} q^{n-r-i} \\
&\quad + \left((pq - p^2)(n-r) + (q^2 - p^2)(n-r)(n-r-1)\right) \binom{n-r-2}{\lceil \frac{n-3}{2} \rceil - r} p^{\lceil \frac{n-3}{2} \rceil - r} q^{\lfloor \frac{n-1}{2} \rfloor} \\
&= \left((pq - p^2)(n-r) + q(q-p)(n-r)\right) \sum_{i=\lceil \frac{n+3}{2} \rceil - r}^{n-r} \binom{n-r-2}{i-2} p^{i-2} q^{n-r-i} \\
&\quad + (q^2 - p^2)(n-r)(n-r-1) \sum_{i=\lceil \frac{n+3}{2} \rceil - r}^{n-r} \binom{n-r-2}{i-2} p^{i-2} q^{n-r-i} \\
&\quad + \left((q^2 - p^2)(n-r)(n-r-1) + (pq - p^2)(n-r)\right) \binom{n-r-2}{\lceil \frac{n-3}{2} \rceil - r} p^{\lceil \frac{n-3}{2} \rceil - r} q^{\lfloor \frac{n-1}{2} \rfloor} \\
&= \left((pq - p^2)(n-r) + q(q-p)(n-r)\right) \sum_{i=\lceil \frac{n+3}{2} \rceil - r}^{n-r} \binom{n-r-2}{i-2} p^{i-2} q^{n-r-i} \\
&\quad + (q^2 - p^2)(n-r)(n-r-1) \sum_{i=\lceil \frac{n+1}{2} \rceil - r}^{n-r} \binom{n-r-2}{i-2} p^{i-2} q^{n-r-i} \\
&\quad + (pq - p^2)(n-r) \binom{n-r-2}{\lceil \frac{n-3}{2} \rceil - r} p^{\lceil \frac{n-3}{2} \rceil - r} q^{\lfloor \frac{n-1}{2} \rfloor}.
\end{aligned}
$$

C Proofs from Section 5

C.1 Lemma 5.1

Case (1):

Clearly,

$$\frac{n}{n-r-2}\,\frac{\prod_{i=0}^{r}(n-2r+2i)}{\prod_{i=1}^{r}(n-r+i)}\,\frac{(n-2r-1)^{1-r}}{(n-r-1)^{2-r}} = \begin{cases} \frac{n}{n-3}, & r=1\\ \frac{n(n-2)}{(n-1)(n-5)}, & r=2\\ \frac{n(n-6)(n-4)^2}{(n-1)(n-5)(n-7)^2}, & r=3\\ \frac{n(n-4)(n-8)(n-5)^2}{(n-1)(n-3)(n-9)^2}, & r=4. \end{cases}$$
$$> 1.$$

Hence, it suffices to show that $\left(\frac{n-2r-1}{n-r-1}\right)^{\frac{n-4}{2}}\left(\frac{n-1}{n-r-1}\right)^{\frac{n-2}{2}} \geq 1$, or equivalently that

$$(n-4)\ln(n-2r-1)-(n-4)\ln(n-r-1)+(n-2)\ln(n-1)-(n-2)\ln(n-r-1) \geq 0.$$

Note that, the derivative of the LHS expression is

$$(n-4)\left(\frac{-r}{(n-r-1)(n-2r-1)}\right)+(n-2)\left(\frac{-r}{(n-1)(n-r-1)}\right)+\ln\left(\frac{n^2-2n-2nr+2r+1}{n^2-2n-2nr+r^2+2r+1}\right) \leq 0.$$

This implies that the expression is non-increasing. The claim follows from the fact that the expression approaches 0 as n approaches ∞.

Case (2):

Define

$$K_r = \frac{n}{n-r-2}\,\frac{\prod_{i=0}^{r}(n-2r+2i)}{\prod_{i=1}^{r}(n-r+i)}\,\frac{(n-1)(n-2r-1)^{1-r}}{(n-r-1)^{3-r}}$$
$$= \begin{cases} \frac{n(n-1)}{(n-3)(n-2)^2}, & r=1\\ \frac{n(n-2)}{(n-3)^2}, & r=2\\ \frac{n(n-4)(n-6)}{(n-5)^3}, & r=3\\ \frac{n(n-4)(n-5)(n-8)}{(n-3)(n-7)^3}, & r=4. \end{cases}$$

and

$$L_r = \left(\frac{n-2r-1}{n-r-1}\right)^{\frac{n-4}{2}}\left(\frac{n-1}{n-r-1}\right)^{\frac{n-4}{2}} = \left(\frac{n^2-2n(r+1)+2r+1}{n^2-2n(r+1)+r^2+2r+1}\right)^{\frac{n-4}{2}} < 1.$$

So, we have to prove that, $n\sqrt{\frac{n}{6}}(1-K_rL_r)+\frac{r(n-r)}{2} \geq \frac{r(n+1)}{4(n-r-1)}$. Note that $\frac{r(n+1)}{4(n-r-1)} \leq 2$ (since $r \leq 4$ and $n \geq 9$). Hence, it suffices to show that $n\sqrt{\frac{n}{6}}(1-K_rL_r)+\frac{r(n-r)}{2} \geq 2$. This is equivalent to $L_r \leq \frac{1}{K_r}\left(\frac{1}{n}\sqrt{\frac{6}{n}}\left(\frac{r(n-r)}{2}-2\right)+1\right)$. Since $L_r < 1$, it suffices to show that $\frac{1}{n}\sqrt{\frac{6}{n}}\left(\frac{r(n-r)}{2}-2\right) \geq K_r - 1$. The last inequality holds trivially for $r=1$ (since $K_1 < 1$). We continue with the remaining cases.

