Facets of the Fully Mixed Nash Equilibrium Conjecture*

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Abstract

In this work, we continue the study of the many facets of the *Fully Mixed Nash Equilibrium Conjecture*, henceforth abbreviated as the FMNE *Conjecture*, in selfish routing for the special case of n identical *users* over two (identical) parallel *links*. We introduce a new measure of *Social Cost*, defined as the expectation of the square of the maximum *congestion* on a link; we call it *Quadratic Maximum Social Cost*. A *Nash equilibrium* is a stable state where no user can improve her (expected) latency by switching her mixed strategy; a *worst-case* Nash equilibrium is one that maximizes Quadratic Maximum Social Cost. In the *fully mixed* Nash equilibrium, all *mixed strategies* achieve full support.

Formulated within this framework is yet another facet of the FMNE *Conjecture*, which states that the fully mixed Nash equilibrium is the worst-case Nash equilibrium. We present an extensive proof of the FMNE *Conjecture*; the proof employs a combination of combinatorial arguments and analytical estimations.

1 Introduction

1.1 Motivation and Framework

We continue the study of the multi-faceted *Fully Mixed Nash Equilibrium Conjecture* in selfish routing, originating from the work of Gairing *et al.* [7] and henceforth abbreviated as the FMNE *Conjecture*. Specifically, we look at a special case of the KP model for selfish routing due to Koutsoupias and Papadimitriou [14]; here, a collection of n (*unweighted*) *users* wish to each transmit one unit of traffic from *source* to *destination*, which are joined through *two* (identical) parallel *links*. The *congestion* on a link is the total number of users choosing it; each user makes her choice using a *mixed strategy*, which is a probability distribution over links; the *latency* on a link is identified with congestion.

In a *Nash equilibrium* [20, 21], no user can improve the expected congestion on the link she chooses by switching to a different (mixed) strategy. Originally considered by Kaplansky back in 1945 [13], *fully mixed Nash equilibria* have all their involved probabilities strictly positive; they were recently coined into the context of selfish routing by Mavronicolas and Spirakis [18]. Roughly speaking, the fully mixed Nash equilibrium maximizes the randomization employed in the mixed strategies of the players; so, it is a natural candidate as a vehicle for the investigation of the effect of randomization on the quality of Nash equilibria.

We introduce a new measure of *Social Cost* [14] for the evaluation of Nash equilibria. The new measure is taken to be the expectation of the square of the maximum congestion on a link; call it *Quadratic Maximum Social Cost* and denote it as QMSC. (The expectatiom is taken over all random choices of the users.) Note that the Quadratic Maximum Social Cost simultaneously generalizes the *Maximum Social Cost* (expectation of maximum latency) proposed in the seminal work of Koutsoupias and Papadimitriou [14] and denoted as MSC, and the *Quadratic Social Cost* (expectation of the sum of the squares of the latencies) proposed in [15] and denoted as QSC.

The motivation to consider the square of the latency comes from the application of scheduling transmissions among nodes on the Euclidean plane. The received power at a receiver is proportional to the power $-\delta$ of the (generalized) Euclidean distance from the sender; δ is the *path-loss exponent*, and it has been empirically assumed that $\delta \geq 2$ (cf. [12]). In many natural cases, the latency is proportional to the (generalized) Euclidean distance;* so, in such cases, the received power is proportional to the power $-\delta$ of the latency. So, investigating the power δ of the expected maximum latency for the initial case where $\delta = 2$ is expected to give insights about the optimization of received power in selfish transmissions.

*The proportionality constant may have to do with external conditions of the medium and the transmission power.

For any particular definition of Social Cost, a facet of the FMNE *Conjecture* states that the fully mixed Nash equilibrium maximizes the Social Cost among all Nash equilibria. The validity of the corresponding facet of the FMNE *Conjecture* implies that computing the worst-case Nash equilibrium (with respect to the particular Social Cost) for a given instance is trivial; it may also allow an approximation to the *Price of Anarchy* [14] in case where there is a polynomial time approximation algorithm for the Social Cost of the fully mixed Nash equilibrium (cf. [6, Section 7]).

1.2 Contribution

In this proposed framework, we formulate a corresponding facet of the FMNE *Conjecture*, called the *Quadratic Maximum Fully Mixed Nash Equilibrium Conjecture* and abbreviated as the QMFMNE *Conjecture*.

Conjecture 1.1 *The fully mixed Nash equilibrium maximizes the Quadratic Maximum Social Cost.*

As our main result, we present an extensive proof of this FMNE *Conjecture* using a combination of combinatorial arguments and analytical estimations (Theorem 4.1). The proof amounts to a delicate comparison of the Quadratic Maximum Social Cost of an *arbitrary* Nash equilibrium to that of the fully mixed Nash equilibrium.

In more detail, the Quadratic Maximum Social Costs of the fully mixed and the arbitrary Nash equilibrium are calculated in Lemmas 3.2 and 3.4, respectively. A more suitable form for the latter is established in Lemma 3.5, involving some parameters A, B and C; simpler expressions and bounds for A, B and C are derived in Claims 3.6, 3.7 and 3.8, respectively. These simpler forms imply a strict lower bound on the difference between the two Quadratic Maximum Social Costs, involving some new parameter D. Further on, we establish a strict lower bound on D by distinguishing the cases where n is even or odd; these give rise to two new parameters G and H, respectively, which are lower-bounded in Lemmas 4.2 and 4.3, respectively. Putting these together yields that the difference between the two Quadratic Maximum Social Costs is strictly positive.

The proof has required some very sharp analytical estimations for various combinatorial functions that entered the analysis; this provides strong evidence that the established inequality among the two compared Quadratic Maximum Social Costs is very tight. The employed analytical estimations may be applicable elsewhere; so, they are interesting on their own right.

Model assumptions	Social Cost	FMNE *Conjecture*?	Reference
$n = 2$, weighted users & identical links	MSC	$\surd$	[6]
unweighted users & related links	MSC	33.06	[6]
weighted users & identical links	MSC	$2h(1+\varepsilon)$, for $\varepsilon > 0$	[9]
$n = 2$, unweighted users & related links	MSC	$\surd$	[16]
$m = 2$, unweighted users & identical links	MSC	$\surd$	[16]
$m = 2, n = 2$ & unrelated links	MSC	$\surd$	[16]
$m = 2, n = 3$ & unrelated links	MSC	$\times$	[16]
unweighted users & identical links	QSC	$\surd$	[15]
unweighted users & links with (identical) non-constant and convex latency functions	$\Sigma_{\mathsf{IC}}\mathsf{SC}$	$\surd$	[9]
unweighted users & identical links	PSC	$\surd$	[10]
weighted users & player-specific links	$\Sigma_{\mathsf{IC}}\mathsf{SC}$	$\surd$	[11]
weighted users & player-specific links	$\mathsf{M}_{\mathsf{IC}}\mathsf{SC}$	$\surd$	[11]
weighted users & identical links	MSC	$\times$	[5]
weighted users with types & identical links	$\Sigma_{\mathsf{IC}}\mathsf{SC}$	$\surd$	[10]
weighted users with types & identical links	$\mathsf{M}_{\mathsf{IC}}\mathsf{SC}$	$\surd$	[10]

Table 1: The status of the studied facets of the FMNE *Conjecture*. A symbol $\surd$ (resp., $\times$) in the third column indicates that the facet of the FMNE *Conjecture* has been proven (resp., refuted) for the corresponding case. A number ρ in the third column indicates that an *approximate* version of the FMNE *Conjecture* has been shown: the Social Cost of an arbitrary Nash equilibrium is at most ρ times the one of the fully mixed. The symbol h denotes the factor by which the largest weight deviates from the average weight (in the case of weighted users). In the case of *related links*, latency is a linear function of the congestion on a link; in the (special) case of *identical links*, the linear function is identity, while in the (more general) case of *player-specific links*, the latency function is specific to each player (cf. [19]). In the (even more general) case of *unrelated links*, there is an additive contribution to latency on a link, which is both player-specific and *link-specific*. The *Polynomial Social Cost*, considered in [10] and denoted as PSC, is the (expectation of the) sum of polynomial functions of the latencies. The *Player-Average Social Cost*, considered in [9, 11] and denoted as $\Sigma_{\mathsf{IC}}\mathsf{SC}$, is the sum of Individual Costs of the players; the *Player-Maximum Social Cost*, considered in [9, 10] and denoted as $\mathsf{M}_{\mathsf{IC}}\mathsf{SC}$, is the maximum Individual Cost of a player.

1.3 Related Work and Comparison

The FMNE *Conjecture* was first stated in [7, Section 1]; there it was motivated by some initial observations in [6, Theorems 4.2 and 6.1]. The fully mixed Nash equilibrium and the (generic) FMNE *Conjecture* have attracted recently a lot of interest and attention; they both have been studied extensively in the last few years for a wide variety of theoretical models for selfish routing and Social Cost measures - see, e.g., [2, 4, 5, 6, 7, 9, 10, 11, 15, 16, 17]. The status of the studied facets of the FMNE *Conjecture* is summarized in Table 1.

The FMNE *Conjecture* has been proved for the Maximum Social Cost for the cases of *(i)* two (*unweighted*) users and non-identical but related links in [16, Theorems 2] and *(ii)* an arbitrary number of (unweighted) users and two (identical) links in [16, Theorems 4]. In fact, our estimation techniques significantly extend those for case *(ii)*; due to the increased complexity of the Quadratic Maximum Social Cost function (over Maximum Social Cost), far more involved estimations have been required in the present proof. Counterexamples to the FMNE *Conjecture* (for the Maximum Social Cost) appeared *(i)* for the case of *unrelated links* in [16, Theorem 7], and *(ii)* for the case of weighted users in [5, Theorem 4].

1.4 Road Map

The rest of this paper is organized as follows. Section 2 collects together some mathematical tools. The theoretical framework and some preliminary calculations are articulated in Section 3. Our main result is presented in Section 4. Some auxiliary estimations and technical claims are deferred to Sections 5 and 6, respectively. We conclude, in Section 7, with a discussion of our result and some open problems.

2 Mathematical Tools

2.1 Notation and Preliminaries

For any integer $n \geq 2$, denote $[n] = \{1, 2, \ldots, n\}$. For a random variable X following the probability distribution $\mathbb{P}$, denote as $\mathbb{E}_{\mathbb{P}}(X)$ the *expectation* of X; $X \sim \mathbb{P}$ signifies that X follows the distribution $\mathbb{P}$. For an integer n, the predicates $\mathsf{Even}(n)$ and $\mathsf{Odd}(n)$ will be 1 if and only if n is even or odd, respectively. For a number x, denote $\mathsf{exp}(x) = e^x$.

In our later proofs, we shall use the following identities between binomial coefficients which holds for all integers $n \geq 1$ and $k \leq n$: $\binom{n}{k} = \binom{n}{n-k}$, $k\binom{n}{k} = n\binom{n-1}{k-1}$, $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, $\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k}$ and $\binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1}$.

2.2 Two Combinatorial Facts

The first fact is an extension of *Stirling's* approximation $n! \approx \sqrt{2\pi} n^{n+\frac{1}{2}} \mathsf{exp}(-n)$ to $n!$. The extension yields a double inequality for $n!$ (cf. [3, Chapter 2, Section 9]).

Lemma 2.1 *For all integers $n \geq 1$,*

$$\sqrt{2\pi} n^{n+\frac{1}{2}} \mathsf{exp}\left(-n + \frac{1}{12n+1}\right) \;\leq\; n! \;\leq\; \sqrt{2\pi} n^{n+\frac{1}{2}} \mathsf{exp}\left(-n + \frac{1}{12n}\right).$$

The second fact is a maximization property of the *Bernstein basis polynomial of order k and degree n* $\mathsf{b}_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, which forms a basis of the vector space of polynomials of degree n [1].

Lemma 2.2 *For each pair of integers k and n with $0 \leq k \leq n$,*

$$\max_{x \in [0,1]} \mathsf{b}_{k,n}(x) \;=\; \binom{n}{k} k^k n^{-n} (n-k)^{n-k},$$

occurring at $x = \frac{k}{n}$.

3 Framework and Preliminaries

Our definitions are based on (and depart from) the standard ones for the KP model [14].

3.1 General

Consider a ***network*** with two parallel *links* 1 and 2 from a ***source*** to a ***destination*** node. Each of $n \geq 2$ *users* $1, 2, \ldots, n$ wishes to route one unit of traffic from source to destination.

A ***pure strategy*** s_i for user $i \in [n]$ is some specific link; a ***mixed strategy*** σ_i is a probability distribution over pure strategies— so, σ_i is a probability distribution over links. The ***support*** of user i in her mixed strategy σ_i, denoted as $\mathsf{support}(\sigma_i)$, is the set of pure strategies to which σ_i assigns strictly positive probability. A ***(pure) profile*** is a vector $\mathbf{s} = \langle s_1, \ldots, s_n \rangle$ of pure strategies, one for each user; a ***mixed profile*** is a vector $\boldsymbol{\sigma} = \langle \sigma_1, \ldots, \sigma_n \rangle$ of mixed strategies, one for each user. Note that all probability distributions in a mixed profile are independent. A mixed profile $\boldsymbol{\sigma}$ induces a (product) probability measure $\mathbb{P}_{\boldsymbol{\sigma}}$ on the space of pure profiles.

A user i is ***pure*** in the mixed profile $\boldsymbol{\sigma}$ if $|\mathsf{support}(\sigma_i)| = 1$; so, a pure profile is the degenerate case of a mixed profile where all users are pure; the user i is mixed in the mixed profile $\boldsymbol{\sigma}$ if she is not pure in $\boldsymbol{\sigma}$. The mixed profile $\boldsymbol{\sigma}$ is ***fully mixed*** if for each user $i \in [n]$ and link $j \in [2]$, $\sigma_i(j) > 0$. The user i is ***fully mixed*** in the mixed profile $\boldsymbol{\sigma}$ if $|\mathsf{support}(\sigma_i)| = 2$; so, a fully mixed profile is the special case of a mixed profile where all users are fully mixed.

Denote as $\mathcal{S}$ the space of all 2^n pure profiles. Denote as $\boldsymbol{\sigma}_{-i} \diamond \sigma'_i$ the mixed profile obtained by substituting the mixed strategy σ_i of player i in $\boldsymbol{\sigma}$ with the mixed strategy σ'_i.

3.2 Cost Measures and Nash Equilibria

The ***congestion*** on link $\ell \in [2]$ in the profile $\mathbf{s}$, denoted as $\mathsf{c}(\ell, \mathbf{s})$, is the number of users choosing link ℓ in $\mathbf{s}$; so,

$$\mathsf{c}(\ell, \mathbf{s}) \quad = \quad |\{i \in [n] : s_i = \ell\}|.$$

The ***Individual Cost*** of user i in the profile $\mathbf{s}$, denoted as $\mathsf{IC}_i(\mathbf{s})$, is the congestion on her chosen link; so, $\mathsf{IC}_i(\mathbf{s}) = \mathsf{c}(s_i, \mathbf{s})$.

The ***expected congestion*** on the link $\ell \in [2]$ in the mixed profile $\boldsymbol{\sigma}$, denoted as $\mathsf{c}(\ell, \boldsymbol{\sigma})$, is the expectation (according to $\boldsymbol{\sigma}$) of the congestion on link ℓ; so,

$$\mathsf{c}(\ell, \boldsymbol{\sigma}) \quad = \quad \mathbb{E}_{\mathbf{s} \sim \mathbb{P}_{\boldsymbol{\sigma}}}(\mathsf{c}(\ell, \mathbf{s})).$$

The ***Expected Individual Cost*** of user i in the mixed profile $\boldsymbol{\sigma}$, denoted as $\mathsf{IC}_i(\boldsymbol{\sigma})$, is the expectation (according to $\boldsymbol{\sigma}$) of her Individual Cost; so,

$$\mathsf{IC}_i(\boldsymbol{\sigma}) \quad = \quad \mathbb{E}_{\mathbf{s} \sim \mathbb{P}_{\boldsymbol{\sigma}}}(\mathsf{IC}_i(\mathbf{s})).$$

The ***Maximum Social Cost*** [14] of the mixed profile $\boldsymbol{\sigma}$, denoted as $\mathsf{MSC}(\boldsymbol{\sigma})$, is the expectation of the maximum congestion; so,

$$\mathsf{MSC}(\boldsymbol{\sigma}) \quad = \quad \mathbb{E}_{\mathbf{s} \sim \mathbb{P}_{\boldsymbol{\sigma}}}\left(\max_{\ell \in [2]} \mathsf{c}(\ell, \mathbf{s})\right).$$

The ***Quadratic Maximum Social Cost*** of the mixed profile $\boldsymbol{\sigma}$, denoted as $\mathsf{QMSC}(\boldsymbol{\sigma})$, is the expectation of the square of the maximum congestion; so,

$$\begin{aligned}
\mathsf{QMSC}(\boldsymbol{\sigma}) \quad &= \quad \mathbb{E}_{\mathbf{s} \sim \mathbb{P}_{\boldsymbol{\sigma}}}\left(\left(\max_{\ell \in [2]} \mathsf{c}(\ell, \mathbf{s})\right)^2\right) \\
&= \quad \sum_{\mathbf{s} \in \mathcal{S}} \mathbb{P}_{\boldsymbol{\sigma}}(\mathbf{s}) \cdot \left(\max_{\ell \in [2]} \mathsf{c}(\ell, \mathbf{s})\right)^2 \\
&= \quad \sum_{\mathbf{s} \in \mathcal{S}} \left(\prod_{k \in [n]} \sigma_k(s_k)\right) \cdot \left(\max_{\ell \in [2]} \mathsf{c}(\ell, \mathbf{s})\right)^2 .
\end{aligned}$$

The mixed profile $\boldsymbol{\sigma}$ is a ***(mixed) Nash equilibrium*** [20, 21] if for each user $i \in [n]$, for each mixed strategy σ'_i of player i, $\mathsf{IC}_i(\boldsymbol{\sigma}) \leq \mathsf{IC}_i(\boldsymbol{\sigma}_{-i} \diamond \sigma'_i)$; so, player i has no incentive to unilaterally change her mixed strategy.

3.3 The Fully Mixed Nash Equilibrium

We are especially interested in the fully mixed Nash equilibrium $\boldsymbol{\phi}$ [18], which is known to exist uniquely in the setting we consider [18, Theorem 4.7]. It is also known that for each pair of a user $i \in [n]$ and a link $\ell \in [2]$, $\boldsymbol{\phi}_i(\ell) = \frac{1}{2}$; so, all 2^n pure profiles are equiprobable, each occurring with probability $\frac{1}{2^n}$ [18, Lemma 15]. A simple expression for the Maximum Social Cost of $\boldsymbol{\phi}$ is given in [16, Lemma 5]:

Lemma 3.1 *For the fully mixed Nash equilibrium $\boldsymbol{\phi}$,*

$$\mathsf{MSC}(\boldsymbol{\phi}) \;=\; \frac{n}{2} + \frac{n}{2^n}\binom{n-1}{\left\lceil \frac{n}{2} \right\rceil - 1}.$$

We now calculate the Quadratic Maximum Social Cost of the fully mixed Nash equilibrium $\boldsymbol{\phi}$.

Lemma 3.2 *For the fully mixed Nash equilibrium $\boldsymbol{\phi}$,*

$$\mathsf{QMSC}(\boldsymbol{\phi}) \;=\; \frac{n}{4} + \frac{n^2}{4} + \frac{n^2}{2^n}\binom{n-1}{\left\lceil \frac{n}{2} \right\rceil - 1}.$$

Proof. Note that the maximum congestion attains the following values:

- $\frac{n^2}{4}$, attained when $\frac{n}{2}$ users are assigned to each of the links 1 and 2; this occurs in $\binom{n}{\frac{n}{2}}$ ways if n is even and cannot occur when n is odd.
- i^2, where $\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n$, attained when i users are assigned to one link and the remaining $n - i < i$ users are assigned to the other link; this occurs in $2\binom{n}{i}$ ways; the factor 2 accounts for exchanging the links where the maximum latency is attained.

