# A Cost Mechanism for Fair Pricing of Resource Usage

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joint work with

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Accepted to the 1st Workshop on Internet and Network Economics (WINE 2005)

# Talk Outline

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# Agents and Resources

- $M = \{ 1, 2, \ldots, m \}$ identical resources
- $N = \{ 1, 2, \ldots, n \}$ agents
- Each agent $i$ has demand $w_i \in \mathrm{R}_+$

  Denote $\mathbf{w}$ the corresponding $n \times 1$ demand vector.

  Denote $W = \sum_{i=1}^{n} w_i$ .

# Strategies and Assignments

- A pure strategy for agent $i$ is some specific resource.

  A mixed strategy for agent $i$ is a probability distribution on the set of pure strategies.

- A pure assignment $\mathbf{L} \in M^n$ is a collection of pure strategies, one per agent.

  A mixed assignment $\mathbf{P} \in \mathrm{R}^{m \times n}$ is a collection of mixed strategies, one per agent.
  - i.e. $p_i^j$ is the probability that agent $i$ selects resource $j$.
  - The support of agent $i$ is $S_i = \{ j \in M : p_i^j > 0 \}$.

# Resource Cost

Fix a pure assignment $\mathbf{L} = < l_1, l_2, \ldots, l_n >$.

- The resource demand on resource $j$ is

$$W^j = \sum_{k \in N : l_k = j} w_k \ .$$

- The resource congestion on resource $j$ is

$$n^j = \sum_{k \in N : l_k = j} 1 \ .$$

- The Resource Cost on resource $j$ is $\mathrm{RC}^j = \dfrac{W^j}{n^j}$.

# Individual Cost

Fix a pure assignment $\mathbf{L} = < l_1, l_2, \ldots, l_n >$.

The Individual Cost for agent $i$ is the Resource Cost of the resource she chooses:

$$\mathrm{IC}_i = \frac{W^{l_i}}{n^{l_i}}.$$

# Expected Individual Cost

Now fix a mixed assignment $\mathbf{P}$.

- The Conditional Expected Individual Cost $\mathrm{IC}_i^j$ of agent $i$ on resource $j$ is the conditional expectation of the Individual Cost of agent $i$ had she been assigned to resource $j$.

- The Expected Individual Cost of agent $i$ is

$$\mathrm{IC}_i = \sum_{j \in M} p_i^j \cdot \mathrm{IC}_i^j \,.$$

# Pure Nash Equilibria

The pure assignment $\mathbf{L} = \langle l_1, l_2, \ldots, l_n \rangle$ is a pure Nash equilibrium if, for all agents $i$, the Individual Cost $\mathrm{IC}_i$ is minimized (given the pure strategies of the other agents).

$ICi$

Thus, in a pure Nash equilibrium, no agent can unilaterally improve her own Individual Cost.

# Mixed Nash Equilibria

The mixed assignment $\mathbf{P}$ is a mixed Nash equilibrium if, for all agents $i$, the Expected Individual Cost $\mathrm{IC}_i$ is minimized (given the mixed strategies of the other agents), or equivalently, for all agents $i$,

$$\mathrm{IC}_i^j = \min_{k \in M} \mathrm{IC}_i^k \quad \forall j : p_i^j > 0$$

$$\mathrm{IC}_i^j \geq \min_{k \in M} \mathrm{IC}_i^k \quad \forall j : p_i^j = 0$$

$\mathbf{P}$ is a fully mixed Nash equilibrium if

$$p_i^j > 0 \quad \forall i \in N, \forall j \in M.$$

# The Price of Anarchy

- Let $\mathbf{w}$ be a demand vector and $\mathbf{P}$ be a Nash equilibrium. The Social Cost is defined as

$$\mathrm{SC}(\mathbf{w},\mathbf{P}) = \mathrm{E}_{\mathbf{P}}\left(\max_{j\in M} \mathrm{RC}^{j}\right).$$

- Let $\mathbf{w}$ be a demand vector. The Optimum is defined as

$$\mathrm{OPT}(\mathbf{w}) = \min_{\mathbf{L}\in M^{n}} \max_{j\in M} \mathrm{RC}^{j}.$$

- The Price of Anarchy is defined to be

$$\mathrm{PA} = \max_{\mathbf{w},\mathbf{P}} \frac{\mathrm{SC}(\mathbf{w},\mathbf{P})}{\mathrm{OPT}(\mathbf{w})}.$$

# The Diffuse Price of Anarchy

Assume demands are chosen according to some joint probability distribution $D$, which comes from some (known) class $\Delta$ of possible distributions.