- $r=2$: We need to prove that $\frac{1}{n}\sqrt{\frac{6}{n}}(n-4) \geq \frac{n(n-2)}{(n-3)^2}-1$. This is equivalent to $2n^3-4n^{5/2}-30n^2+9n^{3/2}+66n-108 \geq 0$. The latter follows since $2n^3-4n^{5/2}-30n^2 \geq 0$ and $9n^{3/2}+66n-108 \geq 0$ for $n \geq 134$.
- $r=3$: We need to prove that $\frac{1}{n}\sqrt{\frac{6}{n}}\left(\frac{3(n-3)}{2}-2\right) \geq \frac{n(n-4)(n-6)}{(n-5)^3}-1$. This is equivalent to $6n^4-10n^{7/2}-174n^3+102n^{5/2}+840n^2-250n^{3/2}-4050n+3250 \geq 0$. The latter follows since $6n^4-10n^{7/2}-174n^3+102n^{5/2} \geq 0$ and $840n^2-250n^{3/2}-4050n \geq 0$, for $n \geq 134$.

- $\underline{r=4}$: We need to prove that $\frac{1}{n}\sqrt{\frac{6}{n}}\left(\frac{4(n-4)}{2}-2\right) \geq \frac{n(n-4)(n-5)(n-8)}{(n-3)(n-7)^3}-1$. This is equivalent to $4n^5-7n^{9/2}-174n^4+118n^{7/2}+1320n^3-624n^{5/2}-11004n^2+1029n^{3/2}+19796n-30870 \geq 0$. The latter follows since $4n^5-7n^{9/2}-174n^4 \geq 0$, $118n^{7/2}-624n^{5/2} \geq 0$, $1320n^3-11004n^2 \geq 0$ and $1029n^{3/2}+19796n-30870 \geq 0$, for $n \geq 134$.

C.2 Lemma 5.2

$$
\begin{aligned}
& \frac{1}{n-r-2}\binom{n-r-2}{\frac{n-2}{2}-r} p^{\frac{n-2}{2}-r} q^{\frac{n-2}{2}} \\
\leq\ & \frac{1}{n-r-2} \max_{0\leq x\leq 1} \binom{n-r-2}{\frac{n-2}{2}-r} x^{\frac{n-2}{2}-r}(1-x)^{\frac{n-2}{2}} \\
\overset{(2.4)}{=}\ & \frac{1}{n-r-2}\binom{n-r-2}{\frac{n-2}{2}-r}\left(\frac{n-2}{2}-r\right)^{\frac{n-2}{2}-r}(n-r-2)^{-n+r+2}\left(\frac{n-2}{2}\right)^{\frac{n-2}{2}} \\
=\ & \frac{1}{n-r-2}\frac{(n-r-2)!}{\left(\frac{n-2}{2}\right)!\left(\frac{n-2}{2}-r\right)!}\left(\frac{n-2}{2}-r\right)^{\frac{n-2}{2}-r}(n-r-2)^{-n+r+2}\left(\frac{n-2}{2}\right)^{\frac{n-2}{2}} \\
\overset{(2.1)}{\leq}\ & \frac{1}{n-r-2}\frac{1}{\sqrt{2\pi}}\frac{(n-r-2)^{n-r-\frac{3}{2}}}{\left(\frac{n-2}{2}\right)^{\frac{n-1}{2}}\left(\frac{n-2}{2}-r\right)^{\frac{n-1}{2}-r}}\exp\left(\frac{1}{12(n-r-2)}-\frac{1}{6n-11}-\frac{1}{6n-12r-11}\right) \\
& \left(\frac{n-2}{2}-r\right)^{\frac{n-2}{2}-r}(n-r-2)^{-n+r+2}\left(\frac{n-2}{2}\right)^{\frac{n-2}{2}} \\
=\ & \frac{1}{n-r-2}\frac{1}{\sqrt{2\pi}}\frac{(n-r-2)^{\frac{1}{2}}}{\left(\frac{n-2}{2}\right)^{\frac{1}{2}}\left(\frac{n-2}{2}-r\right)^{\frac{1}{2}}}\exp\left(\frac{1}{12(n-r-2)}-\frac{1}{6n-11}-\frac{1}{6n-12r-11}\right) \\
=\ & \frac{1}{\sqrt{2\pi(n-r-2)(\frac{n}{2}-1)(\frac{n}{2}-r-1)}}\exp\left(\frac{1}{12(n-r-2)}-\frac{1}{6n-11}-\frac{1}{6n-12r-11}\right).
\end{aligned}
$$

C.3 Lemma 5.3

Case (1):