By the equiprobability of all 2^n pure profiles, it follows that

$$\begin{aligned}
&\mathsf{QMSC}(\boldsymbol{\phi}) \\
&= \frac{1}{2^n}\left(\mathsf{Even}(n)\cdot\frac{n^2}{4}\binom{n}{\frac{n}{2}} + 2\sum_{i=\left\lfloor \frac{n}{2}\right\rfloor+1}^{n} i^2\binom{n}{i}\right) \\
&= \frac{n}{2^n}\left(\mathsf{Even}(n)\cdot\frac{n}{2}\binom{n-1}{\frac{n}{2}-1} + 2\sum_{i=\left\lfloor \frac{n}{2}\right\rfloor+1}^{n} i\binom{n-1}{i-1}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{n}{2^n}\left(\mathsf{Even}(n)\cdot\frac{n}{2}\binom{n-1}{\frac{n}{2}-1} + 2\sum_{i-1=\lfloor\frac{n}{2}\rfloor}^{n-1}((i-1)+1)\binom{n-1}{i-1}\right)\\
&= \frac{n}{2^n}\left(\mathsf{Even}(n)\cdot\frac{n}{2}\binom{n-1}{\frac{n}{2}-1} + 2\sum_{i=\lfloor\frac{n}{2}\rfloor}^{n-1}(i+1)\binom{n-1}{i}\right)\\
&= \frac{n}{2^n}\left(\mathsf{Even}(n)\cdot\frac{n}{2}\binom{n-1}{\frac{n}{2}-1} + 2\sum_{i=\lfloor\frac{n}{2}\rfloor}^{n-1} i\binom{n-1}{i} + 2\sum_{i=\lfloor\frac{n}{2}\rfloor}^{n-1}\binom{n-1}{i}\right)\\
&= \frac{n}{2^n}\left(\mathsf{Even}(n)\cdot\frac{n}{2}\binom{n-1}{\frac{n}{2}-1} + 2(n-1)\sum_{i=\lfloor\frac{n}{2}\rfloor}^{n-1}\binom{n-2}{i-1} + 2\sum_{i=\lfloor\frac{n}{2}\rfloor}^{n-1}\binom{n-1}{i}\right)\\
&= \frac{n}{2^n}\left(\mathsf{Even}(n)\cdot\frac{n}{2}\binom{n-1}{\frac{n}{2}-1} + 2(n-1)\sum_{i-1=\lfloor\frac{n}{2}\rfloor-1}^{n-2}\binom{n-2}{i-1} + 2\sum_{i=\lfloor\frac{n}{2}\rfloor}^{n-1}\binom{n-1}{i}\right)\\
&= \frac{n}{2^n}\left(\mathsf{Even}(n)\cdot\frac{n}{2}\binom{n-1}{\frac{n}{2}-1} + 2(n-1)\sum_{i=\lfloor\frac{n}{2}\rfloor-1}^{n-2}\binom{n-2}{i} + 2\sum_{i=\lfloor\frac{n}{2}\rfloor}^{n-1}\binom{n-1}{i}\right)\\
&= \frac{n}{2^n}\left(\mathsf{Even}(n)\cdot\frac{n}{2}\binom{n-1}{\frac{n}{2}-1}\right.\\
&\quad +(n-1)\left(\sum_{i=0}^{n-2}\binom{n-2}{i} + \mathsf{Even}(n)\cdot\binom{n-2}{\left\lfloor\frac{n}{2}\right\rfloor-1} + \mathsf{Odd}(n)\cdot 2\binom{n-2}{\left\lfloor\frac{n}{2}\right\rfloor-1}\right)\\
&\quad \left.+\sum_{i=0}^{n-1}\binom{n-1}{i} + \mathsf{Odd}(n)\cdot\binom{n-1}{\left\lfloor\frac{n}{2}\right\rfloor}\right)\\
&= \frac{n}{2^n}\left(\mathsf{Even}(n)\cdot\frac{n}{2}\binom{n-1}{\left\lceil\frac{n}{2}\right\rceil-1} + (n-1)\left(2^{n-2} + \mathsf{Even}(n)\cdot\binom{n-2}{\left\lfloor\frac{n}{2}\right\rfloor-1} + \mathsf{Odd}(n)\cdot 2\binom{n-2}{\left\lfloor\frac{n}{2}\right\rfloor-1}\right)\right.\\
&\quad \left.+2^{n-1} + \mathsf{Odd}(n)\cdot\binom{n-1}{\left\lceil\frac{n}{2}\right\rceil-1}\right)\\
&= \mathsf{Even}(n)\cdot\frac{n^2}{2^{n+1}}\binom{n-1}{\left\lceil\frac{n}{2}\right\rceil-1} + \frac{n(n-1)}{4} + \mathsf{Even}(n)\cdot\frac{n(n-1)}{2^n}\binom{n-2}{\left\lfloor\frac{n}{2}\right\rfloor-1}\\
&\quad +\mathsf{Odd}(n)\cdot\frac{2n(n-1)}{2^n}\binom{n-2}{\left\lfloor\frac{n}{2}\right\rfloor-1} + \frac{n}{2} + \mathsf{Odd}(n)\cdot\frac{n}{2^n}\binom{n-1}{\left\lceil\frac{n}{2}\right\rceil-1}\\
&= \frac{n}{4} + \frac{n^2}{4} + \mathsf{Even}(n)\cdot\left(\frac{n^2}{2^{n+1}}\binom{n-1}{\left\lceil\frac{n}{2}\right\rceil-1} + \frac{n(n-1)}{2^n}\binom{n-2}{\left\lceil\frac{n}{2}\right\rceil-1}\right)\\
&\quad +\mathsf{Odd}(n)\cdot\left(\frac{2n(n-1)}{2^n}\binom{n-2}{\left\lceil\frac{n}{2}\right\rceil-2} + \frac{n}{2^n}\binom{n-1}{\left\lceil\frac{n}{2}\right\rceil-1}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{n}{4}+\frac{n^2}{4}+\mathsf{Even}(n)\cdot\left(\frac{n^2}{2^{n+1}}\binom{n-1}{\left\lceil\frac{n}{2}\right\rceil-1}+\frac{n(n-1)}{2^n}\cdot\frac{n-1-\left\lceil\frac{n}{2}\right\rceil+1}{n-1}\binom{n-1}{\left\lceil\frac{n}{2}\right\rceil-1}\right)\\
&\quad+\mathsf{Odd}(n)\cdot\left(\frac{2n(n-1)}{2^n}\cdot\frac{\left\lceil\frac{n}{2}\right\rceil-1}{n-1}\binom{n-1}{\left\lceil\frac{n}{2}\right\rceil-1}+\frac{n}{2^n}\binom{n-1}{\left\lceil\frac{n}{2}\right\rceil-1}\right)\\
&= \frac{n}{4}+\frac{n^2}{4}+\mathsf{Even}(n)\cdot\left(\frac{n^2}{2^{n+1}}\binom{n-1}{\left\lceil\frac{n}{2}\right\rceil-1}+\frac{n^2}{2^{n+1}}\binom{n-1}{\left\lceil\frac{n}{2}\right\rceil-1}\right)\\
&\quad+\mathsf{Odd}(n)\cdot\left(\frac{n(n-1)}{2^n}\binom{n-1}{\left\lceil\frac{n}{2}\right\rceil-1}+\frac{n}{2^n}\binom{n-1}{\left\lceil\frac{n}{2}\right\rceil-1}\right)\\
&= \frac{n}{4}+\frac{n^2}{4}+\mathsf{Even}(n)\cdot\frac{n^2}{2^n}\binom{n-1}{\left\lceil\frac{n}{2}\right\rceil-1}+\mathsf{Odd}(n)\cdot\frac{n^2}{2^n}\binom{n-1}{\left\lceil\frac{n}{2}\right\rceil-1}\\
&= \frac{n}{4}+\frac{n^2}{4}+\frac{n^2}{2^n}\binom{n-1}{\left\lceil\frac{n}{2}\right\rceil-1},
\end{aligned}
$$

as needed. ■

3.4 The Arbitrary Nash Equilibrium $\boldsymbol{\sigma}$

Fix now an arbitrary Nash equilibrium $\boldsymbol{\sigma}$. It is known that $\mathsf{MSC}(\boldsymbol{\phi}) \geq \mathsf{MSC}(\boldsymbol{\sigma})$ [16, Theorem 4] (for the case of unweighted users and two identical links). We consider three sets of users:

- $\mathcal{U}_1 = \{i \mid \mathsf{support}(\sigma_i) = \{1\}\}$ is the set of (pure) users choosing link 1.
- $\mathcal{U}_2 = \{i \mid \mathsf{support}(\sigma_i) = \{2\}\}$ is the set of (pure) users choosing link 2.
- $\mathcal{U}_{12} = \{i \mid \mathsf{support}(\sigma_i) = \{1,2\}\}$ is the set of (fully) mixed users choosing either link 1 or link 2.

Set $u = \min\{|\mathcal{U}_1|, |\mathcal{U}_2|\}$; clearly, $n \geq 2u$. So, there are in total $2u$ (pure) users each of which chooses either link 1 or link 2 (with probability 1). The case where $n = 2u$ is trivial since in that case there are no mixed users, so that $\mathsf{QMSC}(\boldsymbol{\sigma}) = u^2$; by Lemma 3.2, $\mathsf{QMSC}(\boldsymbol{\phi}) > \frac{u}{2} + u^2$, which implies that $\mathsf{QMSC}(\boldsymbol{\phi}) > \mathsf{QMSC}(\boldsymbol{\sigma})$. So, we will henceforth assume that $n > 2u$.

Denote as $\widehat{\boldsymbol{\sigma}}$ the mixed profile derived from $\boldsymbol{\sigma}$ by eliminating those $2u$ users; note that $\widehat{\boldsymbol{\sigma}}$ is a (mixed) Nash equilibrium with $n-2u$ users. Also, denote as $\widehat{\boldsymbol{\phi}}$ the fully mixed Nash equilibrium with $n-2u$ users.

3.5 From $\mathsf{QMSC}(\boldsymbol{\phi}) - \mathsf{QMSC}(\boldsymbol{\sigma})$ to $\mathsf{QMSC}(\widehat{\boldsymbol{\phi}}) - \mathsf{QMSC}(\widehat{\boldsymbol{\sigma}})$

Clearly, $\widehat{\boldsymbol{\sigma}}$ has simpler form than $\boldsymbol{\sigma}$ since in $\widehat{\boldsymbol{\sigma}}$ there is (at least) one link chosen only by the fully mixed users (and no pure users). Hence, it would be simpler to compare $\mathsf{QMSC}(\widehat{\boldsymbol{\phi}})$

and $\mathsf{QMSC}(\widehat{\boldsymbol{\sigma}})$ (instead of comparing $\mathsf{QMSC}(\boldsymbol{\phi})$ and $\mathsf{QMSC}(\boldsymbol{\sigma})$ directly). To do so, we establish a relation between $\mathsf{QMSC}(\boldsymbol{\sigma})$ and $\mathsf{QMSC}(\widehat{\boldsymbol{\sigma}})$, and another relation between $\mathsf{QMSC}(\boldsymbol{\phi})$ and $\mathsf{QMSC}(\widehat{\boldsymbol{\phi}})$. We first compare $\mathsf{QMSC}(\widehat{\boldsymbol{\sigma}})$ and $\mathsf{QMSC}(\boldsymbol{\sigma})$. Clearly,

$$\begin{aligned}
\mathsf{QMSC}(\widehat{\boldsymbol{\sigma}}) &= \mathbb{E}_{\mathbb{P}_{\boldsymbol{\sigma}}}\left((\max\{\mathsf{c}(1,\boldsymbol{\sigma}),\mathsf{c}(2,\boldsymbol{\sigma})\}-u)^2\right)\\
&= \mathbb{E}_{\mathbb{P}_{\boldsymbol{\sigma}}}\left((\max\{\mathsf{c}(1,\boldsymbol{\sigma}),\mathsf{c}(2,\boldsymbol{\sigma})\})^2-2u\max\{\mathsf{c}(1,\boldsymbol{\sigma}),\mathsf{c}(2,\boldsymbol{\sigma})\}+u^2\right)\\
&= \mathbb{E}_{\mathbb{P}_{\boldsymbol{\sigma}}}\left((\max\{\mathsf{c}(1,\boldsymbol{\sigma}),\mathsf{c}(2,\boldsymbol{\sigma})\})^2\right)-2u\mathbb{E}_{\mathbb{P}_{\boldsymbol{\sigma}}}\left(\max\{\mathsf{c}(1,\boldsymbol{\sigma}),\mathsf{c}(2,\boldsymbol{\sigma})\}\right)+u^2\\
&= \mathsf{QMSC}(\boldsymbol{\sigma})-2u\,\mathsf{MSC}(\boldsymbol{\sigma})+u^2,
\end{aligned}$$

We continue to compare $\mathsf{QMSC}(\boldsymbol{\phi})$ and $\mathsf{QMSC}(\widehat{\boldsymbol{\phi}})$. By Lemma 3.2,

$$\begin{aligned}
&\mathsf{QMSC}(\boldsymbol{\phi})-\mathsf{QMSC}(\widehat{\boldsymbol{\phi}})\\
=\ &\frac{n}{4}+\frac{n^2}{4}+\frac{n^2}{2^n}\binom{n-1}{\lceil\frac{n}{2}\rceil-1}-\frac{n-2u}{4}-\frac{(n-2u)^2}{4}-\frac{(n-2u)^2}{2^{n-2u}}\binom{n-2u-1}{\lceil\frac{n-2u}{2}\rceil-1}\\
=\ &-u^2+u\left(n+\frac{1}{2}\right)+\frac{n^2}{2^n}\binom{n-1}{\lceil\frac{n}{2}\rceil-1}-\frac{(n-2u)^2}{2^{n-2u}}\binom{n-2u-1}{\lceil\frac{n-2u}{2}\rceil-1}.
\end{aligned}$$

Hence, by Lemma 3.1,

$$\begin{aligned}
&\mathsf{QMSC}(\boldsymbol{\phi})-\mathsf{QMSC}(\boldsymbol{\sigma})-(\mathsf{QMSC}(\widehat{\boldsymbol{\phi}})-\mathsf{QMSC}(\widehat{\boldsymbol{\sigma}}))\\
=\ &-2u\,\mathsf{MSC}(\boldsymbol{\sigma})+u\left(n+\frac{1}{2}\right)+\frac{n^2}{2^n}\binom{n-1}{\lceil\frac{n}{2}\rceil-1}-\frac{(n-2u)^2}{2^{n-2u}}\binom{n-2u-1}{\lceil\frac{n-2u}{2}\rceil-1}\\
\geq\ &-2u\,\mathsf{MSC}(\boldsymbol{\phi})+u\left(n+\frac{1}{2}\right)+\frac{n^2}{2^n}\binom{n-1}{\lceil\frac{n}{2}\rceil-1}-\frac{(n-2u)^2}{2^{n-2u}}\binom{n-2u-1}{\lceil\frac{n-2u}{2}\rceil-1}\\
=\ &-2u\left(\frac{n}{2}+\frac{n}{2^n}\binom{n-1}{\lceil\frac{n}{2}\rceil-1}\right)+u\left(n+\frac{1}{2}\right)+\frac{n^2}{2^n}\binom{n-1}{\lceil\frac{n}{2}\rceil-1}-\frac{(n-2u)^2}{2^{n-2u}}\binom{n-2u-1}{\lceil\frac{n-2u}{2}\rceil-1}\\
=\ &\frac{u}{2}-2u\frac{n}{2^n}\binom{n-1}{\lceil\frac{n}{2}\rceil-1}+\frac{n^2}{2^n}\binom{n-1}{\lceil\frac{n}{2}\rceil-1}-\frac{(n-2u)^2}{2^{n-2u}}\binom{n-2u-1}{\lceil\frac{n-2u}{2}\rceil-1}\\
=\ &\frac{u}{2}+(n-2u)\frac{n}{2^n}\binom{n-1}{\lceil\frac{n}{2}\rceil-1}-\frac{(n-2u)^2}{2^{n-2u}}\binom{n-2u-1}{\lceil\frac{n-2u}{2}\rceil-1}\\
=\ &\frac{u}{2}+\frac{n-2u}{2^{n-2u}}\left(\frac{n}{2^{2u}}\binom{n-1}{\lceil\frac{n}{2}\rceil-1}-(n-2u)\binom{n-2u-1}{\lceil\frac{n-2u}{2}\rceil-1}\right).
\end{aligned}$$

Now, to prove that $\mathsf{QMSC}(\boldsymbol{\phi})\geq\mathsf{QMSC}(\boldsymbol{\sigma})$, Lemma 6.1 implies that it suffices to prove that $\mathsf{QMSC}(\widehat{\boldsymbol{\phi}})\geq\mathsf{QMSC}(\widehat{\boldsymbol{\sigma}})$.

3.6 The Nash Equilibrium $\widehat{\boldsymbol{\sigma}}$

For notational convenience, rename now the parameters so that both $\widehat{\boldsymbol{\sigma}}$ and $\widehat{\boldsymbol{\phi}}$ refer henceforth to an instance with n users. All n users are fully mixed in $\widehat{\boldsymbol{\phi}}$; assume that in $\widehat{\boldsymbol{\sigma}}$, $r\geq 1$ (pure) users choose link 1 with probability 1 and $n - r$ (fully mixed) users choose both links with positive probability. (The case $r = 0$ is trivial since it yields that $\widehat{\boldsymbol{\sigma}} = \widehat{\boldsymbol{\phi}}$.) We shall recall the following simple claim shown by Lücking *et al.* [16]:

Lemma 3.3 *For the Nash equilibrium $\widehat{\boldsymbol{\sigma}}$, for each mixed user $i \in [n]$, $\widehat{\sigma}_i(1) = \frac{1}{2} - \frac{r}{2(n-r-1)}$. Furthermore, $r \leq \left\lfloor \frac{n-3}{2} \right\rfloor$.*

Henceforth, we shall denote, for each user $i \in [n]$, $p = \widehat{\sigma}_i(1)$ and $q = \widehat{\sigma}_i(2)$, where $p + q = 1$. We now calculate $\mathsf{QMSC}(\widehat{\boldsymbol{\sigma}})$:

Lemma 3.4 *For the Nash equilibrium $\widehat{\boldsymbol{\sigma}}$,*

$$\begin{aligned} & \mathsf{QMSC}(\widehat{\boldsymbol{\sigma}}) \\ = \; & \mathsf{Even}(n) \cdot \frac{n^2}{4} \binom{n-r}{\frac{n}{2}-r} p^{\frac{n}{2}-r} q^{\frac{n}{2}} + \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^{n} i^2 \binom{n-r}{i-r} p^{i-r} q^{n-i} + \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^{n-r} i^2 \binom{n-r}{i} p^{n-r-i} q^{i}. \end{aligned}$$

Proof. Note that the maximum congestion $\max_{\ell \in [2]} \mathsf{c}(\ell, \boldsymbol{\sigma})$ attains the following values:

- $\frac{n^2}{4}$, attained when $\frac{n}{2}$ users are assigned to each of the links 1 and 2. There are $\binom{n-r}{\frac{n}{2}-r}$ such profiles when n is even, and each one occurs with probability $p^{\frac{n}{2}-r} q^{\frac{n}{2}}$. There are no such profiles when n is odd.
- i^2, where $\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n$, attained when i users are assigned to link 1. There are $\binom{n-r}{i-r}$ such profiles, and each one occurs with probability $p^{i-r} q^{n-i}$.
- i^2, when i users are assigned to link 2, where $\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n - r$. There are $\binom{n-r}{i}$ such profiles and each one occurs with probability $p^{n-r-i} q^{i}$.