We define the Diffuse Price of Anarchy to be

$$\mathrm{DPA}_{\boldsymbol{\Delta}} = \max_{D \in \boldsymbol{\Delta}} \left( \mathrm{E}_D \left( \max_{\mathbf{P}} \frac{\mathrm{SC}(\mathbf{w}, \mathbf{P})}{\mathrm{OPT}(\mathbf{w})} \right) \right).$$

# Motivation

- The proposed cost mechanism is used in real life by:
  - Internet service providers
  - Operators in telecommunication networks
  - Restaurants offering an “all-you-can-eat” buffet

- The cost mechanism is *fair* since
  - No resource makes profit
  - Agents sharing the same resource are treated equally

# The Optimum

**Proposition.**

For any demand vector $\mathbf{w}$, $\mathrm{OPT}(\mathbf{w}) = \frac{W}{n}$.

**Proof**

Fix $\mathbf{w}$. The pure assignment where all agents are assigned to the same resource achieves Social Cost $W/n$. Hence

$$\mathrm{OPT}(\mathbf{w}) \leq \frac{W}{n}.$$

# The Optimum

**Proof** (continued)

Consider an arbitrary assignment $\mathbf{L}$ and let $k$ be such that

$$\mathrm{SC}(\mathbf{w},\mathbf{L}) = \frac{W^k}{n^k}.$$

Then, by definition of the Social Cost,

$$\frac{W^j}{n^j} \leq \frac{W^k}{n^k} \Leftrightarrow \frac{n^j}{n^k} \geq \frac{W^j}{W^k}$$

for any resource $j$ such that $n^j > 0$.

# The Optimum

**Proof** (continued)

Summing up over all such resources yields

$$\sum_{j:n^j>0} \frac{n^j}{n^k} \geq \sum_{j:n^j>0} \frac{W^j}{W^k} \Rightarrow \frac{n}{n^k} \geq \frac{W}{W^k} \Rightarrow \frac{W^k}{n^k} \geq \frac{W}{n}.$$

By choice of resource $k$, and since $\mathbf{L}$ was chosen arbitrarily, the above inequality implies that

$$\mathrm{SC}(\mathbf{w},\mathbf{L}) \geq \frac{W}{n} \Rightarrow \min_{\mathbf{L}} \mathrm{SC}(\mathbf{w},\mathbf{L}) \geq \frac{W}{n} \Rightarrow \mathrm{OPT}(\mathbf{w}) \geq \frac{W}{n}.$$

# Pure Nash Equilibria: Inexistence

**Theorem [Inexistence of pure Nash equilibria]**

There is a pure Nash equilibrium if and only if all demands are identical.

**Proof ( if )**

Let $w_i = w \ \ \forall i \in N.$

Then, in any pure assignment $\mathbf{L}$,

$$\mathrm{RC}^j = w \ \ \forall j \in M \quad \Rightarrow \quad \mathrm{IC}_i = w \ \ \forall i \in N.$$

Hence any pure assignment is a pure Nash equilibrium.

# Pure Nash Equilibria: Inexistence

## Proof ( only if )

Assume now that there is a pure Nash equilibrium $\mathbf{L}$.

For each resource $j$, denote $w_1^j, w_2^j, \ldots, w_{n^j}^j$

the demands assigned to resource $j$.

So, $\sum_{k=1}^{n^j} w_k^j = W^j$.

# Pure Nash Equilibria: Inexistence

## Proof ( only if , continued )

Fix now a resource $j$ with $n^j > 0$.

Since $\mathbf{L}$ is a Nash equilibrium, for each agent $k$ assigned to resource $j$ and for each resource $l \neq j$ it holds that

$$\mathrm{IC}_k^j \leq \mathrm{IC}_k^l \Rightarrow \frac{W^j}{n^j} \leq \frac{W^l + w_k^j}{n^l + 1}.$$

Rearranging terms yields $n^l \cdot W^j \leq n^j \cdot W^l$

thus implying that $\dfrac{W^j}{n^j} = \dfrac{W^l}{n^l} \quad \forall j, l \in M : n^j, n^l > 0.$

# Pure Nash Equilibria: Inexistence

## Proof ( only if , continued )

Note that for each agent $k \in \{1, 2, \ldots, n^j\}$,

$$\frac{w_k^j}{n^l + 1} \geq \frac{W^j}{n^j} - \frac{W^l}{n^l + 1}.$$

- Assume that $n^l = 0$. Then

$$w_k^j \geq \frac{W^j}{n^j}.$$

# Pure Nash Equilibria: Inexistence

## Proof ( only if , continued )

- Assume that $n^l > 0$. Then

$$\frac{w_k^j}{n^l + 1} \geq \frac{W^j}{n^j} - \frac{W^l}{n^l + 1} = \frac{W^l}{n^l} - \frac{W^l}{n^l + 1} \Rightarrow$$

$$w_k^j \geq \frac{W^l}{n^l} = \frac{W^j}{n^j}.$$

# Pure Nash Equilibria: Inexistence

## Proof ( only if , continued )

So, in all cases, $w_k^j \geq \frac{W^j}{n^j}$ for all $k \in \{1, \ldots, n^j\}$, implying

$$w_1^j = w_2^j = \cdots = w_{n^j}^j = \frac{W^j}{n^j} \quad \forall j \in M : n^j > 0.$$

Since however $\frac{W^j}{n^j} = \frac{W^l}{n^l} \quad \forall j, l : n^j, n^l > 0,$

it follows that all demands are identical.