Define

$$
\begin{aligned}
K_r &= \frac{n(n-2r-1)^{\frac{n-2}{2}-r}(n^2-n-2nr+r)}{2^{\frac{n-2r-4}{2}}\left(\frac{n-2}{2}-r\right)!(n-2r)}\left(\frac{n-1}{2(n-r-1)}\right)^{\frac{n}{2}} \\
&= \begin{cases}
\frac{6561n(19n-10)}{3840}\left(\frac{n-1}{n+8}\right)^{\frac{n}{2}} \leq \frac{124659n}{3840}\left(\frac{n-1}{n+8}\right)^{\frac{n}{2}}, & r=\left\lfloor\frac{n-3}{2}\right\rfloor-3 \\
\frac{343n(15n-10)}{384}\left(\frac{n-1}{n+6}\right)^{\frac{n}{2}} \leq \frac{5145n}{384}\left(\frac{n-1}{n+6}\right)^{\frac{n}{2}}, & r=\left\lfloor\frac{n-3}{2}\right\rfloor-2 \\
\frac{25n(11n-6)}{48}\left(\frac{n-1}{n+4}\right)^{\frac{n}{2}} \leq \frac{275n}{48}\left(\frac{n-1}{n+4}\right)^{\frac{n}{2}}, & r=\left\lfloor\frac{n-3}{2}\right\rfloor-1 \\
\frac{3n(7n-4)}{16}\left(\frac{n-1}{n+2}\right)^{\frac{n}{2}} \leq \frac{21n}{8}\left(\frac{n-1}{n+2}\right)^{\frac{n}{2}}, & r=\left\lfloor\frac{n-3}{2}\right\rfloor
\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
& L_r \\
=\ & 2r\mathsf{Q}+\frac{n^2}{2(n-r)}-\frac{n^2}{2(n-r-1)} \\
\overset{(2.6)}{\geq}\ & \begin{cases}
\frac{4(n-10)}{7}+\frac{n^2}{n+10}-\frac{n^2}{n+8}=\frac{n^2}{4(n+10)}+\frac{11n}{42}+\frac{3n^2}{4(n+10)}+\frac{13n}{42}-\frac{40}{7}-\frac{n^2}{n+8}, & r=\left\lfloor\frac{n-3}{2}\right\rfloor-3 \\
\frac{3(n-8)}{5}+\frac{n^2}{n+8}-\frac{n^2}{n+6}=\frac{n^2}{4(n+8)}+\frac{7n}{25}+\frac{3n^2}{4(n+8)}+\frac{8n}{25}-\frac{24}{5}-\frac{n^2}{n+6}, & r=\left\lfloor\frac{n-3}{2}\right\rfloor-2 \\
\frac{2(n-6)}{3}+\frac{n^2}{n+6}-\frac{n^2}{n+4}=\frac{n^2}{2(n+6)}+\frac{2n}{18}+\frac{n^2}{2(n+6)}+\frac{10n}{18}-4-\frac{n^2}{n+4}, & r=\left\lfloor\frac{n-3}{2}\right\rfloor-1 \\
\frac{3(n-4)}{4}+\frac{n^2}{n+4}-\frac{n^2}{n+2}=\frac{n^2}{2(n+4)}+\frac{3n}{16}+\frac{n^2}{2(n+4)}+\frac{9n}{16}-3-\frac{n^2}{n+2}, & r=\left\lfloor\frac{n-3}{2}\right\rfloor.
\end{cases}
\end{aligned}
$$

Note that

$$\begin{aligned}
&\frac{3n^2}{4(n+10)}+\frac{13n}{42}-\frac{40}{7}-\frac{n^2}{n+8}\geq\frac{3n^2}{4(n+10)}+\frac{11n}{42}-\frac{n^2}{n+8}=\frac{n(n^2+60n+1760)}{84(n+8)(n+10)}>0\\
&\frac{3n^2}{4(n+8)}+\frac{8n}{25}-\frac{24}{5}-\frac{n^2}{n+6}\geq\frac{3n^2}{4(n+8)}+\frac{7n}{25}-\frac{n^2}{n+6}=\frac{3n(n^2+14n+448)}{100(n+6)(n+8)}>0\\
&\frac{n^2}{2(n+6)}+\frac{10n}{18}-4-\frac{n^2}{n+4}\geq\frac{n^2}{2(n+6)}+\frac{9n}{18}-\frac{n^2}{n+4}=\frac{n(n+12)}{(n+4)(n+6)}>0\\
&\frac{n^2}{2(n+4)}+\frac{9n}{16}-3-\frac{n^2}{n+2}\geq\frac{n^2}{2(n+4)}+\frac{17n}{32}-\frac{n^2}{n+2}=\frac{n(n^2+6n+136)}{32(n+2)(n+4)}>0,
\end{aligned}$$

it follows that

$$L_r\geq\begin{cases}\frac{n^2}{4(n+10)}+\frac{11n}{42}, & r=\left\lfloor\frac{n-3}{2}\right\rfloor-3\\ \frac{n^2}{4(n+8)}+\frac{7n}{25}, & r=\left\lfloor\frac{n-3}{2}\right\rfloor-2\\ \frac{n^2}{2(n+6)}+\frac{2n}{18}, & r=\left\lfloor\frac{n-3}{2}\right\rfloor-1\\ \frac{n^2}{2(n+4)}+\frac{3n}{16}, & r=\left\lfloor\frac{n-3}{2}\right\rfloor.\end{cases}$$

We prove that $K_r \leq L_r$ for all integers r such that $\left\lfloor\frac{n-3}{2}\right\rfloor-3\leq r\leq\left\lfloor\frac{n-3}{2}\right\rfloor$. There are four cases for r:

- $r=\left\lfloor\frac{n-3}{2}\right\rfloor-3$: Note that $K_{\lfloor\frac{n-3}{2}\rfloor-3}\leq L_{\lfloor\frac{n-3}{2}\rfloor-3}$ if and only if $\left(\frac{n+8}{n-1}\right)^{\frac{n}{2}}\geq\frac{2617839(n+10)}{960(43n+220)}$. Note that $\frac{2617839(n+10)}{960(43n+220)}$ is decreasing in n and $\left(\frac{n-8}{n-1}\right)^{\frac{n}{2}}$ is increasing in n. Since $n\geq 134$, the claim follows.
- $r=\left\lfloor\frac{n-3}{2}\right\rfloor-2$: It suffices to prove that $\left(\frac{n+6}{n-1}\right)^{\frac{n}{2}}\geq\frac{128625(n+8)}{96(53n+224)}$. Note that the function $\frac{128625(n+8)}{96(53n+224)}$ is decreasing in n and $\left(\frac{n+6}{n-1}\right)^{\frac{n}{2}}$ is increasing in n. Since $n\geq 134$, the claim follows.
- $r=\left\lfloor\frac{n-3}{2}\right\rfloor-1$: It suffices to prove that $\left(\frac{n+4}{n-1}\right)^{\frac{n}{2}}\geq\frac{2475(n+6)}{48(11n+12)}$. Note that the function $\frac{2475(n+6)}{48(11n+12)}$ is decreasing in n and $\left(\frac{n+4}{n-1}\right)^{\frac{n}{2}}$ is increasing in n. Since $n\geq 134$, the claim follows.
- $r=\left\lfloor\frac{n-3}{2}\right\rfloor$: It suffices to prove that $\left(\frac{n+2}{n-1}\right)^{\frac{n}{2}}\geq\frac{84(n+4)}{22n+24}$. Note that the function $\frac{84(n+4)}{22n+24}$ is decreasing in n and $\left(\frac{n+2}{n-1}\right)^{\frac{n}{2}}$ is increasing in n. Since $n\geq 134$, the claim follows.