Hence,

$$\begin{aligned} & \mathsf{QMSC}(\widehat{\boldsymbol{\sigma}}) \\ = \; & \mathsf{Even}(n) \cdot \frac{n^2}{4} \binom{n-r}{\frac{n}{2}-r} p^{\frac{n}{2}-r} q^{\frac{n}{2}} + \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^{n} i^2 \binom{n-r}{i-r} p^{i-r} q^{n-i} + \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^{n-r} i^2 \binom{n-r}{i} p^{n-r-i} q^{i}, \end{aligned}$$

as needed. ■

We now express $\mathsf{QMSC}(\widehat{\boldsymbol{\sigma}})$ in a different form by adding and subtracting terms. Define the parameters

$$\begin{aligned} & \mathsf{A} \\ = & \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n} (i-r)\binom{n-r}{i-r} p^{i-r-1} q^{n+1-i} + (n-r) \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n-r} \binom{n-r-1}{i-1} p^{n-r-i} q^{i}, \\ & \mathsf{B} \\ = & (n-r)\left(\sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n} (i-r-1)\binom{n-r-1}{i-r-1} p^{i-r-2} q^{n+2-i} + \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n-r} (n-r-1)\binom{n-r-2}{i-2} p^{n-r-i} q^{i} \right) \end{aligned}$$

and

$$\begin{aligned} & \mathsf{C} \\ = & \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n} \binom{n-r}{i-r} \left((i-r)p^{i-r-1}q^{n+1-i} + (i-r)^2 p^{i-r-2} q^{n+2-i} - (i-r)p^{i-r-2}q^{n+2-i} - i^2 p^{i-r} q^{n-i} \right). \end{aligned}$$

We prove that $\mathsf{QMSC}(\widehat{\boldsymbol{\sigma}})$ enjoys a simple form in terms of A, B and C:

Lemma 3.5 *For the Nash equilibrium $\widehat{\boldsymbol{\sigma}}$,*

$$\mathsf{QMSC}(\widehat{\boldsymbol{\sigma}}) \;=\; \mathsf{A} + \mathsf{B} - \mathsf{C} + \mathsf{Even}(n) \cdot \frac{n^2}{4} \binom{n-r}{\frac{n}{2}-r} p^{\frac{n}{2}-r} q^{\frac{n}{2}}.$$

Proof. Clearly,

$$\begin{aligned} & \mathsf{A} + \mathsf{B} - \mathsf{C} \\ = & \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n} (i-r)\binom{n-r}{i-r} p^{i-r-1} q^{n+1-i} + (n-r) \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n-r} \binom{n-r-1}{i-1} p^{n-r-i} q^{i} \\ & + (n-r) \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n} (i-r-1)\binom{n-r-1}{i-r-1} p^{i-r-2} q^{n+2-i} \\ & + (n-r)(n-r-1) \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n-r} \binom{n-r-2}{i-2} p^{n-r-i} q^{i} \\ & - \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n} (i-r)\binom{n-r}{i-r} p^{i-r-1} q^{n+1-i} - \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n} (i-r)^2 \binom{n-r}{i-r} p^{i-r-2} q^{n+2-i} \\ & + \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n} (i-r)\binom{n-r}{i-r} p^{i-r-2} q^{n+2-i} + \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n} i^2 \binom{n-r}{i-r} p^{i-r} q^{n-i} \end{aligned}$$

$$
\begin{aligned}
&= (n-r)\overbrace{\sum_{i=\lfloor\frac{n}{2}\rfloor+1}^{n-r}\binom{n-r-1}{i-1}p^{n-r-i}q^{i}}+(n-r)\underbrace{\sum_{i=\lfloor\frac{n}{2}\rfloor+1}^{n}(i-r-1)\binom{n-r-1}{i-r-1}p^{i-r-2}q^{n+2-i}}\\
&\quad +(n-r)(n-r-1)\overbrace{\sum_{i=\lfloor\frac{n}{2}\rfloor+1}^{n-r}\binom{n-r-2}{i-2}p^{n-r-i}q^{i}}-\underbrace{\sum_{i=\lfloor\frac{n}{2}\rfloor+1}^{n}(i-r)^2\binom{n-r}{i-r}p^{i-r-2}q^{n+2-i}}\\
&\quad +\underbrace{\sum_{i=\lfloor\frac{n}{2}\rfloor+1}^{n}(i-r)\binom{n-r}{i-r}p^{i-r-2}q^{n+2-i}}+\sum_{i=\lfloor\frac{n}{2}\rfloor+1}^{n}i^2\binom{n-r}{i-r}p^{i-r}q^{n-i}.
\end{aligned}
$$

We proceed to calculate separately the two groups of underbraced and overbraced terms we have marked; note first that the three underbraced terms cancel out since for each index i with $\left\lfloor\frac{n}{2}\right\rfloor+1\leq i\leq n$,

$$
\begin{aligned}
& (n-r)(i-r-1)\binom{n-r-1}{i-r-1}-(i-r)^2\binom{n-r}{i-r}+(i-r)\binom{n-r}{i-r}\\
= {}& (n-r)(i-r-1)\binom{n-r-1}{i-r-1}-(i-r)(i-r-1)\binom{n-r}{i-r}\\
= {}& (n-r)(i-r-1)\binom{n-r-1}{i-r-1}-(n-r)(i-r-1)\binom{n-r-1}{i-r-1}\\
= {}& 0.
\end{aligned}
$$

Note now that for the two overbraced terms, for each index i with $\left\lfloor\frac{n}{2}\right\rfloor+1\leq i\leq n-r$,

$$
\begin{aligned}
& (n-r)\binom{n-r-1}{i-1}+(n-r)(n-r-1)\binom{n-r-2}{i-2}\\
= {}& (n-r)\binom{n-r-1}{i-1}+(n-r)(i-1)\binom{n-r-1}{i-1}\\
= {}& (n-r)i\binom{n-r-1}{i-1}\\
= {}& i^2\binom{n-r}{i}.
\end{aligned}
$$

Hence,

$$
\mathsf{A}+\mathsf{B}-\mathsf{C} = \sum_{i=\lfloor\frac{n}{2}\rfloor+1}^{n}i^2\binom{n-r}{i-r}p^{i-r}q^{n-i}+\sum_{i=\lfloor\frac{n}{2}\rfloor+1}^{n-r}i^2\binom{n-r}{i}p^{n-r-i}q^{i}.
$$

The claim follows now from Lemma 3.4. ∎

We express A and B in different forms (Claims 3.6 and 3.7) and provide a lower bound on C (Claim 3.8). We first prove:

Claim 3.6 *It holds that*

$$\mathsf{A} = q(n-r)\left(1 + \mathsf{Odd}(n)\cdot\binom{n-r-1}{\frac{n-1}{2}-r}p^{\frac{n-1}{2}-r}q^{\frac{n-1}{2}}\right).$$

The proof will use the simple fact that for any integer $n \geq 1$, $n = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor$; furthermore, $\left\lfloor \frac{n}{2} \right\rfloor + 1 = \left\lceil \frac{n+1}{2} \right\rceil$ when n is odd, while $\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n+1}{2} \right\rfloor$ when n is even.

Proof. Clearly,

$$\begin{aligned}
&\frac{\mathsf{A}}{q(n-r)} \\
= \quad & \sum_{i=\left\lfloor \frac{n}{2}\right\rfloor+1}^{n} \frac{i-r}{n-r}\binom{n-r}{i-r}p^{i-r-1}q^{n-i} + \sum_{i=\left\lfloor \frac{n}{2}\right\rfloor+1}^{n-r}\binom{n-r-1}{i-1}p^{n-r-i}q^{i-1} \\
= \quad & \sum_{i=\left\lfloor \frac{n}{2}\right\rfloor+1}^{n} \binom{n-r-1}{i-r-1}p^{i-r-1}q^{n-i} + \sum_{i=\left\lfloor \frac{n}{2}\right\rfloor+1}^{n-r}\binom{n-r-1}{i-1}p^{n-r-i}q^{i-1} \\
= \quad & \sum_{i=\left\lfloor \frac{n}{2}\right\rfloor+1}^{n} \binom{n-r-1}{i-r-1}p^{i-r-1}q^{n-i} + \sum_{i=\left\lfloor \frac{n}{2}\right\rfloor+1}^{n-r}\binom{n-r-1}{n-r-i}p^{n-r-i}q^{i-1} \\
= \quad & \sum_{i=\left\lfloor \frac{n}{2}\right\rfloor+1}^{n} \binom{n-r-1}{i-r-1}p^{i-r-1}q^{n-i} + \sum_{n+1-i=n-\left\lfloor \frac{n}{2}\right\rfloor}^{(n+1)-(n-r)}\binom{n-r-1}{n+1-i-(r+1)}p^{n+1-i-(r+1)}q^{n-(n+1-i)} \\
= \quad & \sum_{i=\left\lfloor \frac{n}{2}\right\rfloor+1}^{n} \binom{n-r-1}{i-r-1}p^{i-r-1}q^{n-i} + \sum_{i=r+1}^{n-\left\lfloor \frac{n}{2}\right\rfloor}\binom{n-r-1}{i-r-1}p^{i-r-1}q^{n-i} \\
= \quad & \sum_{i=\left\lfloor \frac{n}{2}\right\rfloor+1}^{n} \binom{n-r-1}{i-r-1}p^{i-r-1}q^{n-i} + \sum_{i=r+1}^{\left\lfloor \frac{n+1}{2}\right\rfloor}\binom{n-r-1}{i-r-1}p^{i-r-1}q^{n-i} \\
= \quad & \sum_{i-(r+1)=\left\lfloor \frac{n}{2}\right\rfloor+1-(r+1)}^{n-(r+1)} \binom{n-r-1}{i-(r+1)}p^{i-(r+1)}q^{n-(i-(r+1))-(r+1)} \\
& + \sum_{i-(r+1)=0}^{\left\lfloor \frac{n+1}{2}\right\rfloor-(r+1)} \binom{n-r-1}{i-(r+1)}p^{i-(r+1)}q^{n-(i-(r+1))-(r+1)} \\
= \quad & \sum_{i=\left\lfloor \frac{n}{2}\right\rfloor+1-(r+1)}^{n-(r+1)} \binom{n-(r+1)}{i}p^{i}q^{n-i-(r+1)} + \sum_{i=0}^{\left\lfloor \frac{n+1}{2}\right\rfloor-(r+1)} \binom{n-(r+1)}{i}p^{i}q^{n-i-(r+1)} \\
= \quad & \sum_{i=0}^{n-(r+1)} \binom{n-(r+1)}{i}p^{i}q^{n-(r+1)-i} + \mathsf{Odd}(n)\cdot\binom{n-(r+1)}{\frac{n+1}{2}-(r+1)}p^{\frac{n+1}{2}-(r+1)}q^{n-(\frac{n+1}{2}-(r+1))-(r+1)} \\
= \quad & 1 + \mathsf{Odd}(n)\cdot\binom{n-r-1}{\frac{n-1}{2}-r}p^{\frac{n-1}{2}-r}q^{\frac{n-1}{2}},
\end{aligned}$$

as needed. ■

We continue to prove:

Claim 3.7 *It holds that*

$$\frac{\mathsf{B}}{q^2(n-r)(n-r-1)} = 1 + \binom{n-r-2}{n-\left\lfloor\frac{n}{2}\right\rfloor - r - 1} p^{n-\lfloor\frac{n}{2}\rfloor - r - 1} q^{\lfloor\frac{n}{2}\rfloor - 1} + \mathsf{Odd}(n)\cdot\binom{n-r-2}{\frac{n-3}{2}-r} p^{\frac{n-3}{2}-r} q^{\frac{n-1}{2}}.$$

Proof. Clearly,

$$\begin{aligned}
&\frac{\mathsf{B}}{q^2(n-r)(n-r-1)}\\
= & \sum_{i=\lfloor\frac{n}{2}\rfloor+1}^{n} \frac{i-r-1}{n-r-1}\binom{n-r-1}{i-r-1} p^{i-r-2}q^{n-i} + \sum_{i=\lfloor\frac{n}{2}\rfloor+1}^{n-r}\binom{n-r-2}{i-2} p^{n-r-i}q^{i-2}\\
= & \sum_{i=\lfloor\frac{n}{2}\rfloor+1}^{n} \binom{n-r-2}{i-r-2} p^{i-r-2}q^{n-i} + \sum_{i=\lfloor\frac{n}{2}\rfloor+1}^{n-r}\binom{n-r-2}{i-2} p^{n-r-i}q^{i-2}\\
= & \sum_{i=\lfloor\frac{n}{2}\rfloor+1}^{n} \binom{n-r-2}{i-r-2} p^{i-r-2}q^{n-i} + \sum_{n-i+2=n-\lfloor\frac{n}{2}\rfloor+1}^{r+2}\binom{n-r-2}{(n-i+2)-r-2} p^{(n-i+2)-r-2}q^{n-(n-i+2)}\\
= & \sum_{i=\lfloor\frac{n}{2}\rfloor+1}^{n} \binom{n-r-2}{i-r-2} p^{i-r-2}q^{n-i} + \sum_{i=r+2}^{n-\lfloor\frac{n}{2}\rfloor+1}\binom{n-r-2}{i-r-2} p^{i-r-2}q^{n-i}\\
= & \sum_{i-r-2=\lfloor\frac{n}{2}\rfloor-r-1}^{n-r-2} \binom{n-r-2}{i-r-2} p^{i-r-2}q^{n-(i-r-2)-r-2} + \sum_{i-r-2=0}^{n-\lfloor\frac{n}{2}\rfloor-r-1}\binom{n-r-2}{i-r-2} p^{i-r-2}q^{n-(i-r-2)-r-2}\\
= & \sum_{i=\lfloor\frac{n}{2}\rfloor-r-1}^{n-r-2} \binom{n-r-2}{i} p^{i}q^{n-r-2-i} + \sum_{i=0}^{n-\lfloor\frac{n}{2}\rfloor-r-1}\binom{n-r-2}{i} p^{i}q^{n-r-2-i}\\
= & \sum_{i=0}^{n-r-2}\binom{n-r-2}{i}p^iq^{n-r-2-i} + \mathsf{Even}(n)\cdot\binom{n-r-2}{n-\left\lfloor\frac{n}{2}\right\rfloor-r-1} p^{n-\lfloor\frac{n}{2}\rfloor-r-1}q^{\lfloor\frac{n}{2}\rfloor-1}\\
& +\mathsf{Odd}(n)\cdot\left(\binom{n-r-2}{n-\left\lfloor\frac{n}{2}\right\rfloor-r-1} p^{n-\lfloor\frac{n}{2}\rfloor-r-1}q^{\lfloor\frac{n}{2}\rfloor-1} + \binom{n-r-2}{\left\lfloor\frac{n}{2}\right\rfloor-r-1} p^{\lfloor\frac{n}{2}\rfloor-r-1}q^{n-\lfloor\frac{n}{2}\rfloor-1}\right)\\
= & \sum_{i=0}^{n-r-2}\binom{n-r-2}{i}p^iq^{n-r-2-i} + (\mathsf{Even}(n)+\mathsf{Odd}(n))\cdot\binom{n-r-2}{n-\left\lfloor\frac{n}{2}\right\rfloor-r-1} p^{n-\lfloor\frac{n}{2}\rfloor-r-1}q^{\lfloor\frac{n}{2}\rfloor-1}\\
& +\mathsf{Odd}(n)\cdot\binom{n-r-2}{\left\lfloor\frac{n}{2}\right\rfloor-r-1} p^{\lfloor\frac{n}{2}\rfloor-r-1}q^{n-\lfloor\frac{n}{2}\rfloor-1}\\
= & \sum_{i=0}^{n-r-2}\binom{n-r-2}{i}p^iq^{n-r-2-i} + \binom{n-r-2}{n-\left\lfloor\frac{n}{2}\right\rfloor-r-1} p^{n-\lfloor\frac{n}{2}\rfloor-r-1}q^{\lfloor\frac{n}{2}\rfloor-1}\\
& +\mathsf{Odd}(n)\cdot\binom{n-r-2}{\left\lfloor\frac{n}{2}\right\rfloor-r-1} p^{\lfloor\frac{n}{2}\rfloor-r-1}q^{n-\lfloor\frac{n}{2}\rfloor-1}
\end{aligned}$$

$$= \quad 1 + \binom{n-r-2}{n-\left\lfloor\frac{n}{2}\right\rfloor-r-1} p^{n-\left\lfloor\frac{n}{2}\right\rfloor-r-1} q^{\left\lfloor\frac{n}{2}\right\rfloor-1} + \mathsf{Odd}(n) \cdot \binom{n-r-2}{\frac{n-3}{2}-r} p^{\frac{n-3}{2}-r} q^{\frac{n-1}{2}},$$

as needed. ■

We finally prove:

Claim 3.8 *It holds that*

$$\mathsf{C} \quad > \quad \frac{r\left((r+4)n^2-2(3r+2)n+r\right)}{4(n-r-1)^2}\binom{n-r-2}{\left\lfloor\frac{n}{2}\right\rfloor-r} p^{\left\lfloor\frac{n}{2}\right\rfloor-r} q^{n-\left\lfloor\frac{n}{2}\right\rfloor-2}$$
$$+\frac{r\left(2n^3-(3r+2)n^2-r\right)}{4(n-r-1)^2}\binom{n-r-2}{\left\lfloor\frac{n}{2}\right\rfloor-r-1} p^{\left\lfloor\frac{n}{2}\right\rfloor-r-1} q^{n-\left\lfloor\frac{n}{2}\right\rfloor-1}.$$

Proof. Clearly,

$$\begin{aligned}
&\mathsf{C}\\
= &\sum_{i=\left\lfloor\frac{n}{2}\right\rfloor+1}^{n} \binom{n-r}{i-r}\left((i-r)p^{i-r-1}q^{n+1-i}+(i-r)^2p^{i-r-2}q^{n+2-i}-(i-r)p^{i-r-2}q^{n+2-i}-i^2p^{i-r}q^{n-i}\right)\\
= &\sum_{i=\left\lfloor\frac{n}{2}\right\rfloor+1}^{n} \binom{n-r}{i-r}p^{i-r-2}q^{n-i}\left((i-r)pq+(i-r)^2q^2-(i-r)q^2-i^2p^2\right)\\
= &\sum_{i-r=\left\lfloor\frac{n}{2}\right\rfloor+1-r}^{n-r} \binom{n-r}{i-r}p^{i-r-2}q^{n-(i-r)-r}\left((i-r)pq+(i-r)^2q^2-(i-r)q^2-((i-r)+r)^2p^2\right)\\
= &\sum_{i=\left\lfloor\frac{n}{2}\right\rfloor+1-r}^{n-r} \binom{n-r}{i}p^{i-2}q^{n-i-r}\left(ipq+i^2q^2-iq^2-(i+r)^2p^2\right)\\
= &\sum_{i=\left\lfloor\frac{n}{2}\right\rfloor+1-r}^{n-r} \binom{n-r}{i}p^{i-2}q^{n-r-i}\left(i(pq-q^2-2rp^2)+i^2(q^2-p^2)-r^2p^2\right)\\
= &(pq-q^2-2rp^2)\sum_{i=\left\lfloor\frac{n}{2}\right\rfloor+1-r}^{n-r} i\binom{n-r}{i}p^{i-2}q^{n-r-i}+(q^2-p^2)\sum_{i=\left\lfloor\frac{n}{2}\right\rfloor+1-r}^{n-r} i^2\binom{n-r}{i}p^{i-2}q^{n-r-i}\\
&-r^2p^2\sum_{i=\left\lfloor\frac{n}{2}\right\rfloor+1-r}^{n-r}\binom{n-r}{i}p^{i-2}q^{n-r-i}\\
= &(pq-q^2-2rp^2)(n-r)\sum_{i=\left\lfloor\frac{n}{2}\right\rfloor+1-r}^{n-r}\binom{n-r-1}{i-1}p^{i-2}q^{n-r-i}\\
&+(q^2-p^2)(n-r)\sum_{i=\left\lfloor\frac{n}{2}\right\rfloor+1-r}^{n-r}((i-1)+1)\binom{n-r-1}{i-1}p^{i-2}q^{n-r-i}\\
&-r^2p^2\sum_{i=\left\lfloor\frac{n}{2}\right\rfloor+1-r}^{n-r}\binom{n-r}{i}p^{i-2}q^{n-r-i}
\end{aligned}$$

$$\begin{aligned}
&= (pq-q^2-2rp^2)(n-r)\sum_{i=\lfloor\frac{n}{2}\rfloor+1-r}^{n-r}\binom{n-r-1}{i-1}p^{i-2}q^{n-r-i}\\
&\quad+(q^2-p^2)(n-r)(n-r-1)\sum_{i=\lfloor\frac{n}{2}\rfloor+1-r}^{n-r}\binom{n-r-2}{i-2}p^{i-2}q^{n-r-i}\\
&\quad+(q^2-p^2)(n-r)\sum_{i=\lfloor\frac{n}{2}\rfloor+1-r}^{n-r}\binom{n-r-1}{i-1}p^{i-2}q^{n-r-i}-r^2p^2\sum_{i=\lfloor\frac{n}{2}\rfloor+1-r}^{n-r}\binom{n-r}{i}p^{i-2}q^{n-r-i}\\
&= (pq-q^2-2rp^2)(n-r)\sum_{i=\lfloor\frac{n}{2}\rfloor+1-r}^{n-r}\left(\binom{n-r-2}{i-1}+\binom{n-r-2}{i-2}\right)p^{i-2}q^{n-r-i}\\
&\quad+(q^2-p^2)(n-r)(n-r-1)\sum_{i=\lfloor\frac{n}{2}\rfloor+1}^{n-r}\binom{n-r-2}{i-2}p^{i-2}q^{n-r-i}\\
&\quad+(q^2-p^2)(n-r)\sum_{i=\lfloor\frac{n}{2}\rfloor+1-r}^{n-r}\left(\binom{n-r-2}{i-1}+\binom{n-r-2}{i-2}\right)p^{i-2}q^{n-r-i}\\
&\quad-r^2p^2\sum_{i=\lfloor\frac{n}{2}\rfloor+1-r}^{n-r}\left(\binom{n-r-2}{i}+2\binom{n-r-2}{i-1}+\binom{n-r-2}{i-2}\right)p^{i-2}q^{n-r-i}\\
&= \big((pq-q^2-2rp^2)(n-r)+(q^2-p^2)(n-r)-2r^2p^2\big)\sum_{i=\lfloor\frac{n}{2}\rfloor+1-r}^{n-r-1}\binom{n-r-2}{i-1}p^{i-2}q^{n-r-i}\\
&\quad+\big((pq-q^2-2rp^2)(n-r)+(q^2-p^2)(n-r)(n-r-1)+(q^2-p^2)(n-r)-r^2p^2\big)\cdot\\
&\quad\sum_{i=\lfloor\frac{n}{2}\rfloor+1-r}^{n-r}\binom{n-r-2}{i-2}p^{i-2}q^{n-r-i}-r^2p^2\sum_{i=\lfloor\frac{n}{2}\rfloor+1-r}^{n-r-2}\binom{n-r-2}{i}p^{i-2}q^{n-r-i}\\
&= \big((pq-p^2-2rp^2)(n-r)-2r^2p^2\big)\sum_{i=\lfloor\frac{n}{2}\rfloor+1-r}^{n-r-1}\binom{n-r-2}{i-1}p^{i-2}q^{n-r-i}\\
&\quad+\big(\big(pq-p^2-2rp^2+(q^2-p^2)(n-r-1)\big)(n-r)-r^2p^2\big)\sum_{i=\lfloor\frac{n}{2}\rfloor+1-r}^{n-r}\binom{n-r-2}{i-2}p^{i-2}q^{n-r-i}\\
&\quad-r^2p^2\sum_{i=\lfloor\frac{n}{2}\rfloor+1-r}^{n-r-2}\binom{n-r-2}{i}p^{i-2}q^{n-r-i}\\
&= \big((pq-p^2-2rp^2)(n-r)-2r^2p^2\big)\sum_{i+1=\lfloor\frac{n}{2}\rfloor+2-r}^{(n-r-1)+1}\binom{n-r-2}{(i+1)-2}p^{(i+1)-3}q^{n-r-(i+1)+1}\\
&\quad+\big(\big(pq-p^2-2rp^2+(q^2-p^2)(n-r-1)\big)(n-r)-r^2p^2\big)\sum_{i=\lfloor\frac{n}{2}\rfloor+1-r}^{n-r}\binom{n-r-2}{i-2}p^{i-2}q^{n-r-i}\\
&\quad-r^2p^2\sum_{i+2=\lfloor\frac{n}{2}\rfloor+3-r}^{(n-r-2)+2}\binom{n-r-2}{(i+2)-2}p^{(i+2)-4}q^{n-r-(i+2)+2}
\end{aligned}$$