# Fully Mixed Nash Equilibria: Existence

**Theorem [Existence of fully mixed Nash equilibria]**

There is always a fully mixed Nash equilibrium.

## Proof

Consider the fully mixed assignment $\mathbf{F}$ with

$$f_i^j = \frac{1}{m} \quad \forall i \in N, \forall j \in M.$$

We will show that $\mathbf{F}$ is a Nash equilibrium.

# Fully Mixed Nash Equilibria: Existence

**Proof** (continued)

In the mixed assignment $\mathbf{F}$, $\forall i \in N, \forall j \in M$

$$\mathrm{IC}_i^j = w_i\left(1-\frac{1}{m}\right)^{n-1} + \sum_{k=2}^{n}\frac{1}{k}\left(\frac{1}{m}\right)^{k-1}\left(1-\frac{1}{m}\right)^{n-k}\left(\binom{n-1}{k-1}w_i + \binom{n-2}{k-2}W_{-i}\right)$$

i.e. independent of $j$, so $\mathbf{F}$ is a fully mixed NE.

# Fully Mixed Nash Equilibria: Uniqueness

**Theorem**

The fully mixed Nash equilibrium $\mathbf{F}$ is the unique Nash equilibrium in the case of 2 agents with non-identical demands.

**Proof**

Consider an arbitrary Nash equilibrium $\mathbf{P}$.

Let $S_1, S_2$ be the support of agent 1, 2 respectively.

W.l.o.g., assume that $w_1 > w_2$.

# Fully Mixed Nash Equilibria: Uniqueness

**Proof** (continued)

- Suppose $S_1 \cap S_2 = \varnothing$. Then, for any $l \in S_2$,

  $$\mathrm{IC}_1 = w_1 > w_1(1 - p_2^l) + \frac{w_2 - w_1}{2} p_2^l = \mathrm{IC}_1^l,$$

  a contradiction to the Nash equilibrium.

- Let $j \in S_1 \cap S_2$. Then

  $$\mathrm{IC}_1 = w_1(1 - p_2^j) + \frac{w_1 + w_2}{2} p_2^j < w_1$$ and

  $$\mathrm{IC}_2 = w_2(1 - p_1^j) + \frac{w_1 + w_2}{2} p_1^j > w_2.$$

# Fully Mixed Nash Equilibria: Uniqueness

**Proof** (continued)

- Assume $\exists k \in S_1 \setminus S_2$. Then $\mathrm{IC}_1^k = w_1 > \mathrm{IC}_1$, a contradiction.

- Assume $\exists k \in S_2 \setminus S_1$. Then $\mathrm{IC}_2^k = w_2 < \mathrm{IC}_2$, a contradiction.

Hence $S_1 = S_2$.

- Assume $\exists k \notin S_1$. Then $\mathrm{IC}_2^k = w_2 < \mathrm{IC}_2$, a contradiction.

Hence $S_1 = S_2 = M$.

# Fully Mixed Nash Equilibria: Uniqueness

**Proof** (continued)

Now fix $j, k \in M$. Then

$$\mathrm{IC}_1^j = \mathrm{IC}_1^k \iff p_2^j = p_2^k \iff p_2^j = \frac{1}{m} \; \forall j \in M \quad \text{and}$$

$$\mathrm{IC}_2^j = \mathrm{IC}_2^k \iff p_1^j = p_1^k \iff p_1^j = \frac{1}{m} \; \forall j \in M.$$

Hence $\mathbf{P}=\mathbf{F}$.

# The Price of Anarchy: Lower Bound

**Theorem**

$$\mathrm{PA} \geq \frac{n}{2e}.$$

**Proof**

First observe that $\mathrm{SC}(\mathbf{w},\mathbf{F}) \geq \left(\frac{1}{m}\right)^n \left(m(m-1)^{n-1} w_1\right).$

Fix a demand vector $\mathbf{w}$ with $w_1 = \Theta\left(2^n\right)$ and $w_i = 1 \quad \forall i \neq 1.$

Then $\frac{w_1}{W} \geq \frac{1}{2}.$

# The Price of Anarchy: Lower Bound

**Proof** (continued)

Now

$$\mathrm{PA} = \max_{\mathbf{w},\mathbf{P}} \left( \frac{n}{W} \cdot \mathrm{SC}(\mathbf{w},\mathbf{P}) \right)$$

$$\geq \max_{\mathbf{w}} \left( \frac{n}{W} \cdot \mathrm{SC}(\mathbf{w},\mathbf{F}) \right)$$

$$\geq \max_{\mathbf{w}} \left( \frac{n w_1}{W} \cdot \left( \frac{m-1}{m} \right)^{n-1} \right)$$

$$\geq \frac{n}{2e}$$

for $m=n$, as needed.