Case (2):

Define

$$\begin{aligned}
K_r &= \frac{n(n+1)}{2(n-r)}\\
&=\begin{cases}\frac{n(n+1)}{n+9}=\frac{2n(n+1)}{5(n+9)}+\frac{3n(n+1)}{5(n+9)}, & r=\left\lfloor\frac{n-3}{2}\right\rfloor-3\\ \frac{n(n+1)}{n+7}=\frac{3n(n+1)}{4(n+7)}+\frac{n(n+1)}{4(n+7)}, & r=\left\lfloor\frac{n-3}{2}\right\rfloor-2\\ \frac{n(n+1)}{n+5}=\frac{n(n+1)}{5(n+5)}+\frac{4n(n+1)}{5(n+5)}, & r=\left\lfloor\frac{n-3}{2}\right\rfloor-1\\ \frac{n(n+1)}{n+3}=\frac{n(n+1)}{5(n+3)}+\frac{4n(n+1)}{5(n+3)}, & r=\left\lfloor\frac{n-3}{2}\right\rfloor\end{cases}
\end{aligned}$$

and

$$L_r = \frac{(n-1)(n+1)}{2(n-r-1)} - 2r\mathsf{Q} + 2(n+1)\binom{n-r-1}{\frac{n-1}{2}-r}\left(\frac{n-2r-1}{2(n-r-1)}\right)^{\frac{n-1}{2}-r}\left(\frac{n-1}{2(n-r-1)}\right)^{\frac{n+1}{2}}$$

$$\stackrel{(2.6)}{\leq} \begin{cases} \frac{(n-1)(n+1)}{n+7} - \frac{8(n-9)}{11} + \frac{64(n+1)^2}{3(n+7)}\left(\frac{n-1}{n+7}\right)^{\frac{n+1}{2}}, & r = \left\lfloor \frac{n-3}{2} \right\rfloor - 3 \\ \frac{(n-1)(n+1)}{n+5} - \frac{7(n-7)}{9} + \frac{9(n+1)^2}{n+5}\left(\frac{n-1}{n+5}\right)^{\frac{n+1}{2}}, & r = \left\lfloor \frac{n-3}{2} \right\rfloor - 2 \\ \frac{(n-1)(n+1)}{n+3} - \frac{6(n-5)}{7} + \frac{4(n+1)^2}{n+3}\left(\frac{n-1}{n+3}\right)^{\frac{n+1}{2}}, & r = \left\lfloor \frac{n-3}{2} \right\rfloor - 1 \\ 2 + 2(n+1)\left(\frac{n-1}{n+1}\right)^{\frac{n+1}{2}}, & r = \left\lfloor \frac{n-3}{2} \right\rfloor. \end{cases}$$

We prove that $K_r \geq L_r$ for each r such that $\left\lfloor \frac{n-3}{2} \right\rfloor - 3 \leq r \leq \left\lfloor \frac{n-3}{2} \right\rfloor$. There are four cases for r.

- $\underline{r = \left\lfloor \frac{n-3}{2} \right\rfloor - 3}$: Note that $K_{\lfloor \frac{n-3}{2} \rfloor - 3} \geq L_{\lfloor \frac{n-3}{2} \rfloor - 3}$ if and only if:

$$\overbrace{\frac{2n(n+1)}{5(n+9)}} + \underbrace{\frac{3n(n+1)}{5(n+9)}} \geq \overbrace{\frac{(n-1)(n+1)}{n+7} - \frac{8(n-9)}{11}} + \underbrace{\frac{64(n+1)^2}{3(n+7)}\left(\frac{n-1}{n+7}\right)^{\frac{n+1}{2}}}.$$

  1. The inequality restricted to the overbraced terms is equivalent to $7n^3 + 21n^2 - 2711n - 12325 \geq 0$, which holds for all integers $n \geq 134$.
  2. The inequality restricted to the underbraced terns is equivalent to $\left(\frac{n+7}{n-1}\right)^{\frac{n+1}{2}} \geq \frac{320(n+9)}{9n}$. Note that the function $\frac{320(n+9)}{9n}$ is decreasing in n, while the function $\left(\frac{n+7}{n-1}\right)^{\frac{n+1}{2}}$ is increasing in n. Since $n \geq 134$, the claim follows.