$$
\begin{aligned}
&= \big((pq-p^2-2rp^2)(n-r)-2r^2p^2\big)\sum_{i=\lfloor\frac{n}{2}\rfloor+2-r}^{n-r}\binom{n-r-2}{i-2}p^{i-3}q^{n-r-i+1} \\
&\quad+\big(\big(pq-p^2-2rp^2+(q^2-p^2)(n-r-1)\big)(n-r)-r^2p^2\big)\sum_{i=\lfloor\frac{n}{2}\rfloor+1-r}^{n-r}\binom{n-r-2}{i-2}p^{i-2}q^{n-r-i} \\
&\quad-r^2p^2\sum_{i=\lfloor\frac{n}{2}\rfloor+3-r}^{n-r}\binom{n-r-2}{i-2}p^{i-4}q^{n-r-i+2} \\
&= \big((pq-p^2-2rp^2)(n-r)-2r^2p^2\big)\cdot\frac{q}{p}\sum_{i=\lfloor\frac{n}{2}\rfloor+2-r}^{n-r}\binom{n-r-2}{i-2}p^{i-2}q^{n-r-i} \\
&\quad+\big(\big(pq-p^2-2rp^2+(q^2-p^2)(n-r-1)\big)(n-r)-r^2p^2\big)\sum_{i=\lfloor\frac{n}{2}\rfloor+1-r}^{n-r}\binom{n-r-2}{i-2}p^{i-2}q^{n-r-i} \\
&\quad-r^2p^2\cdot\frac{q^2}{p^2}\sum_{i=\lfloor\frac{n}{2}\rfloor+3-r}^{n-r}\binom{n-r-2}{i-2}p^{i-2}q^{n-r-i} \\
&= \big((q^2-pq-2rpq)(n-r)-2r^2pq+\big(pq-p^2-2rp^2+(q^2-p^2)(n-r-1)\big)(n-r) \\
&\quad-r^2p^2-r^2q^2\big)\sum_{i=\lfloor\frac{n}{2}\rfloor+3-r}^{n-r}\binom{n-r-2}{i-2}p^{i-2}q^{n-r-i} \\
&\quad+\big((pq-p^2-2rp^2)(n-r)-2r^2p^2\big)\cdot\frac{q}{p}\binom{n-r-2}{\left\lfloor\frac{n}{2}\right\rfloor-r}p^{\lfloor\frac{n}{2}\rfloor-r}q^{n-\lfloor\frac{n}{2}\rfloor-2} \\
&\quad+\big(\big(pq-p^2-2rp^2+(q^2-p^2)(n-r-1)\big)(n-r)-r^2p^2\big)\cdot \\
&\quad\left(\binom{n-r-2}{\left\lfloor\frac{n}{2}\right\rfloor-r-1}p^{\lfloor\frac{n}{2}\rfloor-r-1}q^{n-\lfloor\frac{n}{2}\rfloor-1}+\binom{n-r-2}{\left\lfloor\frac{n}{2}\right\rfloor-r}p^{\lfloor\frac{n}{2}\rfloor-r}q^{n-\lfloor\frac{n}{2}\rfloor-2}\right) \\
&= \big((q^2-pq-2rpq)(n-r)-2r^2pq+\big(pq-p^2-2rp^2+(q^2-p^2)(n-r-1)\big)(n-r) \\
&\quad-r^2p^2-r^2q^2\big)\sum_{i=\lfloor\frac{n}{2}\rfloor+3-r}^{n-r}\binom{n-r-2}{i-2}p^{i-2}q^{n-r-i} \\
&\quad+\left(\big((pq-p^2-2rp^2)(n-r)-2r^2p^2\big)\cdot\frac{q}{p}+\big(pq-p^2-2rp^2+(q^2-p^2)(n-r-1)\big)(n-r)-r^2p^2\right)\cdot \\
&\quad\binom{n-r-2}{\left\lfloor\frac{n}{2}\right\rfloor-r}p^{\lfloor\frac{n}{2}\rfloor-r}q^{n-\lfloor\frac{n}{2}\rfloor-2} \\
&\quad+\big(\big(pq-p^2-2rp^2+(q^2-p^2)(n-r-1)\big)(n-r)-r^2p^2\big)\binom{n-r-2}{\left\lfloor\frac{n}{2}\right\rfloor-r-1}p^{\lfloor\frac{n}{2}\rfloor-r-1}q^{n-\lfloor\frac{n}{2}\rfloor-1}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
& (q^2 - pq - 2rpq)(n-r) - \boxed{2r^2pq} + \big(pq - p^2 - 2rp^2 + (q^2-p^2)(n-r-1)\big)(n-r) - \boxed{r^2p^2} - \boxed{r^2q^2} \\
= \; & \big(q^2 - pq - 2rpq + pq - p^2 - 2rp^2 + (q^2-p^2)(n-r-1)\big)(n-r) - r^2(\underbrace{2pq + p^2 + q^2}_{1}) \\
= \; & \left((q-p)(\underbrace{q+p}_{1}) - 2rp(\underbrace{q+p}_{1}) + (q-p)(\underbrace{q+p}_{1})(n-r-1) \right)(n-r) - r^2 \\
= \; & ((q-p)(n-r) - 2rp)(n-r) - r^2 \\
= \; & \left(\frac{r(n-r)}{n-r-1} - \frac{r(n-2r-1)}{n-r-1} \right)(n-r) - r^2 \\
= \; & \frac{r(n-r)}{n-r-1}(n-r-n+2r+1) - r^2 \\
= \; & \frac{r(r+1)(n-r)}{n-r-1} - r^2 \\
= \; & \frac{r}{n-r-1}((r+1)(n-r) - r(n-r-1)) \\
= \; & \frac{rn}{n-r-1}.
\end{aligned}
$$

Moreover, observe that

$$
\begin{aligned}
& \big((pq - p^2 - 2rp^2)(n-r) - 2r^2p^2\big) \cdot \frac{q}{p} + \big(pq - p^2 - 2rp^2 + (q^2-p^2)(n-r-1)\big)(n-r) - r^2p^2 \\
= \; & (q^2 - pq - 2rpq)(n-r) - 2r^2pq + \big(pq - p^2 - 2rp^2 + (q^2-p^2)(n-r-1)\big)(n-r) - r^2p^2 \\
= \; & \big(q^2 - pq - 2rpq + pq - p^2 - 2rp^2 + (q^2-p^2)(n-r-1)\big)(n-r) - 2r^2pq - r^2p^2 \\
= \; & \left((q-p)(\underbrace{q+p}_{1}) - 2rp(\underbrace{q+p}_{1}) + (q-p)(\underbrace{q+p}_{1})(n-r-1) \right)(n-r) - r^2p(2q+p) \\
= \; & ((q-p)(n-r) - 2rp)(n-r) - r^2p(2q+p) \\
= \; & \frac{r(r+1)(n-r)}{n-r-1} - \frac{r^2(n-2r-1)}{2(n-r-1)} \left(\frac{n-1}{n-r-1} + \frac{n-2r-1}{2(n-r-1)} \right) \\
= \; & \frac{r(r+1)(n-r)}{n-r-1} - \frac{r^2(n-2r-1)(2(n-1)+n-2r-1)}{4(n-r-1)^2} \\
= \; & \frac{r(r+1)(n-r)}{n-r-1} - \frac{r^2(n-2r-1)(3n-2r-3)}{4(n-r-1)^2} \\
= \; & \frac{r}{4(n-r-1)^2}(4(r+1)(n-r)(n-r-1) - r(n-2r-1)(3n-2r-3)) \\
= \; & \frac{r\big((r+4)n^2 - 2(3r+2)n + r\big)}{4(n-r-1)^2}.
\end{aligned}
$$

Finally, note that

$$
\begin{aligned}
& \left(pq - p^2 - 2rp^2 + (q^2 - p^2)(n-r-1)\right)(n-r) - r^2p^2 \\
= & \left(p(q-p) - 2rp^2 + (q-p)\underbrace{(q+p)}_{1}(n-r-1)\right)(n-r) - r^2p^2 \\
= & \left(p(q-p) - 2rp^2 + (q-p)(n-r-1)\right)(n-r) - r^2p^2 \\
= & \left(\frac{r(n-2r-1)}{2(n-r-1)^2} - \frac{r(n-2r-1)^2}{2(n-r-1)^2} + r\right)(n-r) - \frac{r^2(n-2r-1)^2}{4(n-r-1)^2} \\
= & \left(\frac{-r(n-2r-1)(n-2r-2) + 2r(n-r-1)^2}{2(n-r-1)^2}\right)(n-r) - \frac{r^2(n-2r-1)^2}{4(n-r-1)^2} \\
= & \frac{r}{4(n-r-1)^2}\left(-2(n-2r-1)(n-2r-2)(n-r) + 4(n-r-1)^2(n-r) - r(n-2r-1)^2\right) \\
= & \frac{r\left(2n^3 - (3r+2)n^2 - r\right)}{4(n-r-1)^2}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathsf{C} \;=\; & \frac{rn}{n-r-1}\sum_{i=\left\lfloor \frac{n}{2}\right\rfloor+3-r}^{n-r}\binom{n-r-2}{i-2}p^{i-2}q^{n-r-i} \\
& + \frac{r\left((r+4)n^2 - 2(3r+2)n + r\right)}{4(n-r-1)^2}\binom{n-r-2}{\left\lfloor \frac{n}{2}\right\rfloor - r}p^{\left\lfloor \frac{n}{2}\right\rfloor - r}q^{n-\left\lfloor \frac{n}{2}\right\rfloor-2} \\
& + \frac{r\left(2n^3 - (3r+2)n^2 - r\right)}{4(n-r-1)^2}\binom{n-r-2}{\left\lfloor \frac{n}{2}\right\rfloor - r - 1}p^{\left\lfloor \frac{n}{2}\right\rfloor - r - 1}q^{n-\left\lfloor \frac{n}{2}\right\rfloor-1} \\
> \; & \frac{r\left((r+4)n^2 - 2(3r+2)n + r\right)}{4(n-r-1)^2}\binom{n-r-2}{\left\lfloor \frac{n}{2}\right\rfloor - r}p^{\left\lfloor \frac{n}{2}\right\rfloor - r}q^{n-\left\lfloor \frac{n}{2}\right\rfloor-2} \\
& + \frac{r\left(2n^3 - (3r+2)n^2 - r\right)}{4(n-r-1)^2}\binom{n-r-2}{\left\lfloor \frac{n}{2}\right\rfloor - r - 1}p^{\left\lfloor \frac{n}{2}\right\rfloor - r - 1}q^{n-\left\lfloor \frac{n}{2}\right\rfloor-1}.
\end{aligned}
$$

as needed. ■

4 The QMFMNE Conjecture is Valid

As our main result, we show:

Theorem 4.1 *For the fully mixed Nash equilibrium* $\widehat{\boldsymbol{\phi}}$ *and the Nash equilibrium* $\widehat{\boldsymbol{\sigma}}$*,*

$$\mathsf{QMSC}(\widehat{\boldsymbol{\phi}}) \;>\; \mathsf{QMSC}(\widehat{\boldsymbol{\sigma}}).$$

The proof will use some estimations and technical claims which have been deferred to Sections 5 and 6, respectively.

Proof. Lemma 3.2, 3.5 and Claims 3.6, 3.7 and 3.8 imply that

$$\mathsf{QMSC}(\widehat{\boldsymbol{\phi}}) - \mathsf{QMSC}(\widehat{\boldsymbol{\sigma}}) \quad > \quad \frac{n}{4} + \frac{n^2}{4} + \frac{n^2}{2^n}\binom{n-1}{\left\lceil \frac{n}{2} \right\rceil - 1} - q(n-r) - q^2(n-r)(n-r-1) + \mathsf{D},$$

where

$$\begin{aligned}
\mathsf{D} \quad = \quad & -q^2(n-r)(n-r-1)\binom{n-r-2}{n-\left\lfloor \frac{n}{2} \right\rfloor - r - 1} p^{n-\lfloor \frac{n}{2} \rfloor - r - 1} q^{\lfloor \frac{n}{2} \rfloor - 1} \\
& + \frac{r((r+4)n^2 - 2(3r+2)n + r)}{4(n-r-1)^2}\binom{n-r-2}{\left\lfloor \frac{n}{2} \right\rfloor - r} p^{\lfloor \frac{n}{2} \rfloor - r} q^{n-\lfloor \frac{n}{2} \rfloor - 2} \\
& + \frac{r(2n^3 - (3r+2)n^2 - r)}{4(n-r-1)^2}\binom{n-r-2}{\left\lfloor \frac{n}{2} \right\rfloor - r - 1} p^{\lfloor \frac{n}{2} \rfloor - r - 1} q^{n-\lfloor \frac{n}{2} \rfloor - 1} \\
& - \mathsf{Odd}(n) \cdot q(n-r)\left(\binom{n-r-1}{\frac{n-1}{2} - r} p^{\frac{n-1}{2} - r} q^{\frac{n-1}{2}} + q(n-r-1)\binom{n-r-2}{\frac{n-3}{2} - r} p^{\frac{n-3}{2} - r} q^{\frac{n-1}{2}}\right) \\
& - \mathsf{Even}(n) \cdot \frac{n^2}{4}\binom{n-r}{\frac{n}{2} - r} p^{\frac{n}{2} - r} q^{\frac{n}{2}} \\
= \quad & -\frac{(n-1)^2(n-r)}{4(n-r-1)}\binom{n-r-2}{n-\left\lfloor \frac{n}{2} \right\rfloor - r - 1} p^{n-\lfloor \frac{n}{2} \rfloor - r - 1} q^{\lfloor \frac{n}{2} \rfloor - 1} \\
& + \frac{r((r+4)n^2 - 2(3r+2)n + r)}{4(n-r-1)^2}\binom{n-r-2}{\left\lfloor \frac{n}{2} \right\rfloor - r} p^{\lfloor \frac{n}{2} \rfloor - r} q^{n-\lfloor \frac{n}{2} \rfloor - 2} \\
& + \frac{r(2n^3 - (3r+2)n^2 - r)}{4(n-r-1)^2}\binom{n-r-2}{\left\lfloor \frac{n}{2} \right\rfloor - r - 1} p^{\lfloor \frac{n}{2} \rfloor - r - 1} q^{n-\lfloor \frac{n}{2} \rfloor - 1} \\
& - \mathsf{Odd}(n) \cdot \frac{(n-1)(n-r)}{2(n-r-1)}\binom{n-r-1}{\frac{n-1}{2} - r} p^{\frac{n-1}{2} - r} q^{\frac{n-1}{2}} \\
& - \mathsf{Odd}(n) \cdot \frac{(n-1)^2(n-r)}{4(n-r-1)}\binom{n-r-2}{\frac{n-3}{2} - r} p^{\frac{n-3}{2} - r} q^{\frac{n-1}{2}} - \mathsf{Even}(n) \cdot \frac{n^2}{4}\binom{n-r}{\frac{n}{2} - r} p^{\frac{n}{2} - r} q^{\frac{n}{2}}.
\end{aligned}$$

We now prove a lower bound on D. We proceed by case analysis on whether n is even or odd.

$\boxed{\textbf{Case 1: } n \textbf{ is even}}$ Then,

$$\begin{aligned}
& \mathsf{D} \\
= \quad & -\frac{(n-1)^2(n-r)}{4(n-r-1)}\binom{n-r-2}{\frac{n}{2} - r - 1} p^{\frac{n}{2} - r - 1} q^{\frac{n}{2} - 1} \\
& + \frac{r((r+4)n^2 - 2(3r+2)n + r)}{4(n-r-1)^2}\binom{n-r-2}{\frac{n}{2} - r} p^{\frac{n}{2} - r} q^{\frac{n}{2} - 2} \\
& + \frac{r(2n^3 - (3r+2)n^2 - r)}{4(n-r-1)^2}\binom{n-r-2}{\frac{n}{2} - r - 1} p^{\frac{n}{2} - r - 1} q^{\frac{n}{2} - 1} - \frac{n^2}{4}\binom{n-r}{\frac{n}{2} - r} p^{\frac{n}{2} - r} q^{\frac{n}{2}}
\end{aligned}$$

$$
\begin{aligned}
= \quad & -\frac{(n-1)^2(n-r)}{4(n-r-1)}\binom{n-r-2}{\frac{n}{2}-r-1}p^{\frac{n}{2}-r-1}q^{\frac{n}{2}-1} \\
& +\frac{r((r+4)n^2-2(3r+2)n+r)}{4(n-r-1)^2}\cdot\frac{n-r-2-\left(\frac{n}{2}-r\right)+1}{\frac{n}{2}-r}\binom{n-r-2}{\frac{n}{2}-r-1}p^{\frac{n}{2}-r}q^{\frac{n}{2}-2} \\
& +\frac{r(2n^3-(3r+2)n^2-r)}{4(n-r-1)^2}\binom{n-r-2}{\frac{n}{2}-r-1}p^{\frac{n}{2}-r-1}q^{\frac{n}{2}-1} \\
& -\frac{n^2}{4}\,\frac{n-r}{\frac{n}{2}-r}\cdot\frac{n-r-1}{(n-r-1)-\left(\frac{n}{2}-r-1\right)}\binom{n-r-2}{\frac{n}{2}-r-1}p^{\frac{n}{2}-r}q^{\frac{n}{2}} \\
= \quad & \Bigg(-\frac{(n-1)^2(n-r)}{4(n-r-1)}+\frac{r((r+4)n^2-2(3r+2)n+r)}{4(n-r-1)^2}\cdot\frac{n-2}{n-2r}\cdot\frac{p}{q} \\
& +\frac{r(2n^3-(3r+2)n^2-r)}{4(n-r-1)^2}-\frac{n^2(n-r)(n-r-1)}{(n-2r)n}\cdot pq\Bigg)\binom{n-r-2}{\frac{n}{2}-r-1}p^{\frac{n}{2}-r-1}q^{\frac{n}{2}-1} \\
= \quad & \Bigg(-\frac{(n-1)^2(n-r)}{4(n-r-1)}+\frac{r((r+4)n^2-2(3r+2)n+r)}{4(n-r-1)^2}\cdot\frac{n-2}{n-2r}\cdot\frac{n-2r-1}{n-1} \\
& +\frac{r(2n^3-(3r+2)n^2-r)}{4(n-r-1)^2}-\frac{n(n-r)(n-r-1)}{(n-2r)}\cdot\frac{(n-2r-1)(n-1)}{4(n-r-1)^2}\Bigg)\binom{n-r-2}{\frac{n}{2}-r-1}p^{\frac{n}{2}-r-1}q^{\frac{n}{2}-1} \\
= \quad & -\frac{n(n-2r-1)^2(n^3-(r+2)n^2+n+2r)}{2(n-1)(n-2r)(n-r-1)^2}\binom{n-r-2}{\frac{n}{2}-r-1}p^{\frac{n}{2}-r-1}q^{\frac{n}{2}-1} \\
\overset{n>2r}{>} \quad & -\frac{n(n-2r-1)^2(n^3-(r+2)n^2+n+n)}{2(n-1)(n-2r)(n-r-1)^2}\binom{n-r-2}{\frac{n}{2}-r-1}p^{\frac{n}{2}-r-1}q^{\frac{n}{2}-1} \\
= \quad & -\frac{n^2(n-2r-1)^2(n^2-(r+2)n+2)}{2(n-1)(n-2r)(n-r-1)^2}\binom{n-r-2}{\frac{n}{2}-r-1}p^{\frac{n}{2}-r-1}q^{\frac{n}{2}-1} \\
= \quad & -\frac{n^2(n-2r-1)(n^2-(r+2)n+2)}{(n-1)(n-2r)(n-r-1)}\cdot\underbrace{\frac{n-2r-1}{2(n-r-1)}}_{p}\binom{n-r-2}{\frac{n}{2}-r-1}p^{\frac{n}{2}-r-1}q^{\frac{n}{2}-1} \\
= \quad & -\frac{n^2(n-2r-1)(n^2-(r+2)n+2)}{(n-1)(n-2r)(n-r-1)}\binom{n-r-2}{\frac{n}{2}-r-1}p^{\frac{n}{2}-r}q^{\frac{n}{2}-1}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \mathsf{QMSC}(\widehat{\boldsymbol{\phi}})-\mathsf{QMSC}(\widehat{\boldsymbol{\sigma}}) \\
> \quad & \boxed{\frac{n}{4}+\frac{n^2}{4}}+\frac{n^2}{2^{n+1}}\binom{n}{\frac{n}{2}}\boxed{-\frac{(n-1)(n-r)}{2(n-r-1)}-\frac{(n-1)^2(n-r)}{4(n-r-1)}} \\
& -\frac{n^2(n-2r-1)(n^2-(r+2)n+2)}{(n-1)(n-2r)(n-r-1)}\binom{n-r-2}{\frac{n}{2}-r-1}p^{\frac{n}{2}-r}q^{\frac{n}{2}-1} \\
= \quad & -\frac{r(n+1)}{4(n-r-1)}+\underbrace{\frac{n^2}{2^{n+1}}\binom{n}{\frac{n}{2}}-\frac{n^2(n-2r-1)(n^2-(r+2)n+2)}{(n-1)(n-2r)(n-r-1)}\binom{n-r-2}{\frac{n}{2}-r-1}p^{\frac{n}{2}-r}q^{\frac{n}{2}-1}}_{\mathsf{G}}.
\end{aligned}
$$

We prove:

Lemma 4.2 *For any pair of an even integer $n \geq 80$ and an integer r such that $1 \leq r \leq \left\lfloor \frac{n-3}{2} \right\rfloor$,*

$$\mathsf{G} \;>\; \frac{r(n+1)}{4(n-r-1)}.$$

The proof will use the simple fact that for all numbers $x \geq 0$, $\exp(x) \geq 1 + x$.