# The Price of Anarchy: Upper Bounds

**Theorem**

Assume that $n=2$. Then $\mathrm{PA} < 2 - \frac{1}{m}$.

**Proof**

- If $w_1 = w_2 = w$ then
  - any assignment has Social Cost $w$,
  - Optimum equals to $w$,
  - hence $\mathrm{PA} = 1$.

# The Price of Anarchy: Upper Bounds

**Proof** (continued)

- Else, w.l.o.g., assume that $w_1 > w_2$.

  In that case, $\mathbf{F}$ is the unique Nash equilibrium.

  Observe that $\mathrm{SC}(\mathbf{w}, \mathbf{F}) = \left(\frac{1}{m}\right)^2 \left(m(m-1)w_1 + m\frac{w_1 + w_2}{2}\right)$.

  Since $\mathrm{OPT}(\mathbf{w}) = \frac{w_1 + w_2}{2}$, we can easily derive

  $\mathrm{PA} < 2 - \frac{1}{m}$, as needed.

# The Price of Anarchy: Upper Bounds

**Theorem**

$$\mathrm{PA} \le \frac{n \cdot w_1}{W}.$$

**Proof**

Fix any $\mathbf{w}$. For any pure assignment, $\frac{W^j}{n^j} \le w_1 \quad \forall j \in M : n^j > 0.$

Hence, for any Nash equilibrium $\mathbf{P}$,

$$\mathrm{SC}(\mathbf{w}, \mathbf{P}) = \mathrm{E}_{\mathbf{P}}\left( \max_j \frac{W^j}{n^j} \right) \le w_1 \quad \Rightarrow \quad \mathrm{PA} \le \frac{n \cdot w_1}{W}.$$

# The Diffuse Price of Anarchy

**Definition** [Bounded, Independent Probability Distributions]

The class of bounded, independent probability distributions $\Delta$ includes all probability distributions $D$ for which the demands $w_i$ are i.i.d. random variables such that:

- There is some parameter $\delta_D(n) < \infty$ such that

$$w_i \in \left[0, \delta_D(n)\right] \quad \forall i \in N.$$

- There is some (universal) constant $\ell_\Delta > 0$ such that

$$\frac{\delta_D(n)}{\mathrm{E}_D(w_i)} \leq \ell_\Delta \quad \forall i \in N.$$

# The Diffuse Price of Anarchy

**Theorem**

Consider the class $\Delta$ of bounded, independent probability distributions. Then:

1. $\mathrm{DPA}_\Delta \leq \dfrac{\ell_\Delta}{1 - \ell_\Delta \sqrt{1/2 \ln n}} + n \exp\left(-\dfrac{n}{\ln n}\right)$

2. $\lim_{n \to \infty} \mathrm{DPA}_\Delta \leq \ell_\Delta$.

# The Diffuse Price of Anarchy

## Proof

Follows from the subsequent version of *Hoeffding bound* :

Corollary

Let $w_1, \ldots, w_n$ be i.i.d. with $0 \le w_i \le \delta_D(n)$.

Denote $\overline{W} = \frac{1}{n}\sum_{i=1}^{n} w_i$ and $\overline{\mu} = \mathrm{E}\left(\overline{W}\right)$.

Then, for any $\varepsilon > 0$,

$$\Pr\left\{\overline{W} \le (1-\varepsilon)\overline{\mu}\right\} \le \exp\left(\frac{-2n\varepsilon^2\overline{\mu}^2}{\delta_D^2(n)}\right).$$

# The Diffuse Price of Anarchy

Consider the class $\Delta_{sym} \subseteq \Delta$ of bounded, independent, expectation-symmetric probability distributions:

$\forall D \in \Delta_{sym}$, each $w_i$ is distributed symmetrically around its expectation.

Hence $\ell_{\Delta_{sym}} = 2$ so the previous theorem implies:

**Corollary**

$$\lim_{n \to \infty} \mathrm{DPA}_{\Delta_{sym}} \leq 2\,.$$

# Discussion and Future Directions

## Summary

- Intuitive, pragmatic and fair cost mechanism for pricing the competitive usage of resources by selfish agents

## Future Research

- More general pricing functions
- Heterogeneous cases of selfish agents
- The proposed Diffuse Price of Anarchy could be of general applicability (e.g. in congestion games)

Thank you