- $\underline{r = \left\lfloor \frac{n-3}{2} \right\rfloor - 2}$: Note that $K_{\lfloor \frac{n-3}{2} \rfloor - 2} \geq L_{\lfloor \frac{n-3}{2} \rfloor - 2}$ if and only if

$$\overbrace{\frac{3n(n+1)}{4(n+7)}} + \underbrace{\frac{n(n+1)}{4(n+7)}} \geq \overbrace{\frac{(n-1)(n+1)}{n+5} - \frac{7(n-7)}{9}} + \underbrace{\frac{9(n+1)^2}{n+5}\left(\frac{n-1}{n+5}\right)^{\frac{n+1}{2}}}.$$

  1. The inequality restricted to the overbraced terms is equivalent to $19n^3 + 50n^2 - 1201n - 6608 \geq 0$. This holds for all $n \geq 134$.
  2. The inequality restricted to the underbraced terns is equivalent to $\left(\frac{n+5}{n-1}\right)^{\frac{n+1}{2}} \geq \frac{36(n+7)}{n}$. Note that the function $\frac{36(n+7)}{n}$ is decreasing in n, while the function $\left(\frac{n+5}{n-1}\right)^{\frac{n+1}{2}}$ is increasing in n. Since $n \geq 134$, the claim follows.

- $\underline{r = \left\lfloor \frac{n-3}{2} \right\rfloor - 1}$: Note that $K_{\lfloor \frac{n-3}{2} \rfloor - 1} \geq L_{\lfloor \frac{n-3}{2} \rfloor - 1}$ if and only if

$$\overbrace{\frac{n(n+1)}{5(n+5)}} + \underbrace{\frac{4n(n+1)}{5(n+5)}} \geq \overbrace{\frac{(n-1)(n+1)}{n+3} - \frac{6(n-5)}{7}} + \underbrace{\frac{4(n+1)^2}{n+3}\left(\frac{n-1}{n+3}\right)^{\frac{n+1}{2}}}.$$

  1. The inequality restricted to the overbraced terms is equivalent to the inequality $2n^3 - 57n^2 - 694n - 2075 \geq 0$. The latter follows from the facts that *(i)* $2n^3 - 57n^2 = n^2(n-57) > 0$ and *(ii)* $n^3 - 694n - 2075 > 0$, for $n \geq 134$.
  2. The inequality restricted to the underbraced terms is equivalent to the inequality $\left(\frac{n+3}{n-1}\right)^{\frac{n+1}{2}} \geq \frac{5(n+5)}{n}$. Note that the function $\frac{5(n+5)}{n}$ is decreasing in n, while the function $\left(\frac{n+3}{n-1}\right)^{\frac{n+1}{2}}$ is increasing in n. Since $n \geq 134$, the claim follows.

- $r = \left\lfloor \frac{n-3}{2} \right\rfloor$: Note that $K_{\lfloor \frac{n-3}{2} \rfloor} \geq L_{\lfloor \frac{n-3}{2} \rfloor}$ if and only if

$$\overbrace{\frac{n(n+1)}{5(n+3)}} + \underbrace{\frac{4n(n+1)}{5(n+3)}} \quad \geq \quad \overbrace{2} + \underbrace{2(n+1)\left(\frac{n-1}{n+1}\right)^{\frac{n+1}{2}}}.$$

  1. The inequality restricted to the overbraced terms holds for all $n \geq 134$.
  2. For the inequality restricted to the underbraced terms, note that the function $\frac{2n}{5(n+3)}$ is increasing in n. Since $n \geq 135$, it suffices to show that $\frac{27}{69} \geq \left(\frac{n-1}{n+1}\right)^{\frac{n+1}{2}}$ or equivalently that $\ln(n+1) - \ln(n-1) - \frac{2}{n+1}\ln\left(\frac{69}{27}\right) \geq 0$. Note that the derivative of the LHS expression is $\frac{\left(-2+\ln\left(\frac{69}{27}\right)^2\right)n - \ln\left(\frac{69}{27}\right)^2 - 2}{(n-1)(n+1)^2} < 0$. Hence, the expression is decreasing in n. The claim follows since $\lim_{n\to\infty}\left(\ln(n+1) - \ln(n-1) - \frac{1}{n+1}\ln\left(\frac{69}{27}\right)^2\right) = 0$.

C.4 Lemma 5.4

Case (1):

Note that

$$\frac{\prod_{i=0}^{r}(n-2r+1+2i)}{\prod_{i=0}^{r}(n-r+i)} \frac{(n-2r-1)^{-r}}{(n-r-1)^{-r}} = \begin{cases} \frac{(n-2)(n+1)}{n(n-3)}, & r=1 \\ \frac{(n+1)(n-3)^3}{n(n-2)(n-5)^2}, & r=2 \\ \frac{(n-5)(n+1)(n-4)^3}{n(n-2)(n-7)^3}, & r=3. \end{cases}$$
$$> 1.$$

Hence, it suffices to show that $\left(\frac{n-2r-1}{n-r-1}\right)^{\frac{n-1}{2}}\left(\frac{n-1}{n-r-1}\right)^{\frac{n+1}{2}} \geq 1$, or, equivalently, that

$$(n-1)\ln(n-2r-1) - (n-1)\ln(n-r-1) + (n+1)\ln(n-1) - (n+1)\ln(n-r-1) \quad \geq \quad 0.$$

Clearly, the partial derivative in n of the LHS expression is

$$\begin{aligned} &\frac{n-1}{n-2r-1} + \ln(n-2r-1) - \frac{n-1}{n-r-1} - \ln(n-r-1) + \frac{n+1}{n-1} + \ln(n-1) - \frac{n+1}{n-r-1} - \ln(n-r-1) \\ = \quad &(n-1)\left(-\frac{r}{(n-r-1)(n-2r-1)}\right) + (n+1)\left(-\frac{r}{(n-1)(n-r-1)}\right) + \ln\left(\frac{n^2-2n-2nr+2r+1}{n^2-2n-2nr+r^2+2r+1}\right), \end{aligned}$$

which is non-positive. This implies that the LHS expression is non-increasing in n. The claim follows from the fact that the expression approaches 0 as n approaches ∞.