Proof. We proceed by case analysis on the range of values of r.

$\boxed{1. \quad r = 1:}$ Note that in this case $p = \frac{n-3}{2(n-2)}$ and $q = \frac{n-1}{2(n-2)}$. So,

$$\begin{aligned}
& \mathsf{G} \\
= \quad & \frac{n^2}{2^{n+1}}\binom{n}{\frac{n}{2}} - \frac{n^2(n-3)(n-1)(n-2)}{(n-1)(n-2)^2}\binom{n-3}{\frac{n}{2}-2}\left(\frac{n-3}{2(n-2)}\right)^{\frac{n}{2}-1}\left(\frac{n-1}{2(n-2)}\right)^{\frac{n}{2}-1} \\
= \quad & \frac{n^2}{2^{n+1}}\binom{n}{\frac{n}{2}} - \frac{n^2(n-3)(n-1)(n-2)}{(n-1)(n-2)^2}\binom{n-3}{\frac{n}{2}-2}\cdot\frac{1}{2^{n-2}}\left(\frac{n-3}{n-2}\right)^{\frac{n}{2}-1}\left(\frac{n-1}{n-2}\right)^{\frac{n}{2}-1} \\
= \quad & \frac{n^2}{2^{n+1}}\binom{n}{\frac{n}{2}} - \frac{n^2(n-3)}{(n-2)}\cdot\frac{\frac{n}{2}-1}{n-2}\cdot\frac{\frac{n}{2}}{n-1}\cdot\frac{n-\frac{n}{2}}{n}\binom{n}{\frac{n}{2}}\cdot\frac{1}{2^{n-2}}\left(\frac{n-3}{n-2}\right)^{\frac{n}{2}-1}\left(\frac{n-1}{n-2}\right)^{\frac{n}{2}-1} \\
= \quad & \frac{n^2}{2^{n+1}}\binom{n}{\frac{n}{2}} - \frac{n^2(n-3)}{(n-2)}\cdot\frac{n}{8(n-1)}\binom{n}{\frac{n}{2}}\frac{1}{2^{n-2}}\left(\frac{n-3}{n-2}\right)^{\frac{n}{2}-1}\left(\frac{n-1}{n-2}\right)^{\frac{n}{2}-1} \\
= \quad & \frac{n^2}{2^{n+1}}\binom{n}{\frac{n}{2}}\left(1 - \frac{(n-3)n}{(n-2)(n-1)}\left(\frac{n-3}{n-2}\right)^{\frac{n}{2}-1}\left(\frac{n-1}{n-2}\right)^{\frac{n}{2}-1}\right) \\
> \quad & \frac{n^2}{2^{n+1}}\binom{n}{\frac{n}{2}}\left(1 - \left(\frac{n-3}{n-2}\right)^{\frac{n}{2}-1}\left(\frac{n-1}{n-2}\right)^{\frac{n}{2}-1}\right) \\
\stackrel{(5.1)}{\geq} \quad & n\sqrt{\frac{n}{2\pi}}\exp\left(\frac{1}{12n+1} - \frac{1}{3n}\right)\left(1 - \left(\frac{n-3}{n-2}\right)^{\frac{n}{2}-1}\left(\frac{n-1}{n-2}\right)^{\frac{n}{2}-1}\right) \\
\stackrel{\exp(x)\geq 1+x}{\geq} \quad & n\sqrt{\frac{n}{2\pi}}\left(1 + \frac{1}{12n+1} - \frac{1}{3n}\right)\left(1 - \left(\frac{n-3}{n-2}\right)^{\frac{n}{2}-1}\left(\frac{n-1}{n-2}\right)^{\frac{n}{2}-1}\right) \\
> \quad & n\sqrt{\frac{n}{2\pi}}\left(1 - \frac{1}{3n}\right)\left(1 - \left(\frac{n-3}{n-2}\right)^{\frac{n}{2}-1}\left(\frac{n-1}{n-2}\right)^{\frac{n}{2}-1}\right) \\
\stackrel{n\geq 3}{>} \quad & \frac{8n}{9}\sqrt{\frac{n}{2\pi}}\left(1 - \left(\frac{n-3}{n-2}\right)^{\frac{n}{2}-1}\left(\frac{n-1}{n-2}\right)^{\frac{n}{2}-1}\right) \\
> \quad & \frac{n\sqrt{n}}{3}\left(1 - \left(\frac{n-3}{n-2}\right)^{\frac{n}{2}-1}\left(\frac{n-1}{n-2}\right)^{\frac{n}{2}-1}\right) \\
= \quad & \frac{n\sqrt{n}}{3}\left(1 - \left(\frac{n-3}{n-2}\cdot\frac{n-1}{n-2}\right)^{\frac{n}{2}-1}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{n\sqrt{n}}{3}\left(1-\left(1-\frac{1}{(n-2)^2}\right)^{\frac{n}{2}-1}\right)\\
&\overset{(6.2)}{>} \frac{n\sqrt{n}}{3}\left(1-\left(1-\frac{1}{(n-2)^2}\binom{\frac{n}{2}-1}{1}+\frac{1}{(n-2)^4}\binom{\frac{n}{2}-1}{2}\right)\right)\\
&= \frac{n\sqrt{n}}{3}\left(\frac{1}{(n-2)^2}\binom{\frac{n}{2}-1}{1}-\frac{1}{(n-2)^4}\binom{\frac{n}{2}-1}{2}\right)\\
&> \frac{n\sqrt{n}}{3}\left(\frac{1}{2(n-2)}-\frac{1}{8(n-2)^2}\right)\\
&= \frac{n\sqrt{n}(4n-9)}{24(n-2)^2}\\
&\overset{n\geq 4}{>} \frac{n\sqrt{n}(n+1)}{24(n-2)^2}\\
&\overset{n\geq 32}{>} \frac{n+1}{4(n-2)}.
\end{aligned}
$$

$\boxed{2.\quad 2\leq r\leq \left\lfloor \frac{n-3}{2}\right\rfloor :}$ We shall use the estimations from Lemmas 5.1, 5.3, 6.3, 6.4 and 6.5. Clearly,

$$
\begin{aligned}
&\mathsf{G}\\
&= \frac{n^2}{2^{n+1}}\binom{n}{\frac{n}{2}}-\frac{n^2(n-2r-1)(n^2-(r+2)n+2)}{(n-1)(n-2r)(n-r-1)}\binom{n-r-2}{\frac{n}{2}-r-1}p^{\frac{n}{2}-r}q^{\frac{n}{2}-1}\\
&\overset{n\geq 4}{>} \frac{n^2}{2^{n+1}}\binom{n}{\frac{n}{2}}-\frac{n^2(n-2r-1)\left(n^2-(r+2)n+\frac{n}{2}\right)}{(n-1)(n-2r)(n-r-1)}\binom{n-r-2}{\frac{n}{2}-r-1}p^{\frac{n}{2}-r}q^{\frac{n}{2}-1}\\
&= \frac{n^2}{2^{n+1}}\binom{n}{\frac{n}{2}}-\frac{n^3(n-2r-1)\left(n-r-\frac{3}{2}\right)}{(n-1)(n-2r)(n-r-1)}\binom{n-r-2}{\frac{n}{2}-r-1}p^{\frac{n}{2}-r}q^{\frac{n}{2}-1}\\
&\overset{(5.1)}{\geq} n\sqrt{\frac{n}{2\pi}}\mathsf{exp}\left(\frac{1}{12n+1}-\frac{1}{3n}\right)-\frac{n^3(n-2r-1)\left(n-r-\frac{3}{2}\right)}{(n-1)(n-2r)(n-r-1)}\binom{n-r-2}{\frac{n}{2}-r-1}p^{\frac{n}{2}-r}q^{\frac{n}{2}-1}\\
&\overset{(5.3)}{\geq} n\sqrt{\frac{n}{2\pi}}\mathsf{exp}\left(\frac{1}{12n+1}-\frac{1}{3n}\right)-\frac{n^3(n-2r-1)\left(n-r-\frac{3}{2}\right)}{(n-1)(n-2r)(n-r-1)}\cdot\\
&\qquad\sqrt{\frac{(n-r-2)(n-2r-1)^2}{2\pi(n-2)(n-2r-2)(n-r-1)^2}}\mathsf{exp}\left(\frac{1}{12(n-r-2)}-\frac{1}{6n-11}-\frac{1}{6n-12r-11}\right)\\
&= n\sqrt{\frac{n}{2\pi}}\mathsf{exp}\left(\frac{1}{12n+1}-\frac{1}{3n}\right)\cdot\\
&\qquad\left(1-\frac{1}{n}\sqrt{\frac{2\pi}{n}}\mathsf{exp}\left(-\frac{1}{12n+1}+\frac{1}{3n}\right)\frac{n^3(n-2r-1)\left(n-r-\frac{3}{2}\right)}{(n-1)(n-2r)(n-r-1)}\cdot\right.\\
&\qquad\left.\sqrt{\frac{(n-r-2)(n-2r-1)^2}{2\pi(n-2)(n-2r-2)(n-r-1)^2}}\mathsf{exp}\left(\frac{1}{12(n-r-2)}-\frac{1}{6n-11}-\frac{1}{6n-12r-11}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= n\sqrt{\frac{n}{2\pi}}\mathsf{exp}\left(\frac{1}{12n+1}-\frac{1}{3n}\right)\cdot \\
&\quad \left(1-\sqrt{\frac{n^3\left(n-r-\frac{3}{2}\right)^2(n-r-2)(n-2r-1)^4}{(n-1)^2(n-2)(n-2r-2)(n-2r)^2(n-r-1)^4}}\cdot\right. \\
&\quad \left.\mathsf{exp}\left(\frac{1}{12(n-r-2)}-\frac{1}{6n-11}-\frac{1}{6n-12r-11}-\frac{1}{12n+1}+\frac{1}{3n}\right)\right) \\
&\overset{(6.3)}{>} n\sqrt{\frac{n}{2\pi}}\mathsf{exp}\left(\frac{1}{12n+1}-\frac{1}{3n}\right)\left(1-\sqrt{\frac{\underbrace{n^3\left(n-r-\frac{3}{2}\right)^2(n-r-2)(n-2r-1)^4}_{\mathsf{A}}}{\underbrace{(n-1)^2(n-2)(n-2r-2)(n-2r)^2(n-r-1)^4}_{\mathsf{B}}}}\right) \\
&\overset{(6.4)}{\geq} n\sqrt{\frac{n}{2\pi}}\left(1+\frac{1}{12n+1}-\frac{1}{3n}\right)\left(1-\sqrt{\frac{\mathsf{A}}{\mathsf{B}}}\right) \\
&> n\sqrt{\frac{n}{2\pi}}\left(1-\frac{1}{3n}\right)\left(1-\sqrt{\frac{\mathsf{A}}{\mathsf{B}}}\right) \\
&\overset{n\geq 3}{>} \frac{8n}{9}\sqrt{\frac{n}{2\pi}}\left(1-\sqrt{\frac{\mathsf{A}}{\mathsf{B}}}\right) \\
&> \frac{n\sqrt{n}}{3}\left(1-\sqrt{\frac{\mathsf{A}}{\mathsf{B}}}\right) \\
&= \frac{n\sqrt{n}}{3}\cdot\frac{\mathsf{B}-\mathsf{A}}{\sqrt{\mathsf{B}}(\sqrt{\mathsf{B}}+\sqrt{\mathsf{A}})} \\
&\overset{(6.4)}{\geq} \frac{n\sqrt{n}}{3}\cdot\frac{\mathsf{B}-\mathsf{A}}{\sqrt{\mathsf{B}}(\sqrt{\mathsf{B}}+\sqrt{\mathsf{B}})} \\
&= \frac{n\sqrt{n}}{6}\cdot\frac{\mathsf{B}-\mathsf{A}}{\mathsf{B}} \\
&\overset{(6.5)}{\geq} \frac{n\sqrt{n}}{6}\cdot\frac{3r(n+1)(n-1)^2(n-2r-2)(n-2r)^2(n-r-1)^3}{2\sqrt{n}\cdot(n-1)^2(n-2)(n-2r-2)(n-2r)^2(n-r-1)^4} \\
&= \frac{nr(n+1)}{4(n-2)(n-r-1)} \\
&> \frac{r(n+1)}{4(n-r-1)},
\end{aligned}
$$

and the claim follows. ■

The proof for the case where n is even is now complete.

$\boxed{\textbf{Case 2: } n \textbf{ is odd}}$ Then,

$$
\begin{aligned}
& \mathsf{D} \\
= \; & -\frac{(n-1)^2(n-r)}{4(n-r-1)}\binom{n-r-2}{\frac{n-1}{2}-r}p^{\frac{n-1}{2}-r}q^{\frac{n-3}{2}} \\
& +\frac{r\left((r+4)n^2-2(3r+2)n+r\right)}{4(n-r-1)^2}\binom{n-r-2}{\frac{n-1}{2}-r}p^{\frac{n-1}{2}-r}q^{\frac{n-3}{2}} \\
& +\frac{r\left(2n^3-(3r+2)n^2-r\right)}{4(n-r-1)^2}\binom{n-r-2}{\frac{n-3}{2}-r}p^{\frac{n-3}{2}-r}q^{\frac{n-1}{2}} \\
& -\frac{(n-1)(n-r)}{2(n-r-1)}\binom{n-r-1}{\frac{n-1}{2}-r}p^{\frac{n-1}{2}-r}q^{\frac{n-1}{2}}-\frac{(n-1)^2(n-r)}{4(n-r-1)}\binom{n-r-2}{\frac{n-3}{2}-r}p^{\frac{n-3}{2}-r}q^{\frac{n-1}{2}} \\
= \; & -\frac{(n-1)^2(n-r)}{4(n-r-1)}\binom{n-r-2}{\frac{n-1}{2}-r}p^{\frac{n-1}{2}-r}q^{\frac{n-3}{2}} \\
& +\frac{r\left((r+4)n^2-2(3r+2)n+r\right)}{4(n-r-1)^2}\binom{n-r-2}{\frac{n-1}{2}-r}p^{\frac{n-1}{2}-r}q^{\frac{n-3}{2}} \\
& +\frac{r\left(2n^3-(3r+2)n^2-r\right)}{4(n-r-1)^2}\cdot\frac{\frac{n-1}{2}-r}{n-r-2-\left(\frac{n-1}{2}-r\right)+1}\binom{n-r-2}{\frac{n-1}{2}-r}p^{\frac{n-3}{2}-r}q^{\frac{n-1}{2}} \\
& -\frac{(n-1)(n-r)}{2(n-r-1)}\cdot\frac{n-r-1}{n-r-1-\left(\frac{n-1}{2}-r\right)}\binom{n-r-2}{\frac{n-1}{2}-r}p^{\frac{n-1}{2}-r}q^{\frac{n-1}{2}} \\
& -\frac{(n-1)^2(n-r)}{4(n-r-1)}\cdot\frac{\frac{n-1}{2}-r}{n-r-2-\left(\frac{n-1}{2}-r\right)+1}\binom{n-r-2}{\frac{n-1}{2}-r}p^{\frac{n-3}{2}-r}q^{\frac{n-1}{2}} \\
= \; & \Bigg(-\frac{(n-1)^2(n-r)}{4(n-r-1)}+\frac{r\left((r+4)n^2-2(3r+2)n+r\right)}{4(n-r-1)^2}+\frac{r\left(2n^3-(3r+2)n^2-r\right)}{4(n-r-1)^2}\cdot\underbrace{\frac{n-2r-1}{n-1}\cdot\frac{q}{p}}_{1} \\
& -\frac{(n-1)(n-r)}{2(n-r-1)}\cdot\underbrace{\frac{2(n-r-1)}{n-1}\cdot q}_{1}-\frac{(n-1)^2(n-r)}{4(n-r-1)}\cdot\underbrace{\frac{n-2r-1}{n-1}\cdot\frac{q}{p}}_{1}\Bigg)\binom{n-r-2}{\frac{n-1}{2}-r}p^{\frac{n-1}{2}-r}q^{\frac{n-3}{2}} \\
= \; & \Bigg(-\frac{(n-1)^2(n-r)}{4(n-r-1)}+\frac{r\left((r+4)n^2-2(3r+2)n+r\right)}{4(n-r-1)^2}+\frac{r\left(2n^3-(3r+2)n^2-r\right)}{4(n-r-1)^2} \\
& -\frac{(n-1)(n-r)}{2(n-r-1)}-\frac{(n-1)^2(n-r)}{4(n-r-1)}\Bigg)\binom{n-r-2}{\frac{n-1}{2}-r}p^{\frac{n-1}{2}-r}q^{\frac{n-3}{2}} \\
= \; & -\frac{n(n-2r-1)\left(n^2-(r+1)n-r\right)}{2(n-r-1)^2}\binom{n-r-2}{\frac{n-1}{2}-r}p^{\frac{n-1}{2}-r}q^{\frac{n-3}{2}} \\
= \; & -\frac{n\left(n^2-(r+1)n-r\right)}{n-r-1}\cdot\underbrace{\frac{n-2r-1}{2(n-r-1)}}_{p}\binom{n-r-2}{\frac{n-1}{2}-r}p^{\frac{n-1}{2}-r}q^{\frac{n-3}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{n\left(n^2-(r+1)n-r\right)}{n-r-1}\binom{n-r-2}{\frac{n-1}{2}-r}p^{\frac{n+1}{2}-r}q^{\frac{n-3}{2}}\\
&\overset{r\geq 0}{>} -\frac{n\left(n^2-(r+1)n\right)}{n-r-1}\binom{n-r-2}{\frac{n-1}{2}-r}p^{\frac{n+1}{2}-r}q^{\frac{n-3}{2}}\\
&= -n^2\binom{n-r-2}{\frac{n-1}{2}-r}p^{\frac{n+1}{2}-r}q^{\frac{n-3}{2}}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\mathsf{QMSC}(\widehat{\boldsymbol{\phi}})-\mathsf{QMSC}(\widehat{\boldsymbol{\sigma}})\\
>\; &\boxed{\frac{n}{4}+\frac{n^2}{4}}+\frac{n(n+1)}{2^{n+1}}\binom{n}{\frac{n+1}{2}}\boxed{-\frac{(n-1)(n-r)}{2(n-r-1)}-\frac{(n-1)^2(n-r)}{4(n-r-1)}}-n^2\binom{n-r-2}{\frac{n-1}{2}-r}p^{\frac{n+1}{2}-r}q^{\frac{n-3}{2}}\\
=\; &-\frac{r(n+1)}{4(n-r-1)}+\underbrace{\frac{n(n+1)}{2^{n+1}}\binom{n}{\frac{n+1}{2}}-n^2\binom{n-r-2}{\frac{n-1}{2}-r}p^{\frac{n+1}{2}-r}q^{\frac{n-3}{2}}}_{\mathsf{H}}.
\end{aligned}
$$

We prove:

Lemma 4.3 *For any pair of an odd integer $n \geq 81$ and an integer r such that $1 \leq r \leq \left\lfloor \frac{n-3}{2} \right\rfloor$,*

$$
\mathsf{H} \quad > \quad \frac{r(n+1)}{4(n-r-1)}.
$$

Proof. We proceed by case analysis on the range of values of r.