Case (2):

Define

$$\begin{aligned} K_r &= \frac{\prod_{i=0}^{r}(n-2r+1+2i)}{\prod_{i=0}^{r}(n-r+i)} \cdot \frac{(n-1)(n-r-1)^{r-1}}{(n-2r-1)^r} \\ &= \begin{cases} \frac{(n-1)(n+1)}{n(n-3)}, & r=1 \\ \frac{(n-1)(n+1)(n-3)^2}{n(n-2)(n-5)^2} < \frac{(n-3)(n-1)(n+1)}{n(n-5)^2}, & r=2 \\ \frac{(n-5)(n-1)(n+1)(n-4)^2}{n(n-2)(n-7)^3} < \frac{(n-1)(n+1)(n-4)^2}{n(n-7)^3}, & r=3. \end{cases} \end{aligned}$$

and

$$L_r = \left(\frac{n-2r-1}{n-r-1}\right)^{\frac{n-1}{2}}\left(\frac{n-1}{n-r-1}\right)^{\frac{n-1}{2}} = \left(\frac{n^2-2n(r+1)+2r+1}{n^2-2n(r+1)+r^2+2r+1}\right)^{\frac{n-1}{2}} < 1.$$

So, we need to prove that $(n-1)\left(\frac{n}{n-1}\right)^n\sqrt{\frac{n}{6}}(1-K_rL_r)+\frac{r(n-r)}{2}\geq\frac{r(n+1)}{4(n-r-1)}$. Note that $\frac{r(n+1)}{4(n-r-1)}\leq 1$ (since $r\leq 3$ and $n\geq 19$). Hence it suffices to prove that $(n-1)\left(\frac{n}{n-1}\right)^n\sqrt{\frac{n}{6}}(1-K_rL_r)+\frac{r(n-r)}{2}\geq 1$. This is equivalent to $L_r\leq\frac{1}{n-1}\left(\frac{n-1}{n}\right)^n\sqrt{\frac{6}{n}}\left(\frac{r(n-r)}{2}-1\right)\frac{1}{K_r}+\frac{1}{K_r}$. Since $L_r<1$, it suffices to show that $\frac{1}{n-1}\left(\frac{n-1}{n}\right)^n\sqrt{\frac{6}{n}}\left(\frac{r(n-r)}{2}-1\right)\frac{1}{K_r}+\frac{1}{K_r}\geq 1$, or that $n\ln\left(\frac{n-1}{n}\right)\geq\ln\left(\sqrt{\frac{n}{6}}\frac{2(n-1)}{(r(n-r)-2)}(K_r-1)\right)$. We observe that the LHS expression is increasing in n. Hence, it suffices to show that $135\ln\left(\frac{135}{134}\right)\geq\ln\left(\sqrt{\frac{n}{6}}\frac{2(n-1)}{(r(n-r)-2)}(K_r-1)\right)$. We consider subcases.

- $\underline{r=1}$: Note that

$$\ln\left(\sqrt{\frac{n}{6}}\frac{2(n-1)}{n-3}\left(\frac{(n-1)(n+1)}{n(n-3)}-1\right)\right)=\ln\left(2\sqrt{n}(n-1)(3n-1)\right)-\ln\left(n\sqrt{6}(n-3)^2\right).$$

  We observe that this is a decreasing function in n whose value at $n=135$ is $>135\ln\left(\frac{135}{134}\right)$.

- $\underline{r=2}$: Note that

$$\ln\left(\sqrt{\frac{n}{6}}\frac{n-1}{n-3}\left(\frac{(n-3)(n-1)(n+1)}{n(n-5)^2}-1\right)\right)=\ln\left(\sqrt{n}(n-1)(7n^2-26n+3)\right)-\ln\left(n\sqrt{6}(n-3)(n-5)^2\right).$$

  We observe that this is a decreasing function in n whose value at $n=135$ is $>135\ln\left(\frac{135}{134}\right)$.

- $\underline{r=3}$: Note that

$$\begin{aligned}&\ln\left(\sqrt{\frac{n}{6}}\frac{2(n-1)}{3n-11}\left(\frac{(n-1)(n+1)(n-4)^2}{n(n-7)^3}-1\right)\right)\\ =\;&\ln\left(2\sqrt{n}(n-1)(13n^3-132n^2+351n-16)\right)-\ln\left(n\sqrt{6}(3n-11)(n-7)^3\right).\end{aligned}$$

  We observe that this is a decreasing function in n whose value at $n=135$ is $>135\ln\left(\frac{135}{134}\right)$. Since $n\geq 135$ the claim follows.

C.5 Lemma 5.5

$$
\begin{aligned}
& \binom{n-r-1}{\frac{n-1}{2}-r} p^{\frac{n-1}{2}-r} q^{\frac{n+1}{2}} \exp\left(-\left(\frac{1}{12(n-r-1)}-\frac{1}{6n-5}-\frac{1}{6n-12r-5}\right)\right) \\
= \quad & q\binom{n-r-1}{\frac{n-1}{2}-r} p^{\frac{n-1}{2}-r} q^{\frac{n-1}{2}} \exp\left(-\left(\frac{1}{12(n-r-1)}-\frac{1}{6n-5}-\frac{1}{6n-12r-5}\right)\right) \\
\leq \quad & q \max_{0\leq x\leq 1} \binom{n-r-1}{\frac{n-1}{2}-r} x^{\frac{n-1}{2}-r}(1-x)^{\frac{n-1}{2}} \exp\left(-\left(\frac{1}{12(n-r-1)}-\frac{1}{6n-5}-\frac{1}{6n-12r-5}\right)\right) \\
\overset{(2.4)}{=} \quad & q\binom{n-r-1}{\frac{n-1}{2}-r} \left(\frac{n-1}{2}-r\right)^{\frac{n-1}{2}-r} (n-r-1)^{-n+r+1} \left(\frac{n-1}{2}\right)^{\frac{n-1}{2}} \\
& \exp\left(-\left(\frac{1}{12(n-r-1)}-\frac{1}{6n-5}-\frac{1}{6n-12r-5}\right)\right) \\
= \quad & q\frac{(n-r-1)!}{\left(\frac{n-1}{2}\right)!\left(\frac{n-1}{2}-r\right)!} \left(\frac{n-1}{2}-r\right)^{\frac{n-1}{2}-r} (n-r-1)^{-n+r+1} \left(\frac{n-1}{2}\right)^{\frac{n-1}{2}} \\
& \exp\left(-\left(\frac{1}{12(n-r-1)}-\frac{1}{6n-5}-\frac{1}{6n-12r-5}\right)\right) \\
\overset{(2.1)}{\leq} \quad & q\frac{1}{\sqrt{2\pi}} \frac{(n-r-1)^{n-r-\frac{1}{2}}}{\left(\frac{n-1}{2}\right)^{\frac{n}{2}}\left(\frac{n-1}{2}-r\right)^{\frac{n}{2}-r}} \left(\frac{n-1}{2}-r\right)^{\frac{n-1}{2}-r} (n-r-1)^{-n+r+1} \left(\frac{n-1}{2}\right)^{\frac{n-1}{2}} \\
\overset{(3.4)}{=} \quad & \frac{n-1}{2(n-r-1)}\frac{1}{\sqrt{2\pi}} \frac{(n-r-1)^{n-r-\frac{1}{2}}}{\left(\frac{n-1}{2}\right)^{\frac{n}{2}}\left(\frac{n-1}{2}-r\right)^{\frac{n}{2}-r}} \left(\frac{n-1}{2}-r\right)^{\frac{n-1}{2}-r} (n-r-1)^{-n+r+1} \left(\frac{n-1}{2}\right)^{\frac{n-1}{2}} \\
= \quad & \frac{n-1}{\sqrt{2\pi(n-1)(n-r-1)(n-2r-1)}}.
\end{aligned}
$$

D Proofs from Section 6

D.1 Lemma 6.1

Clearly,

$$
\begin{aligned}
\frac{1}{6(n-r-2)}-\frac{1}{12(n-r-2)}-\frac{1}{6(n-2r)-11}-\frac{1}{6n-11} \quad & \leq \quad -\frac{1}{12(n-r-2)}-\frac{1}{6n-11} \\
& < \quad -\frac{1}{12n-1}-\frac{2}{12n-1} \\
& = \quad -\frac{1}{12n-1}-\frac{36n+1}{144n^2-1}+\frac{1}{12n+1} \\
& < \quad -\frac{1}{12n-1}-\frac{1}{4n}+\frac{1}{12n+1} \\
& < \quad -\frac{1}{3n}+\frac{1}{12n+1}.
\end{aligned}
$$

D.2 Lemma 6.2

We first prove the following two intermediate milestones.

Lemma D.1 *For even* $n \geq 134$ *and* $5 \leq r \leq \left\lfloor \frac{n-3}{2} \right\rfloor - 4$, $\frac{(n-2)(n-r-2)(n-2r-2)}{n-r} - n(n-r) \geq -r\sqrt{6}(n-r-2)$.

Proof: The claim is equivalent to $(r(\sqrt{6}-1)-6)n^2 + (r^2(-2\sqrt{6}+1)+r(-2\sqrt{6}+12))n + \sqrt{6}r^3 \geq 0$. The first and the second derivative in n, of the LHS function $f(n)$, namely $f'(n)$ and $f''(n)$, are $2n(r(\sqrt{6}-1)-6)+$

$r^2(-2\sqrt{6}+1) + r(-2\sqrt{6}+12)$ and $2r(\sqrt{6}-1) - 12$, respectively. Clearly, $f''(n) \geq 0$, for $r \geq 5$; hence, $f'(n)$ is increasing in n. It is verified that $f'(134) \geq 0$; hence, $f'(n) \geq 0$. It follows that $f(n)$ is increasing in n. Since $f(134) \geq 0$, the claim follows. ■

Lemma D.2 *For even* $n \geq 134$ *and* $5 \leq r \leq \left\lfloor \frac{n-3}{2} \right\rfloor - 4$,

$$\frac{(n-2)(n-r-2)(n-2r-2)}{n-r} + \sqrt{n(n-2)(n-r-2)(n-2r-2)} \quad \geq \quad 4\sqrt{n}(n-r-1).$$

Proof: The claim is equivalent to

$$\overbrace{(n-2)(n-r-2)(n-2r-2)}\underbrace{\left((n-2)(n-r-2)(n-2r-2)+n(n-r)^2\right)} \quad \geq \quad \overbrace{16(n-r-1)^2}\underbrace{n(n-r)^2}.$$

Since the underbraced terms satisfy the inequality, it suffices to show that the overbraced terms do. This is equivalent to $n^3 - (3r+22)n^2 + (2r^2+44r+44)n - 20r^2 - 44r - 24 \geq 0$. The first derivative (in n) of the LHS function $f(n)$ is $f'(n) = 3n^2 - 2(3r+22)n + 2r^2 + 44r + 44$. Let Δ be the discriminant of $f'(n)$. Since $\Delta = 4(3r+22)^2 - 12(2r^2+44r+44) = 12r^2 + 1408 > 0$ there are two distinct roots $n_1 \leq n_2$. Recall that $r \leq \left\lfloor \frac{n-3}{2} \right\rfloor - 4 = \frac{n-12}{2}$, which implies that $n \geq 2r+12$. We calculate that $n_2 \leq 2r+12$ if and only if $r^2 + 14r - 26 \geq 0$, which holds. It follows that $f'(n) \geq 0$ for the given range of values of r, so that $f(n)$ is increasing and is verified that $f(134) \geq 0$ for the given range of values of r. It follows that $f(n) \geq 0$ for the given range of values of r. ■