$\boxed{1.\quad 1 \leq r \leq 2:}$ Note that $\frac{r(n+1)}{4(n-r-1)} = \frac{n+1}{4(n-2)}$ if $r=1$ and $\frac{n+1}{2(n-3)}$ if $r=2$. By substituting p and q from Lemma 3.3, we get that

$$
\mathsf{H} \;=\; \frac{n(n+1)}{2^{n+1}}\binom{n}{\frac{n+1}{2}}-n^2\binom{n-r-2}{\frac{n-1}{2}-r}\left(\frac{n-2r-1}{2(n-r-1)}\right)^{\frac{n+1}{2}-r}\left(\frac{n-1}{2(n-r-1)}\right)^{\frac{n-3}{2}}.
$$

By repeated application of the combinatorial identity $\binom{n}{k}=\frac{k+1}{n+1}\binom{n+1}{k+1}$, we obtain that

$$
\binom{n-r-2}{\frac{n-1}{2}-r} \;=\; \left(\prod_{i=1}^{r+1}\frac{\frac{n-1}{2}-r+i}{n-r-2+i}\right)\binom{n-1}{\frac{n+1}{2}}.
$$

Hence,

$$
\begin{aligned}
&\mathsf{H}-\frac{n(n+1)}{2^{n+1}}\binom{n}{\frac{n+1}{2}}\\
= \; &-n^2\left(\prod_{i=1}^{r+1}\frac{\frac{n-1}{2}-r+i}{n-r-2+i}\right)\binom{n-1}{\frac{n+1}{2}}\left(\frac{n-2r-1}{2(n-r-1)}\right)^{\frac{n+1}{2}-r}\left(\frac{n-1}{2(n-r-1)}\right)^{\frac{n-3}{2}}\\
= \; &-n^2\left(\prod_{i=1}^{r+1}\frac{\frac{n-1}{2}-r+i}{n-r-2+i}\right)\cdot\frac{n-\frac{n+1}{2}}{n}\binom{n}{\frac{n+1}{2}}\left(\frac{n-2r-1}{2(n-r-1)}\right)^{\frac{n+1}{2}-r}\left(\frac{n-1}{2(n-r-1)}\right)^{\frac{n-3}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&= -n^2 \cdot \frac{1}{2^{r+1}} \left(\prod_{i=1}^{r+1} \frac{n-1-2r+2i}{n-r-2+i}\right) \cdot \frac{n-1}{2n} \binom{n}{\frac{n+1}{2}} \frac{1}{2^{n-r-1}} \left(\frac{n-2r-1}{n-r-1}\right)^{\frac{n+1}{2}-r} \left(\frac{n-1}{n-r-1}\right)^{\frac{n-3}{2}} \\
&= -\frac{n(n+1)}{2^{n+1}} \binom{n}{\frac{n+1}{2}} \cdot \\
&\quad \underbrace{\frac{n-1}{n+1} \left(\prod_{i=1}^{r+1} \frac{n-1-2r+2i}{n-r-2+i}\right) \left(\frac{n-1}{n-r-1}\right)^{r-2}}_{\mathsf{A}} \left(\frac{n-2r-1}{n-r-1}\right)^{\frac{n+1}{2}-r} \left(\frac{n-1}{n-r-1}\right)^{\frac{n+1}{2}-r}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \mathsf{H} \\
= \quad & \frac{n(n+1)}{2^{n+1}} \binom{n}{\frac{n+1}{2}} \left(1 - \mathsf{A} \left(\frac{n-2r-1}{n-r-1}\right)^{\frac{n+1}{2}-r} \left(\frac{n-1}{n-r-1}\right)^{\frac{n+1}{2}-r}\right) \\
= \quad & \frac{n(n+1)}{2^{n+1}} \binom{n}{\frac{n+1}{2}} \left(1 - \mathsf{A} \left(\frac{n-2r-1}{n-r-1} \cdot \frac{n-1}{n-r-1}\right)^{\frac{n+1}{2}-r}\right) \\
= \quad & \frac{n(n+1)}{2^{n+1}} \binom{n}{\frac{n+1}{2}} \left(1 - \mathsf{A} \left(1 - \frac{r^2}{(n-r-1)^2}\right)^{\frac{n+1}{2}-r}\right) \\
\overset{(6.8)}{>} \quad & \frac{n(n+1)}{2^{n+1}} \binom{n}{\frac{n+1}{2}} \left(1 - \mathsf{A} \left(1 - \frac{r^2}{(n-r-1)^2} \binom{\frac{n+1}{2}-r}{1} + \frac{r^4}{(n-r-1)^4} \binom{\frac{n+1}{2}-r}{2}\right)\right) \\
= \quad & \frac{n(n+1)}{2^{n+1}} \binom{n}{\frac{n+1}{2}} \cdot \\
& \left(1 - \mathsf{A} \left(1 - \frac{r^2}{(n-r-1)^2} \left(\frac{n+1}{2} - r\right) + \frac{r^4}{(n-r-1)^4} \frac{\left(\frac{n+1}{2}-r\right)\left(\frac{n+1}{2}-r-1\right)}{2}\right)\right) \\
= \quad & \frac{n(n+1)}{2^{n+1}} \binom{n}{\frac{n+1}{2}} \cdot \\
& \left(1 - \mathsf{A} \left(1 - \frac{r^2}{(n-r-1)^2} \frac{n-2r+1}{2} + \frac{r^4}{(n-r-1)^4} \frac{(n-2r+1)(n-2r-1)}{8}\right)\right) \\
\overset{r\in\{1,2\}}{\geq} \quad & \frac{n(n+1)}{2^{n+1}} \binom{n}{\frac{n+1}{2}} \cdot \\
& \left(1 - \mathsf{A} \left(1 - \frac{r^2}{(n-r-1)^2} \frac{n-r-1}{2} + \frac{r^4}{(n-r-1)^4} \frac{(n-2r+1)(n-2r-1)}{8}\right)\right) \\
\overset{(6.9)}{>} \quad & \frac{n(n+1)}{2^{n+1}} \binom{n}{\frac{n+1}{2}} \cdot \\
& \left(1 - \mathsf{A} \left(1 - \frac{r^2}{(n-r-1)^2} \frac{n-r-1}{2} + \frac{r^4}{(n-r-1)^4} \frac{(n-r-1)^2}{8}\right)\right) \\
= \quad & \frac{n(n+1)}{2^{n+1}} \binom{n}{\frac{n+1}{2}} \left(1 - \mathsf{A} \left(1 - \frac{r^2}{2(n-r-1)} + \frac{r^4}{8(n-r-1)^2}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= && \frac{n(n+1)}{2^{n+1}}\binom{n}{\frac{n+1}{2}} \cdot \begin{cases} \dfrac{4n-9}{8(n-2)^2}, & r=1 \\ \dfrac{n^2-4n-1}{(n-2)(n-3)^2}, & r=2 \end{cases} \\
&\overset{(5.2)}{\geq} && n\sqrt{\frac{n}{2\pi}}\,\mathsf{exp}\left(\frac{1}{12n+1}-\frac{1}{3n-3}\right) \cdot \begin{cases} \dfrac{4n-9}{8(n-2)^2}, & r=1 \\ \dfrac{n^2-4n-1}{(n-2)(n-3)^2}, & r=2 \end{cases} \\
&\overset{\mathsf{exp}(x)\geq 1+x}{\geq} && n\sqrt{\frac{n}{2\pi}}\left(1+\frac{1}{12n+1}-\frac{1}{3n-3}\right) \cdot \begin{cases} \dfrac{4n-9}{8(n-2)^2}, & r=1 \\ \dfrac{n^2-4n-1}{(n-2)(n-3)^2}, & r=2 \end{cases} \\
&> && n\sqrt{\frac{n}{2\pi}}\left(1-\frac{1}{3n-3}\right) \cdot \begin{cases} \dfrac{4n-9}{8(n-2)^2}, & r=1 \\ \dfrac{n^2-4n-1}{(n-2)(n-3)^2}, & r=2 \end{cases} \\
&\overset{n\geq 4}{>} && \frac{8n}{9}\sqrt{\frac{n}{2\pi}} \cdot \begin{cases} \dfrac{4n-9}{8(n-2)^2}, & r=1 \\ \dfrac{n^2-4n-1}{(n-2)(n-3)^2}, & r=2 \end{cases} \\
&> && \frac{n\sqrt{n}}{3} \cdot \begin{cases} \dfrac{4n-9}{8(n-2)^2}, & r=1 \\ \dfrac{n^2-4n-1}{(n-2)(n-3)^2}, & r=2 \end{cases} \\
&\overset{n\geq 7}{>} && \frac{n\sqrt{n}}{6} \cdot \begin{cases} \dfrac{n+1}{4(n-2)^2}, & r=1 \\ \dfrac{2(n+1)}{(n+1)(n-3)}, & r=2 \end{cases} \\
&\overset{n\geq 4}{>} && \begin{cases} \dfrac{n+1}{4(n-2)}, & r=1 \\ \dfrac{n+1}{2(n-3)}, & r=2. \end{cases}
\end{aligned}
$$

$\boxed{2.\quad 3 \leq r \leq \left\lfloor \frac{n-3}{2} \right\rfloor :}$ We shall use the estimations from Lemmas 5.2, 5.4, 6.10, 6.11 and 6.12. Clearly,

$$
\begin{aligned}
& && \mathsf{H} \\
&= && \frac{n(n+1)}{2^{n+1}}\binom{n}{\frac{n+1}{2}} - n^2\binom{n-r-2}{\frac{n-1}{2}-r} p^{\frac{n+1}{2}-r} q^{\frac{n-3}{2}} \\
&\overset{(5.2)}{\geq} && n\sqrt{\frac{n}{2\pi}}\mathsf{exp}\left(\frac{1}{12n+1}-\frac{1}{3n-3}\right) - n^2\binom{n-r-2}{\frac{n-1}{2}-r} p^{\frac{n+1}{2}-r} q^{\frac{n-3}{2}} \\
&\overset{(5.4)}{\geq} && n\sqrt{\frac{n}{2\pi}}\mathsf{exp}\left(\frac{1}{12n+1}-\frac{1}{3n-3}\right) \cdot \\
& && -n^2\sqrt{\frac{(n-r-2)(n-2r-1)}{2\pi(n-3)(n-r-1)^2}}\,\mathsf{exp}\left(\frac{1}{12(n-r-2)}-\frac{1}{6n-17}-\frac{1}{6n-12r-5}\right) \\
&= && n\sqrt{\frac{n}{2\pi}}\,\mathsf{exp}\left(\frac{1}{12n+1}-\frac{1}{3n-3}\right)\left(1-\frac{1}{n}\sqrt{\frac{2\pi}{n}}\mathsf{exp}\left(-\frac{1}{12n+1}+\frac{1}{3n-3}\right)\right. \\
& && \left. n^2\sqrt{\frac{(n-r-2)(n-2r-1)}{2\pi(n-3)(n-r-1)^2}}\,\mathsf{exp}\left(\frac{1}{12(n-r-2)}-\frac{1}{6n-17}-\frac{1}{6n-12r-5}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= && n\sqrt{\frac{n}{2\pi}}\,\mathsf{exp}\left(\frac{1}{12n+1}-\frac{1}{3n-3}\right)\left(1-\sqrt{\frac{n(n-r-2)(n-2r-1)}{(n-3)(n-r-1)^2}}\cdot\right.\\
& && \left.\mathsf{exp}\left(\frac{1}{12(n-r-2)}-\frac{1}{6n-17}-\frac{1}{6n-12r-5}+\frac{1}{12n+1}-\frac{1}{3n-3}\right)\right)\\
&\overset{(6.10)}{>} && n\sqrt{\frac{n}{2\pi}}\,\mathsf{exp}\left(\frac{1}{12n+1}-\frac{1}{3n-3}\right)\left(1-\sqrt{\frac{n(n-r-2)(n-2r-1)}{(n-3)(n-r-1)^2}}\right)\\
&\overset{(6.11)}{>} && n\sqrt{\frac{n}{2\pi}}\left(1+\frac{1}{12n+1}-\frac{1}{3n-3}\right)\left(1-\sqrt{\frac{n(n-r-2)(n-2r-1)}{(n-3)(n-r-1)^2}}\right)\\
&> && n\sqrt{\frac{n}{2\pi}}\left(1-\frac{1}{3n-3}\right)\left(1-\sqrt{\frac{n(n-r-2)(n-2r-1)}{(n-3)(n-r-1)^2}}\right)\\
&\overset{n\geq 4}{>} && \frac{8n}{9}\sqrt{\frac{n}{2\pi}}\left(1-\sqrt{\frac{n(n-r-2)(n-2r-1)}{(n-3)(n-r-1)^2}}\right)\\
&> && \frac{n\sqrt{n}}{3}\left(1-\sqrt{\frac{n(n-r-2)(n-2r-1)}{(n-3)(n-r-1)^2}}\right)\\
&= && \frac{n\sqrt{n}}{3}\cdot\frac{(n-3)(n-r-1)^2-n(n-r-2)(n-2r-1)}{\sqrt{(n-3)(n-r-1)^2}\left(\sqrt{(n-3)(n-r-1)^2}+\sqrt{n(n-r-2)(n-2r-1)}\right)}\\
&\overset{(6.11)}{\geq} && \frac{n\sqrt{n}}{3}\cdot\frac{(n-3)(n-r-1)^2-n(n-r-2)(n-2r-1)}{\sqrt{(n-3)(n-r-1)^2}\left(\sqrt{(n-3)(n-r-1)^2}+\sqrt{(n-3)(n-r-1)^2}\right)}\\
&= && \frac{n\sqrt{n}}{6}\cdot\frac{(n-3)(n-r-1)^2-n(n-r-2)(n-2r-1)}{(n-3)(n-r-1)^2}\\
&\overset{(6.12)}{\geq} && \frac{n\sqrt{n}}{6}\cdot\frac{3r(n+1)(n-r-1)}{2\sqrt{n}\cdot(n-3)(n-r-1)^2}\\
&= && \frac{nr(n+1)}{4(n-3)(n-r-1)}\\
&> && \frac{r(n+1)}{4(n-r-1)}.
\end{aligned}
$$

as needed. The proof is now complete. ∎

The proof for the case where n is odd is complete, and this completes the proof. ∎

5 Estimations

We collect together all estimations (and their proofs) used in the proof of Theorem 4.1. Some of these estimations refer to the probabilities p and q introduced in Lemma 3.3. The proofs of the estimations rely on Lemmas 2.1 and 2.2. We first prove:

Lemma 5.1 *For all even integers $n \geq 2$,*

$$n\sqrt{\frac{n}{2\pi}}\,\mathsf{exp}\left(\frac{1}{12n+1}-\frac{1}{3n}\right) \;\leq\; \frac{n^2}{2^{n+1}}\binom{n}{\frac{n}{2}}.$$

Proof. By Lemma 2.1,

$$\begin{aligned}
\frac{n^2}{2^{n+1}}\binom{n}{\frac{n}{2}} &= \frac{n^2}{2^{n+1}}\frac{n!}{\left(\left(\frac{n}{2}\right)!\right)^2} \\
&\geq \frac{n^2}{2^{n+1}}\frac{\sqrt{2\pi}\,n^{n+\frac{1}{2}}\,\mathsf{exp}\left(-n+\frac{1}{12n+1}\right)}{\left(\sqrt{2\pi}\left(\frac{n}{2}\right)^{\frac{n+1}{2}}\mathsf{exp}\left(-\frac{n}{2}+\frac{1}{6n}\right)\right)^2} \\
&= n\sqrt{\frac{n}{2\pi}}\,\mathsf{exp}\left(\frac{1}{12n+1}-\frac{1}{3n}\right),
\end{aligned}$$

as needed. ■

We continue to prove:

Lemma 5.2 *For all odd integers $n \geq 3$,*

$$n\sqrt{\frac{n}{2\pi}}\mathsf{exp}\left(\frac{1}{12n+1}-\frac{1}{3n-3}\right) \;\leq\; \frac{n(n+1)}{2^{n+1}}\binom{n}{\frac{n+1}{2}}.$$

The proof will use the fact that for all $n \geq 3$, $\left(\frac{n}{n-1}\right)^n > \mathsf{exp}(1)$.

Proof. By Lemma 2.1,

$$\begin{aligned}
\frac{n(n+1)}{2^{n+1}}\binom{n}{\frac{n+1}{2}} &= \frac{n(n+1)}{2^{n+1}}\frac{n!}{\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!} \\
&= \frac{n}{2^n}\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2} \\
&\geq \frac{n}{2^n}\frac{\sqrt{2\pi}\,n^{n+\frac{1}{2}}\,\mathsf{exp}\left(-n+\frac{1}{12n+1}\right)}{\left(\sqrt{2\pi}\left(\frac{n-1}{2}\right)^{\frac{n}{2}}\mathsf{exp}\left(-\frac{n-1}{2}+\frac{1}{6n-6}\right)\right)^2} \\
&= n\left(\frac{n}{n-1}\right)^n\sqrt{\frac{n}{2\pi}}\,\mathsf{exp}\left(\frac{1}{12n+1}-\frac{1}{3n-3}-1\right) \\
&> n\,\mathsf{exp}(1)\sqrt{\frac{n}{2\pi}}\,\mathsf{exp}\left(\frac{1}{12n+1}-\frac{1}{3n-3}-1\right) \\
&= n\sqrt{\frac{n}{2\pi}}\,\mathsf{exp}\left(\frac{1}{12n+1}-\frac{1}{3n-3}\right),
\end{aligned}$$

as needed. ■

We now prove:

Lemma 5.3 *For all pairs of even integers $n \geq 4$ and integers r such that $1 \leq r \leq \left\lfloor \frac{n-3}{2} \right\rfloor$,*

$$\begin{aligned}
&\binom{n-r-2}{\frac{n-2}{2}-r} p^{\frac{n}{2}-r} q^{\frac{n-2}{2}} \\
\leq \ & \sqrt{\frac{(n-r-2)(n-2r-1)^2}{2\pi(n-2)(n-2r-2)(n-r-1)^2}} \,\mathsf{exp}\left(\frac{1}{12(n-r-2)} - \frac{1}{6n-11} - \frac{1}{6n-12r-11}\right).
\end{aligned}$$

Proof. By Lemma 2.2,

$$\begin{aligned}
&\binom{n-r-2}{\frac{n-2}{2}-r} p^{\frac{n}{2}-r} q^{\frac{n-2}{2}} \\
\leq \ & p \max_{0 \leq x \leq 1} \binom{n-r-2}{\frac{n-2}{2}-r} x^{\frac{n-2}{2}-r}(1-x)^{n-r-2-\left(\frac{n-2}{2}-r\right)} \\
= \ & p \max_{0 \leq x \leq 1} \mathsf{b}_{\frac{n-2}{2}-r, n-r-2}(x) \\
= \ & p \binom{n-r-2}{\frac{n-2}{2}-r} \left(\frac{n-2}{2}-r\right)^{\frac{n-2}{2}-r} (n-r-2)^{-(n-r-2)} \left(n-r-2-\left(\frac{n-2}{2}-r\right)\right)^{n-r-2-\left(\frac{n-2}{2}-r\right)} \\
= \ & p \frac{(n-r-2)!}{\left(\frac{n-2}{2}\right)! \left(\frac{n-2}{2}-r\right)!} \left(\frac{n-2}{2}-r\right)^{\frac{n-2}{2}-r} (n-r-2)^{-n+r+2} \left(\frac{n-2}{2}\right)^{\frac{n-2}{2}}.
\end{aligned}$$

Hence, by Lemma 2.1,

$$\begin{aligned}
&\binom{n-r-2}{\frac{n-2}{2}-r} p^{\frac{n}{2}-r} q^{\frac{n-2}{2}} \\
\leq \ & p \frac{\sqrt{2\pi}\,(n-r-2)^{n-r-2+\frac{1}{2}} \,\mathsf{exp}\left(-(n-r-2) + \frac{1}{12(n-r-2)}\right)}{\sqrt{2\pi}\left(\frac{n-2}{2}\right)^{\frac{n-2}{2}+\frac{1}{2}} \,\mathsf{exp}\left(-\left(\frac{n-2}{2}\right) + \frac{1}{12\left(\frac{n-2}{2}\right)+1}\right)} \cdot \\
& \frac{1}{\sqrt{2\pi}\left(\frac{n-2}{2}-r\right)^{\frac{n-2}{2}-r+\frac{1}{2}} \,\mathsf{exp}\left(-\left(\frac{n-2}{2}-r\right) + \frac{1}{12\left(\frac{n-2}{2}-r\right)+1}\right)} \cdot \\
& \left(\frac{n-2}{2}-r\right)^{\frac{n-2}{2}-r} (n-r-2)^{-n+r+2} \left(\frac{n-2}{2}\right)^{\frac{n-2}{2}} \\
= \ & p \frac{1}{\sqrt{2\pi}} \frac{(n-r-2)^{n-r-\frac{3}{2}}}{\left(\frac{n-2}{2}\right)^{\frac{n-1}{2}} \left(\frac{n-2}{2}-r\right)^{\frac{n-1}{2}-r}} \,\mathsf{exp}\left(\frac{1}{12(n-r-2)} - \frac{1}{6n-11} - \frac{1}{6n-12r-11}\right) \cdot \\
& \left(\frac{n-2}{2}-r\right)^{\frac{n-2}{2}-r} (n-r-2)^{-n+r+2} \left(\frac{n-2}{2}\right)^{\frac{n-2}{2}}
\end{aligned}$$

$$\begin{aligned}
&= p\,\frac{1}{\sqrt{2\pi}}\frac{(n-r-2)^{n-r-\frac{3}{2}-n+r+2}}{\left(\frac{n-2}{2}\right)^{\frac{n-1}{2}-\frac{n-2}{2}}\left(\frac{n-2}{2}-r\right)^{\frac{n-1}{2}-r-\left(\frac{n-2}{2}-r\right)}}\,. \\
&\quad \exp\left(\frac{1}{12(n-r-2)}-\frac{1}{6n-11}-\frac{1}{6n-12r-11}\right) \\
&= p\sqrt{\frac{n-r-2}{2\pi\left(\frac{n-2}{2}\right)\left(\frac{n-2}{2}-r\right)}}\exp\left(\frac{1}{12(n-r-2)}-\frac{1}{6n-11}-\frac{1}{6n-12r-11}\right) \\
&= \frac{n-2r-1}{2(n-r-1)}\sqrt{\frac{n-r-2}{2\pi\left(\frac{n-2}{2}\right)\left(\frac{n-2}{2}-r\right)}}\exp\left(\frac{1}{12(n-r-2)}-\frac{1}{6n-11}-\frac{1}{6n-12r-11}\right) \\
&= \sqrt{\frac{(n-r-2)(n-2r-1)^2}{2\pi(n-2)(n-2r-2)(n-r-1)^2}}\exp\left(\frac{1}{12(n-r-2)}-\frac{1}{6n-11}-\frac{1}{6n-12r-11}\right),
\end{aligned}$$

as needed. ∎

We continue to prove:

Lemma 5.4 *For all pairs of odd integers n and integers r such that $1 \le r \le \left\lfloor \frac{n-3}{2} \right\rfloor$,*

$$\begin{aligned}
&\binom{n-r-2}{\frac{n-1}{2}-r} p^{\frac{n+1}{2}-r} q^{\frac{n-3}{2}} \\
\le\ &\sqrt{\frac{(n-r-2)(n-2r-1)}{2\pi(n-3)(n-r-1)^2}}\exp\left(\frac{1}{12(n-r-2)}-\frac{1}{6n-17}-\frac{1}{6n-12r-5}\right).
\end{aligned}$$

Proof. By Lemma 2.2,

$$\begin{aligned}
&\binom{n-r-2}{\frac{n-1}{2}-r} p^{\frac{n+1}{2}-r} q^{\frac{n-3}{2}} \\
\le\ & p \max_{0\le x\le 1} \binom{n-r-2}{\frac{n-1}{2}-r} x^{\frac{n-1}{2}-r}(1-x)^{n-r-2-\left(\frac{n-1}{2}\right)-r} \\
=\ & p \max_{0\le x\le 1} \mathsf{b}_{\frac{n-1}{2}-r,n-r-2}(x) \\
=\ & p\binom{n-r-2}{\frac{n-1}{2}-r}\left(\frac{n-1}{2}-r\right)^{\frac{n-1}{2}-r}(n-r-2)^{-(n-r-2)}\left(n-r-2-\left(\frac{n-1}{2}-r\right)\right)^{n-r-2-\left(\frac{n-1}{2}-r\right)} \\
=\ & p\frac{(n-r-2)!}{\left(\frac{n-3}{2}\right)!\left(\frac{n-1}{2}-r\right)!}\left(\frac{n-1}{2}-r\right)^{\frac{n-1}{2}-r}(n-r-2)^{-n+r+2}\left(\frac{n-3}{2}\right)^{\frac{n-3}{2}}.
\end{aligned}$$

Hence, by Lemma 2.1,

$$
\begin{aligned}
& \binom{n-r-2}{\frac{n-1}{2}-r} p^{\frac{n+1}{2}-r} q^{\frac{n-3}{2}} \\
\leq \;& p \, \frac{\sqrt{2\pi}\,(n-r-2)^{n-r-2+\frac{1}{2}} \operatorname{exp}\left(-(n-r-2)+\frac{1}{12(n-r-2)}\right)}{\sqrt{2\pi}\left(\frac{n-3}{2}\right)^{\frac{n-3}{2}+\frac{1}{2}} \operatorname{exp}\left(-\left(\frac{n-3}{2}\right)+\frac{1}{12\left(\frac{n-3}{2}\right)+1}\right)} \cdot \\
& \frac{1}{\sqrt{2\pi}\left(\frac{n-1}{2}-r\right)^{\frac{n-1}{2}-r+\frac{1}{2}} \operatorname{exp}\left(-\left(\frac{n-1}{2}-r\right)+\frac{1}{12\left(\frac{n-1}{2}-r\right)+1}\right)} \\
& \left(\frac{n-1}{2}-r\right)^{\frac{n-1}{2}-r} (n-r-2)^{-n+r+2} \left(\frac{n-3}{2}\right)^{\frac{n-3}{2}} \\
= \;& p \, \frac{1}{\sqrt{2\pi}} \frac{(n-r-2)^{n-r-\frac{3}{2}}}{\left(\frac{n-3}{2}\right)^{\frac{n-2}{2}}\left(\frac{n-1}{2}-r\right)^{\frac{n}{2}-r}} \operatorname{exp}\left(\frac{1}{12(n-r-2)}-\frac{1}{6n-17}-\frac{1}{6n-12r-5}\right) \\
& \left(\frac{n-1}{2}-r\right)^{\frac{n-1}{2}-r} (n-r-2)^{-n+r+2} \left(\frac{n-3}{2}\right)^{\frac{n-3}{2}} \\
= \;& p \, \frac{1}{\sqrt{2\pi}} \frac{(n-r-2)^{n-r-\frac{3}{2}-n+r+2}}{\left(\frac{n-3}{2}\right)^{\frac{n-2}{2}-\frac{n-3}{2}}\left(\frac{n-1}{2}-r\right)^{\frac{n}{2}-r-\left(\frac{n-1}{2}-r\right)}} \cdot \\
& \operatorname{exp}\left(\frac{1}{12(n-r-2)}-\frac{1}{6n-17}-\frac{1}{6n-12r-5}\right) \\
= \;& p \sqrt{\frac{n-r-2}{2\pi\left(\frac{n-3}{2}\right)\left(\frac{n-1}{2}-r\right)}} \operatorname{exp}\left(\frac{1}{12(n-r-2)}-\frac{1}{6n-17}-\frac{1}{6n-12r-5}\right) \\
= \;& \frac{n-2r-1}{2(n-r-1)} \sqrt{\frac{n-r-2}{2\pi\left(\frac{n-3}{2}\right)\left(\frac{n-1}{2}-r\right)}} \operatorname{exp}\left(\frac{1}{12(n-r-2)}-\frac{1}{6n-17}-\frac{1}{6n-12r-5}\right) \\
= \;& \sqrt{\frac{(n-r-2)(n-2r-1)}{2\pi(n-3)(n-r-1)^2}} \operatorname{exp}\left(\frac{1}{12(n-r-2)}-\frac{1}{6n-17}-\frac{1}{6n-12r-5}\right),
\end{aligned}
$$

as needed. ∎

6 Technical Claims

We collect together the simple technical claims (together with their proofs) that were used before. We first prove:

Lemma 6.1 *For each pair of integers $n \geq 2$ and u such that $n > 2u$,*

$$\frac{n}{2^{2u}}\binom{n-1}{\left\lceil \frac{n}{2}\right\rceil - 1} - (n-2u)\binom{n-2u-1}{\left\lceil \frac{n-2u}{2}\right\rceil - 1} \quad > \quad 0.$$

Proof. We consider separately the two cases where n is even or odd. Assume first that n is even. Then,

$$\begin{aligned}
\frac{n}{2^{2u}}\binom{n-1}{\left\lceil \frac{n}{2}\right\rceil - 1} &= \frac{n}{2^{2u}}\binom{n-1}{\frac{n}{2}-1} \\
&= \frac{n}{2^{2u}}\frac{(n-1)!}{\frac{n}{2}!\left(\frac{n}{2}-1\right)!} \\
&= \frac{n}{2^{2u}}\frac{(n-1)(n-2)\cdot\ldots\cdot(n-2u+1)(n-2u)}{\frac{n}{2}\cdot\ldots\cdot\left(\frac{n}{2}-u+1\right).\left(\frac{n}{2}-1\right)\cdot\ldots\cdot\left(\frac{n}{2}-u\right)}\frac{(n-2u-1)!}{\left(\frac{n}{2}-u\right)!\left(\frac{n}{2}-u-1\right)!} \\
&= \frac{n}{2^{2u}}\frac{(n-1)(n-2)\cdot\ldots\cdot(n-2u+1)(n-2u)}{\frac{n}{2}\cdot\ldots\cdot\left(\frac{n}{2}-u+1\right).\left(\frac{n}{2}-1\right)\cdot\ldots\cdot\left(\frac{n}{2}-u\right)}\binom{n-2u-1}{\frac{n}{2}-u-1} \\
&= \frac{n}{2^{2u}}\frac{2^{2u}(n-1)(n-3)\cdot\ldots\cdot(n-2u+3)(n-2u+1)}{n(n-2)\cdot\ldots\cdot(n-2u+4)(n-2u+2)}\binom{n-2u-1}{\frac{n}{2}-u-1} \\
&> (n-2u+1)\binom{n-2u-1}{\frac{n}{2}-u-1} \\
&> (n-2u)\binom{n-2u-1}{\frac{n}{2}-u-1} = (n-2u)\binom{n-2u-1}{\left\lceil \frac{n-2u}{2}\right\rceil - 1}.
\end{aligned}$$

Assume now that n is odd. Then,

$$\begin{aligned}
&\frac{n}{2^{2u}}\binom{n-1}{\left\lceil \frac{n}{2}\right\rceil - 1} \\
= \;& \frac{n}{2^{2u}}\binom{n-1}{\frac{n+1}{2}-1} \\
= \;& \frac{n}{2^{2u}}\frac{(n-1)!}{\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}-1\right)!} \\
= \;& \frac{n}{2^{2u}}\frac{(n-1)(n-2)\cdot\ldots\cdot(n-2u+1)(n-2u)}{\left(\frac{n-1}{2}\right)\cdot\ldots\cdot\left(\frac{n-1}{2}-u+1\right).\left(\frac{n+1}{2}-1\right)\cdot\ldots\cdot\left(\frac{n+1}{2}-u\right)}\frac{(n-2u-1)!}{\left(\frac{n-1}{2}-u\right)!\left(\frac{n+1}{2}-u-1\right)} \\
= \;& \frac{n}{2^{2u}}\frac{(n-1)(n-2)\cdot\ldots\cdot(n-2u+1)(n-2u)}{\left(\frac{n-1}{2}\right)\cdot\ldots\cdot\left(\frac{n-1}{2}-u+1\right).\left(\frac{n+1}{2}-1\right)\cdot\ldots\cdot\left(\frac{n+1}{2}-u\right)}\binom{n-2u-1}{\frac{n+1}{2}-u-1} \\
= \;& \frac{n}{2^{2u}}\frac{2^{2u}(n-2)(n-4)\cdot\ldots\cdot(n-2u+2)(n-2u)}{(n-1)(n-3)\cdot\ldots\cdot(n-2u+3)(n-2u+1)}\binom{n-2u-1}{\frac{n+1}{2}-u-1} \\
> \;& \frac{n(n-2u)}{n-1}\binom{n-2u-1}{\frac{n+1}{2}-u-1} \\
> \;& (n-2u)\binom{n-2u-1}{\frac{n+1}{2}-u-1} = (n-2u)\binom{n-2u-1}{\left\lceil \frac{n-2u}{2}\right\rceil - 1}.
\end{aligned}$$

This completes the proof. $\blacksquare$

We now prove:

Lemma 6.2 *For all even integers $n \geq 4$, it holds that*

$$\left(1 - \frac{1}{(n-2)^2}\right)^{\frac{n}{2}-1} \quad < \quad 1 - \frac{1}{(n-2)^2}\binom{\frac{n}{2}-1}{1} + \frac{1}{(n-2)^4}\binom{\frac{n}{2}-1}{2}.$$

Proof. Consider the binomial expansion of $\left(1 - \frac{1}{(n-2)^2}\right)^{\frac{n}{2}-1}$. Note that if $\frac{n}{2} - 1$ is odd, the last term in the binomial expansion is negative; so, consider the binomial expansion without any such last term. Consider any pair of consecutive terms

$$-\left(\frac{1}{(n-2)^2}\right)^i \binom{\frac{n}{2}-1}{i} \text{ and } \left(\frac{1}{(n-2)^2}\right)^{i+1} \binom{\frac{n}{2}-1}{i+1},$$

where $3 \leq i \leq \frac{n}{2} - 1$ and i is odd, in the binomial expansion of $\left(1 - \frac{1}{(n-2)^2}\right)^{\frac{n}{2}-1}$. Clearly,

$$\begin{aligned}
& -\left(\frac{1}{(n-2)^2}\right)^i \binom{\frac{n}{2}-1}{i} + \left(\frac{1}{(n-2)^2}\right)^{i+1} \binom{\frac{n}{2}-2}{i+1} \\
= \quad & \left(\frac{1}{(n-2)^2}\right)^i \left(-\binom{\frac{n}{2}-1}{i} + \frac{1}{(n-2)^2}\binom{\frac{n}{2}-1}{i+1}\right) \\
= \quad & \left(\frac{1}{(n-2)^2}\right)^i \left(-\binom{\frac{n}{2}-1}{i} + \frac{1}{(n-2)^2}\frac{(n-2)-2i}{2(i+1)}\binom{\frac{n}{2}-1}{i}\right) \\
\stackrel{i\geq 3}{<} \quad & \left(\frac{1}{(n-2)^2}\right)^i \binom{\frac{n}{2}-1}{i} \left(-1 + \frac{1}{(n-2)^2} \cdot \frac{n-2}{8}\right) \\
= \quad & \left(\frac{1}{(n-2)^2}\right)^i \binom{\frac{n}{2}-1}{i} \left(-1 + \frac{1}{8(n-2)}\right) \\
\stackrel{n\geq 4}{<} \quad & 0.
\end{aligned}$$

It follows that

$$\begin{aligned}
& \left(1 - \frac{1}{(n-2)^2}\right)^{\frac{n}{2}-1} \\
= \quad & 1 - \frac{1}{(n-2)^2}\binom{\frac{n}{2}-1}{1} + \frac{1}{(n-2)^4}\binom{\frac{n}{2}-1}{2} + \sum_{i=3}^{\frac{n}{2}-1}(-1)^i\left(\frac{1}{(n-2)^2}\right)^i\binom{\frac{n}{2}-1}{i} \\
< \quad & 1 - \frac{1}{(n-2)^2}\binom{\frac{n}{2}-1}{1} + \frac{1}{(n-2)^4}\binom{\frac{n}{2}-1}{2},
\end{aligned}$$

as needed. $\blacksquare$

We continue to prove:

Lemma 6.3 *For all pairs of integers $n \geq 4$ and $r \geq 1$,*

$$\frac{1}{12(n-r-2)} - \frac{1}{6n-11} - \frac{1}{6n-12r-11} - \frac{1}{12n+1} + \frac{1}{3n} \quad < \quad 0.$$

Proof. Clearly,

$$\begin{aligned}
& \frac{1}{12(n-r-2)}-\frac{1}{6n-11}-\frac{1}{6n-12r-11}-\frac{1}{12n+1}+\frac{1}{3n} \\
= \quad & \frac{1}{12(n-r-2)}-\frac{1}{6n-11}-\frac{1}{6(n-r-2)-6r+1}-\frac{1}{12n+1}+\frac{1}{3n} \\
\overset{r\geq 1}{<} \quad & \frac{1}{12(n-r-2)}-\frac{1}{6n-11}-\frac{1}{6(n-r-2)}-\frac{1}{12n+1}+\frac{1}{3n} \\
= \quad & -\frac{1}{12(n-r-2)}-\frac{1}{6n-11}-\frac{1}{12n+1}+\frac{1}{3n} \\
\overset{r\geq 1}{\leq} \quad & -\frac{1}{12n-36}-\frac{1}{6n-11}-\frac{1}{12n+1}+\frac{1}{3n} \\
= \quad & \frac{-474n^2+1119n+132}{3n(4n-9)(6n-11)(12n+1)} \\
= \quad & \frac{-474n\left(n-\frac{1119}{474}\right)+132}{3n(4n-9)(6n-11)(12n+1)} \\
\overset{n\geq 4}{<} \quad & 0,
\end{aligned}$$

as needed. ■

We now prove:

Lemma 6.4 *For all pairs of integers $n \geq 80$ and r such that $2 \leq r \leq \left\lfloor \frac{n-3}{2} \right\rfloor$,*

$$n^3\left(n-r-\frac{3}{2}\right)^2(n-r-2)(n-2r-1)^4 \quad \leq \quad (n-1)^2(n-2)(n-2r-2)(n-2r)^2(n-r-1)^4.$$

Proof. Since $n-r-\frac{3}{2} \; < \; n-r-1$, it suffices to prove that

$$n^3(n-r-2)(n-2r-1)^4 \quad \leq \quad (n-1)^2(n-2)(n-2r-2)(n-2r)^2(n-r-1)^2.$$

Define the polynomial $f_r(n)$ (with parameter r) with

$$f_r(n) \quad = \quad (n-1)^2(n-2)(n-2r-2)(n-2r)^2(n-r-1)^2-n^3(n-r-2)(n-2r-1)^4.$$

Then,

$$\begin{aligned}
f_r(n) \quad = \quad & (r-2)n^7-(\underbrace{7r^2-14r-12}_{\mathsf{A}})n^6+2(\underbrace{9r^3-21r^2-46r-14}_{\mathsf{B}})n^5 \\
& -(\underbrace{20r^4-72r^3-273r^2-198r-32}_{\mathsf{C}})n^4+(\underbrace{8r^5-72r^4-382r^3-488r^2-197r-18}_{\mathsf{D}})n^3 \\
& +4(\underbrace{8r^5+59r^4+123r^3+94r^2+23r+1}_{\mathsf{E}})n^2-8(\underbrace{5r^5+22r^4+31r^3+16r^2+2r}_{\mathsf{F}})n \\
& +16r^2(r+1)^3.
\end{aligned}$$

Note that for $n = 80$, $2 \leq r \leq 38$. We shall prove that $f_r(n) \geq 0$. To do so, we calculate the first seven derivatives of $f_r(n)$:

$$\begin{aligned}
f_r^{(1)}(n) &= 7(r-2)n^6 - 6\mathsf{A}n^5 + 10\mathsf{B}n^4 - 4\mathsf{C}n^3 + 3\mathsf{D}n^2 + 8\mathsf{E}n - 8\mathsf{F}, \\
f_r^{(2)}(n) &= 42(r-2)n^5 - 30\mathsf{A}n^4 + 40\mathsf{B}n^3 - 12\mathsf{C}n^2 + 6\mathsf{D}n + 8\mathsf{E}, \\
f_r^{(3)}(n) &= 210(r-2)n^4 - 120\mathsf{A}n^3 + 120\mathsf{B}n^2 - 24\mathsf{C}n + 6\mathsf{D}, \\
f_r^{(4)}(n) &= 840(r-2)n^3 - 360\mathsf{A}n^2 + 240\mathsf{B}n - 24\mathsf{C}, \\
f_r^{(5)}(n) &= 2520(r-2)n^2 - 720\mathsf{A}n + 240\mathsf{B}, \\
f_r^{(6)}(n) &= 5040(r-2)n - 720\mathsf{A}, \\
f_r^{(7)}(n) &= 5040(r-2).
\end{aligned}$$

- Since $r \geq 2$, $f_r^{(7)}(n) \geq 0$; hence, $f^{(6)}(n)$ is non-decreasing in n for $n \geq 80$.
- Note that for $n = 80$ $f_r^{(6)}(80) = -5040r^2 + 413280r - 797760$; since the discriminant of $f_r^{(6)}(80)$ is $413280^2 - 4 \cdot 5040 \cdot 797760 > 0$, there are two distinct roots $r_1 < r_2$; $r_1 < 2$ and $r_2 > 38$. Since the quadratic power coefficient -5040 of $f^{(6)}(80)$ is negative, it follows that $f_r^{(6)}(80) > 0$ for $2 \leq r \leq 38$. Thus, $f_r^{(6)}(n) > 0$; hence, $f_r^{(5)}(n)$ is increasing in n for $n \geq 80$.
- It is verified that $f_r^{(5)}(80) = 2520 \cdot 80^2 \cdot (r-2) - 57600\mathsf{A} + 240\mathsf{B} > 0$ for $2 \leq r \leq 38$. Thus, $f_r^{(5)}(n) > 0$; hence, $f_r^{(4)}(n)$ is increasing in n for $n \geq 80$.
- It is verified that $f_r^{(4)}(80) = 840 \cdot 80^3 \cdot (r-2) - 360 \cdot 80^2 \cdot \mathsf{A} + 19200\mathsf{B} - 24\mathsf{C} > 0$ for $2 \leq r \leq 38$. Thus, $f_r^{(4)}(n) > 0$; hence, $f_r^{(3)}(n)$ is increasing in n for $n \geq 80$.
- It is verified that $f_r^{(3)}(80) = 210 \cdot 80^4 \cdot (r-2) - 120 \cdot 80^3 \cdot \mathsf{A} + 768000\mathsf{B} - 1920\mathsf{C} + 6\mathsf{D} > 0$ for $2 \leq r \leq 38$. Thus, $f_r^{(3)}(n) > 0$; hence, $f_r^{(2)}(n)$ is increasing in n for $n \geq 80$.
- It is verified that $f_r^{(2)}(80) = 42 \cdot 80^5 \cdot (r-2) - 30 \cdot 80^4 \cdot \mathsf{A} + 40 \cdot 80^3 \cdot \mathsf{B} - 76800\mathsf{C} + 480\mathsf{D} + 8\mathsf{E} > 0$ for $2 \leq r \leq 38$. Thus, $f_r^{(2)}(n) > 0$; hence, $f_r^{(1)}(n)$ is increasing in n for $n \geq 80$.
- It is verified that $f_r^{(1)}(80) = 7 \cdot 80^6 \cdot (r-2) - 6 \cdot 80^5 \cdot \mathsf{A} + 10 \cdot 80^4 \cdot \mathsf{B} - 4 \cdot 80^3 \cdot \mathsf{C} + 19200\mathsf{D} + 640\mathsf{E} - 8\mathsf{F} > 0$ for $2 \leq r \leq 38$. Thus, $f_r^{(1)}(n) > 0$; hence, $f_r(n)$ is increasing in n for $n \geq 80$.
- Since $f_r(80) = 80^7 \cdot (r-2) - 80^6 \cdot \mathsf{A} + 2 \cdot 80^5 \cdot \mathsf{B} - 80^4 \cdot \mathsf{C} + 80^3 \cdot \mathsf{D} + 25600\mathsf{E} - 640\mathsf{F} + 16r^2(r+1)^3 > 0$ for $2 \leq r \leq 38$, the claim follows. ∎