We continue to complete the proof of Lemma 6.2. Note that the claim is equivalent to

$$\sqrt{\frac{n}{6}}\left(1 - \sqrt{\frac{n(n-r)^2}{(n-2)(n-r-2)(n-2r-2)}}\right) \quad \geq \quad -\frac{r}{4}\left(\frac{n-r-2}{n-r-1}\right).$$

Since $1 - \sqrt{a} = \dfrac{1-a}{1+\sqrt{a}}$ we get that

$$1 - \sqrt{\frac{n(n-r)^2}{(n-2)(n-r-2)(n-2r-2)}} = \frac{\dfrac{(n-2)(n-r-2)(n-2r-2)}{n-r} - n(n-r)}{\dfrac{(n-2)(n-r-2)(n-2r-2)}{n-r} + \sqrt{n(n-2)(n-r-2)(n-2r-2)}}.$$

The claim follows immediately from Lemma D.1 and Lemma D.2.

D.3 Lemma 6.3

Clearly,

$$\begin{aligned}
& \frac{1}{12(n-r-1)} - \frac{1}{6n-5} - \frac{1}{6n-12r-5} - \frac{1}{12n+1} + \frac{1}{3n-3} \\
< \quad & -\frac{1}{6n-5} - \frac{1}{12n-24r-10} - \frac{1}{12n+1} + \frac{1}{3n-3} \\
\overset{r\geq 3}{<} \quad & -\frac{1}{6n-5} - \frac{1}{12n-82} - \frac{1}{12n+1} + \frac{1}{3n-3} \\
= \quad & \frac{-954n^2 - 1335n + 1379}{(6n-5)(12n-82)(12n+1)(3n-3)} \\
< \quad & 0.
\end{aligned}$$

D.4 Lemma 6.4

First, we prove two intermediate technical claims.

Lemma D.3 *For odd $n \geq 135$ and $4 \leq r \leq \left\lfloor \frac{n-3}{2} \right\rfloor - 4$,* $\frac{n(n-1)(n-r-1)(n-2r-1)}{(n+1)(n-r)} - (n+1)(n-r) \geq -r\sqrt{6}(n-r-2)$.

Proof: The claim is equivalent to $(r(\sqrt{6}-1)-5)n^2 + (r^2(-2\sqrt{6}+1) + r(-\sqrt{6}+10))n + \sqrt{6}r^3 - 4r^2 - r(2\sqrt{6}+1) - 1 \geq 0$. The first and second derivatives of the LHS function $f(n)$ are $f'(n) = 2n(r(\sqrt{6}-1)-5) + r^2(-2\sqrt{6}+1) + r(-\sqrt{6}+10) \geq 0$ and $f''(n) = 2r(\sqrt{6}-1) - 10$. Clearly, $f''(n) \geq 0$, for $r \geq 5$; hence, $f'(n)$ is increasing in n. It is verified that $f'(135) \geq 0$; hence, $f'(n) \geq 0$. It follows that $f(n)$ is increasing in n. Since $f(135) \geq 0$, the claim follows. ∎

Lemma D.4 *For odd $n \geq 135$ and $4 \leq r \leq \left\lfloor \frac{n-3}{2} \right\rfloor - 4$,*

$$\frac{n(n-1)(n-r-1)(n-2r-1)}{(n+1)(n-r)} + \sqrt{n(n-1)(n-r-1)(n-2r-1)} \quad \geq \quad 4\sqrt{n}(n-r-1).$$

Proof: The claim is equivalent to

$$\overbrace{(n-1)(n-2r-1)}\underbrace{\left(n(n-1)(n-r-1)(n-2r-1) + (n+1)^2(n-r)^2\right)} \quad \geq \quad \overbrace{16(n-r-1)}\underbrace{(n+1)^2(n-r)^2}.$$

Note that the underbraced terms satisfy the inequality trivially. So, it suffices to prove that the overbraced terms do. This is equivalent to $n^2 - (2r+18)n + 18r + 17 \geq 0$. Clearly, $f'(n) = 2(n-r-9) \geq 0$ (for the given range of values of r); hence, $f(n)$ is increasing in n. It is verified that $f(135) \geq 0$; hence, $f(n) \geq 0$. ∎

We continue to complete the proof of Lemma 6.4. Note that the claim is equivalent to

$$\sqrt{\frac{n}{6}}\left(1 - \sqrt{\frac{(n+1)^2(n-r)^2}{n(n-1)(n-r-1)(n-2r-1)}}\right) \quad \geq \quad -\frac{r}{4}\left(\frac{n-r-2}{n-r-1}\right).$$

Since $1 - \sqrt{a} = \frac{1-a}{1+\sqrt{a}}$ we get that

$$1 - \sqrt{\frac{(n+1)^2(n-r)^2}{n(n-1)(n-r-1)(n-2r-1)}} = \frac{\frac{n(n-1)(n-r-1)(n-2r-1)}{(n+1)(n-r)} - (n+1)(n-r)}{\frac{n(n-1)(n-r-1)(n-2r-1)}{(n+1)(n-r)} + \sqrt{n(n-1)(n-r-1)(n-2r-1)}}.$$

The claim follows now from Lemma D.3 and Lemma D.4.