We continue to prove:

Lemma 6.5 *For all pairs of integers $n \geq 80$ and r such that $2 \leq r \leq \left\lfloor \frac{n-3}{2} \right\rfloor$,*

$$\begin{aligned} & 3r(n+1)(n-1)^2(n-2r-2)(n-2r)^2(n-r-1)^3 \\ \leq\ & 2\sqrt{n}\left((n-1)^2(n-2)(n-2r-2)(n-2r)^2(n-r-1)^4 - n^3\left(n-r-\frac{3}{2}\right)^2(n-r-2)(n-2r-1)^4\right). \end{aligned}$$

Proof. We proceed by case analysis on the range of values of r. Define the function $f_r(n)$ (with parameter r) with

$$\begin{aligned} & f_r(n) \\ =\ & 2\sqrt{n}\left((n-1)^2(n-2)(n-2r-2)(n-2r)^2(n-r-1)^4 - n^3\left(n-r-\frac{3}{2}\right)^2(n-r-2)(n-2r-1)^4\right) \\ & -3r(n+1)(n-1)^2(n-2r-2)(n-2r)^2(n-r-1)^3 \\ =\ & 2(r-1)n^{\frac{19}{2}} - 3rn^9 - \left(18r^2-12r-\frac{35}{2}\right)n^{\frac{17}{2}} + 9r(3r+2)n^8 + \left(66r^3-30r^2-\frac{343r}{2}-65\right)n^{\frac{15}{2}} \\ & -3r(33r^2+47r+13)n^7 - (126r^4-64r^3-740r^2-658r-133)n^{\frac{13}{2}} + 3r(63r^3+146r^2+86r+10)n^6 \\ & +(132r^5-156r^4-1792r^3-2624r^2-1263r-162)n^{\frac{11}{2}} - 3r(66r^4+227r^3+212r^2+46r-5)n^5 \\ & -\left(72r^6-256r^5-2546r^4-5272r^3-4302r^2-1366r-\frac{235}{2}\right)n^{\frac{9}{2}} \\ & +3r(36r^5+182r^4+233r^3+46r^2-51r-14)n^4 \\ & +\left(16r^7-208r^6-2020r^5-5576r^4-6746r^3-3754r^2-\frac{1671r}{2}-47\right)n^{\frac{7}{2}} \\ & -3r(8r^6+68r^5+98r^4-63r^3-177r^2-85r-9)n^3 + 8(r+1)^3(8r^4+71r^3+124r^2+30r+1)n^{\frac{5}{2}} \\ & +6r(r+1)^2(4r^4-10r^3-51r^2-20r-1)n^2 - 16r(r+1)^4(5r^2+16r+2)n^{\frac{3}{2}} \\ & +12r^2(r+1)^3(2r^2+11r+2)n + 32r^2(r+1)^5\sqrt{n} - 24r^3(r+1)4. \end{aligned}$$

First, we consider the case where $r \in \{2,3,4\}$. For $r=2$,

$$\begin{aligned} & f_2(n) \\ =\ & 2n^{\frac{19}{2}} - 6n^9 - \frac{61}{2}n^{\frac{17}{2}} + 144n^8 - 1434n^7 + 2905n^{\frac{13}{2}} + 7620n^6 - 25792n^{\frac{11}{2}} - 22842n^5 \\ & +\frac{213107}{2}n^{\frac{9}{2}} + 35976n^4 - 235822n^{\frac{7}{2}} - 17190n^3 + 270648n^{\frac{5}{2}} - 28188n^2 - 139968n^{\frac{3}{2}} \\ & +41472n + 31104\sqrt{n} - 15552 \\ =\ & (\sqrt{n}-6)n^9 + \left(n-\frac{61}{2}\right)n^{\frac{17}{2}} + (144n-1434)n^7 + (2905n-25792)n^{\frac{11}{2}} + (7620n-22842)n^5 \\ & +\left(\frac{213107}{2}n-235822\right)n^{\frac{7}{2}} + (35976n-17190)n^3 + (270648n-28188\sqrt{n}-139968)n^{\frac{3}{2}} \\ & +41472n + 31104\sqrt{n} - 15552 \\ >\ & 0, \end{aligned}$$

since all parenthesized coefficients are positive for $n \geq 80$.

For $r = 3$,

$$\begin{aligned}
& f_3(n) \\
= & \frac{1}{2}\Big(8n^{\frac{19}{2}} - 18n^9 - 217n^{\frac{17}{2}} + 594n^8 + 1865n^{\frac{15}{2}} - 8118n^7 + 578n^{\frac{13}{2}} + 59094n^6 - 113022n^{\frac{11}{2}} \\
& -243288n^5 \ + 802447n^{\frac{9}{2}} + 540504n^4 - 2555275n^{\frac{7}{2}} - 481248n^3 + 3862528n^{\frac{5}{2}} - 268416n^2 \\
& -2334720n^{\frac{3}{2}} + 732672n + 589824\sqrt{n} - 331776\Big) \\
= & \frac{1}{2}\Big((3\sqrt{n} - 18)n^9 + (5n - 217)\, n^{\frac{17}{2}} + (594n - 8118)n^7 + (1865n^2 - 113022)n^{\frac{11}{2}} + 578n^{\frac{13}{2}} \\
& +(59094n - 243288)n^5 + (802447n - 2555275)\, n^{\frac{7}{2}} + (540404n - 481248)n^3 \\
& +(3862528n - 268416\sqrt{n} - 2334720)n^{\frac{3}{2}} + 732672n + 589824\sqrt{n} - 331766\Big) \\
> & \ 0,
\end{aligned}$$

since all parenthesized coefficients are positive for $n \geq 80$.

For $r = 4$,

$$\begin{aligned}
& f_4(n) \\
= & \ 6n^{\frac{19}{2}} - 12n^9 - \frac{445}{2}n^{\frac{17}{2}} + 504n^8 + 2993n^{\frac{15}{2}} - 8748n^7 - 13555n^{\frac{13}{2}} + 80664n^6 - 66654n^{\frac{11}{2}} \\
& -419940n^5 + \frac{2061659}{2}n^{\frac{9}{2}} + 1186632n^4 - 4580957n^{\frac{7}{2}} - 1443300n^3 + 8697000n^{\frac{5}{2}} - 307800n^2 \\
& -5840000n^{\frac{3}{2}} + 1872000n + 1600000\sqrt{n} - 960000 \\
= & \ (2\sqrt{n} - 12)n^9 + \left(4n - \frac{445}{2}\right) n^{\frac{17}{2}} + (504n - 8748)n^7 + (2993n - 13555)n^{\frac{13}{2}} \\
& +(74664\sqrt{n} - 66654)n^{\frac{11}{2}} + (6000n - 419940)n^5 + \left(\frac{2061659}{2}n - 4580957\right) n^{\frac{7}{2}} \\
& +(1186632n - 1443300)n^3 + (8697000n - 307800\sqrt{n} - 5840000)n^{\frac{3}{2}} + 1872000n + 1600000\sqrt{n} \\
& -960000 \\
> & \ 0,
\end{aligned}$$

since all parenthesized coefficients are positive for $n \geq 80$.

We continue to prove the claim for $5 \leq r \leq \left\lfloor \frac{n-3}{2} \right\rfloor$. Since $n - r - \frac{3}{2} < n - r - 1$, it suffices to prove that

$$\begin{aligned}
& 2\sqrt{n}((n-1)^2(n-2)(n-2r-2)(n-r-1)^2 - n^3(n-r-2)(n-2r-1)^2) \\
\geq & \ 3r(n+1)(n-1)^2(n-2r-2)(n-r-1).
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
& \Big(2(r-4)n^3 - 3rn^{\frac{5}{2}} - 6(r^2 - 4r - 7)n^2 + 3r(3r+4)n^{\frac{3}{2}} + 2(2r^3 - 14r^2 - 57r - 42)n \\
& -3r(2r^2 + 7r + 4)\sqrt{n} + 2(8r^3 + 49r^2 + 82r + 41)\Big)\, n^{\frac{5}{2}} + \Big(3r(2r^2 + r - 2)n^{\frac{3}{2}} \\
& -20(r^3 + 4r^2 + 5r + 2)n\Big)\sqrt{n} + 3r(2r^2 + 7r + 5)n + 8(r+1)^3\sqrt{n} - 6r(r+1)^2 \\
\geq & \ 0.
\end{aligned}$$

It suffices to prove that

$$\begin{aligned}
&\Big(\overbrace{2(r-4)n^3-3rn^{\frac{5}{2}}-6(r^2-4r-7)n^2+3r(3r+4)n^{\frac{3}{2}}+2(2r^3-14r^2-57r-42)n}\\
&\overbrace{-3r(2r^2+7r+4)\sqrt{n}+2(8r^3+49r^2+82r+41)}\Big)\, n^{\frac{5}{2}}\\
&+\Big(\underbrace{3r(2r^2+r-2)n^{\frac{3}{2}}-20(r^3+4r^2+5r+2)n-6r(r+1)^2}\Big)\sqrt{n}\\
\geq\ & 0.
\end{aligned}$$

Define the function $f_r(n)$ (with parameter r) restricted to the overbraced terms as

$$\begin{aligned}
f_r(n) \;=\;& 2(r-4)n^3-3rn^{\frac{5}{2}}-6(r^2-4r-7)n^2+3r(3r+4)n^{\frac{3}{2}}+2(2r^3-14r^2-57r-42)n\\
&-3r(2r^2+7r+4)\sqrt{n}+2(8r^3+49r^2+82r+41).
\end{aligned}$$

Define also the function $g_r(n)$ (with parameter r) restricted to the underbraced terms as

$$g_r(n) \;=\; 3r(2r^2+r-2)n^{\frac{3}{2}}-20(r^3+4r^2+5r+2)n-6r(r+1)^2.$$

We shall prove that both $f_r(n)\geq 0$ and $g_r(n)\geq 0$ in the ranges of n and r. We first prove:

Claim 6.6 *For all pairs of integers $n\geq 80$ and r such that $5\leq r\leq \left\lfloor \frac{n-3}{2}\right\rfloor$, $f_r(n)\geq 0$.*

Proof. The first and second derivatives of $f(n)$ are

$$\begin{aligned}
f_r^{(1)}(n) \;=\;& 6(r-4)n^2-\frac{15r}{2}n^{\frac{3}{2}}-12(r^2-4r-7)n+\frac{9r}{2}(3r+4)\sqrt{n}+2(2r^3-14r^2-57r-42)\\
&-\frac{3r}{2\sqrt{n}}(2r^2+7r+4),
\end{aligned}$$

$$\begin{aligned}
f_r^{(2)}(n) \;&=\; 12(r-4)n-\frac{45r}{4}\sqrt{n}-12(r^2-4r-7)+\frac{9r}{4\sqrt{n}}(3r+4)+\frac{3r}{4}(2r^2+7r+4)n^{-\frac{3}{2}}\\
&>\; 12(r-4)n-12r\sqrt{n}-12(r^2-4r-7)\\
&=\; 12\left(\underbrace{(r-4)n-r\sqrt{n}-r^2+4r+7}_{h_r(n)}\right).
\end{aligned}$$

Note that for the first derivative $h_r^{(1)}(n)$, it holds that

$$\begin{aligned}
h^{(1)}(n) \;&=\; r-4-\frac{r}{2\sqrt{n}}\\
&\overset{n>2r}{>}\; r-4-\frac{\sqrt{r}}{2\sqrt{2}}\\
&=\; \frac{2r\sqrt{2}-8\sqrt{2}-\sqrt{r}}{2\sqrt{2}}\\
&=\; \frac{\sqrt{r}\left(2\sqrt{2r}-1\right)-8\sqrt{2}}{2\sqrt{2}}\\
&\overset{r\geq 5}{>}\; 0.
\end{aligned}$$

Hence, $h_r(n)$ is increasing in n. Note that $h(80) = -r^2 + (84 - \sqrt{80})r - 313$. Note that the discriminant of $h(80)$ is positive; hence, there are two distinct roots $r_1 < r_2$, and it is verified that that $r_1 < 5$ and $r_2 > 38$. Since the quadratic power coefficient of $h_r(80)$ is negative, it follows that $h(80) > 0$ for $5 \le r \le 38$. Hence $f_r^{(2)}(n) > 0$, which implies that $f_r^{(1)}(n)$ is increasing in n. It is verified that $f_r^{(1)}(80) \ge 0$; hence, $f_r^{(1)}(n) \ge 0$ for $n \ge 80$. This implies that $f_r(n)$ is non-decreasing in n for $n \ge 80$. It is verified that $f_r(80) \ge 0$ for $5 \ge r \ge 38$. Hence, $f_r(n) \ge 0$ for $n \ge 80$, as needed. ■

We now prove:

Claim 6.7 *For all pairs of integers $n \ge 80$ and r such that $5 \le r \le \left\lfloor \frac{n-3}{2} \right\rfloor$, $g_r(n) \ge 0$.*

Proof. The first and second derivatives of $g_r(n)$ are

$$g_r^{(1)}(n) = \frac{9r}{2}\left(2r^2 + r - 2\right)\sqrt{n} - 20\left(r^3 + 4r^2 + 5r + 2\right)$$

and

$$g^{(2)}(n) = \frac{9r}{4\sqrt{n}}(2r^2 + r - 2).$$

Since $r \ge 5$, $g_r^{(2)}(n) \ge 0$; hence, $g_r^{(1)}(n)$ is non-decreasing in n. It is verified that $g_r^{(1)}(80) \ge 0$; hence, $g_r^{(1)}(n) \ge 0$ for all $n \ge 80$. It follows that $g_r(n)$ is non-decreasing in n. Since $g_r(80) \ge 0$, the claim follows. ■

The claim follows now from Claims 6.6 and 6.7. ■

We continue to prove:

Lemma 6.8 *For all odd integers $n \ge 5$ and $r \in \{1, 2\}$, it holds that*

$$\left(1 - \frac{r^2}{(n-r-1)^2}\right)^{\frac{n+1}{2}-r} < 1 - \frac{r^2}{(n-r-1)^2}\binom{\frac{n+1}{2} - r}{1} + \frac{r^4}{(n-r-1)^4}\binom{\frac{n+1}{2} - r}{2}.$$

Proof. Consider the binomial expansion of $\left(1 - \frac{r^2}{(n-r-1)^2}\right)^{\frac{n+1}{2}-r}$. Note that if $\frac{n+1}{2} - r$ is odd, the last term in the binomial expansion is negative; so, consider the binomial expansion without any such last term. Consider any pair of consecutive terms

$$-\left(\frac{r^2}{(n-r-1)^2}\right)^i \binom{\frac{n+1}{2} - r}{i} \text{ and } \left(\frac{r^2}{(n-r-1)^2}\right)^{i+1} \binom{\frac{n+1}{2} - r}{i+1},$$

with $3 \le i \le \dfrac{n+1}{2} - r$ and i is odd, in the binomial expansion of $\left(1 - \dfrac{r^2}{(n-r-1)^2}\right)^{\frac{n+1}{2}-r}$.
Clearly,

$$\begin{aligned}
& -\left(\frac{r^2}{(n-r-1)^2}\right)^i \binom{\frac{n+1}{2}-r}{i} + \left(\frac{r^2}{(n-r-1)^2}\right)^{i+1} \binom{\frac{n+1}{2}-r}{i+1} \\
= \quad & \left(\frac{r^2}{(n-r-1)^2}\right)^i \left(-\binom{\frac{n+1}{2}-r}{i} + \frac{r^2}{(n-r-1)^2}\binom{\frac{n+1}{2}-r}{i+1}\right) \\
= \quad & \left(\frac{r^2}{(n-r-1)^2}\right)^i \left(-\binom{\frac{n+1}{2}-r}{i} + \frac{r^2}{(n-r-1)^2}\frac{(n-2r+1)-2i}{2(i+1)}\binom{\frac{n+1}{2}-r}{i}\right) \\
\overset{i\geq 3}{<} \quad & \left(\frac{r^2}{(n-r-1)^2}\right)^i \binom{\frac{n+1}{2}-r}{i} \left(-1 + \frac{r^2}{(n-r-1)^2}\cdot\frac{n-2r+1}{8}\right) \\
\overset{r\in\{1,2\}}{\leq} \quad & \left(\frac{r^2}{(n-r-1)^2}\right)^i \binom{\frac{n+1}{2}-r}{i} \left(-1 + \frac{1}{2(n-3)}\right) \\
\overset{n\geq 5}{<} \quad & 0.
\end{aligned}$$

It follows that

$$\begin{aligned}
\left(1 - \frac{r^2}{(n-r-1)^2}\right)^{\frac{n+1}{2}-r} \quad = \quad & 1 - \frac{r^2}{(n-r-1)^2}\binom{\frac{n+1}{2}-r}{1} + \frac{r^4}{(n-r-1)^4}\binom{\frac{n+1}{2}-r}{2} \\
& + \sum_{i=3}^{\frac{n+1}{2}-r} (-1)^i \left(\frac{r^2}{(n-r-1)^2}\right)^i \binom{\frac{n+1}{2}-r}{i} \\
< \quad & 1 - \frac{r^2}{(n-r-1)^2}\binom{\frac{n+1}{2}-r}{1} + \frac{r^4}{(n-r-1)^4}\binom{\frac{n+1}{2}-r}{2},
\end{aligned}$$

as needed. ■

We now prove:

Lemma 6.9 *For $n \ge 4$ and $r \in \{1, 2\}$, it holds that*

$$(n-r-1)^2 \quad > \quad (n-2r+1)(n-2r-1).$$

Proof. The claim is equivalent to

$$(n-r-1)^2 - (n-2r+1)(n-2r-1) \quad > \quad 0.$$

Clearly,

$$\begin{aligned}
(n-r-1)^2 - (n-2r+1)(n-2r-1) \quad & = \quad 2rn - 2n - 3r^2 + 2r + 2 \\
& = \quad 2n(r-1) - 3r^2 + 2r + 2 \\
& \overset{n\geq 4}{\geq} \quad 8(r-1) - 3r^2 + 2r + 2 \\
& \overset{r\in\{1,2\}}{>} \quad 0.
\end{aligned}$$

The claim follows. ■

We continue to prove:

Lemma 6.10 *For all pairs of integers $n \geq 5$ and $r \geq 2$,*

$$\frac{1}{12(n-r-2)} - \frac{1}{6n-17} - \frac{1}{6n-12r-5} - \frac{1}{12n+1} + \frac{1}{3n-3} \;<\; 0.$$

Proof. Clearly,

$$\begin{aligned}
& \frac{1}{12(n-r-2)} - \frac{1}{6n-17} - \frac{1}{6n-12r-5} - \frac{1}{12n+1} + \frac{1}{3n-3} \\
= \quad & \frac{1}{12(n-r-2)} - \frac{1}{6n-17} - \frac{1}{6(n-r-2)-6r+7} - \frac{1}{12n+1} + \frac{1}{3n-3} \\
\overset{r\geq 2}{<} \quad & \frac{1}{12(n-r-2)} - \frac{1}{6n-17} - \frac{1}{6(n-r-2)} - \frac{1}{12n+1} + \frac{1}{3n-3} \\
= \quad & -\frac{1}{12(n-r-2)} - \frac{1}{6n-17} - \frac{1}{12n+1} + \frac{1}{3n-3} \\
\overset{r\geq 2}{\leq} \quad & -\frac{1}{(12n-48)} - \frac{1}{6n-17} - \frac{1}{12n+1} + \frac{1}{3n-3} \\
= \quad & \frac{-402n^2 + 1095n + 1023}{(4n-16)(6n-17)(12n+1)(3n-3)} \\
= \quad & \frac{-402n\left(n - \frac{1095}{402}\right) + 1023}{(4n-16)(6n-17)(12n+1)(3n-3)} \\
\overset{n\geq 5}{<} \quad & 0,
\end{aligned}$$

as needed. ■

We now prove:

Lemma 6.11 *For all pairs of integers $n \geq 1$ and r such that $3 \leq r \leq \left\lfloor \frac{n-3}{2} \right\rfloor$,*

$$n(n-r-2)(n-2r-1) \;\leq\; (n-3)(n-r-1)^2.$$

Proof. Since $n-r-2 < n-r-1$, it suffices to prove that $n(n-2r-1) \leq (n-3)(n-r-1)$, or equivalently that

$$(n-3)(n-r-1) - n(n-2r-1) \;\geq\; 0.$$

Clearly,

$$\begin{aligned}
(n-3)(n-r-1) - n(n-2r-1) \quad &= \quad n^2 - rn - 4n + 3r + 3 - n^2 + 2rn + n \\
&= \quad (r-3)n + 3r + 3 \\
&\overset{r\geq 3}{\geq} \quad 0,
\end{aligned}$$

as needed. ■

We finally prove:

Lemma 6.12 *For all pairs of integers $n \geq 81$ and r such that $3 \leq r \leq \left\lfloor \frac{n-3}{2} \right\rfloor$,*

$$3r(n+1)(n-r-1) \quad \leq \quad 2\sqrt{n}\left((n-3)(n-r-1)^2 - n(n-r-2)(n-2r-1)\right).$$

Proof. Define the function $f_r(n)$ (with parameter r) with

$$\begin{aligned} f_r(n) &= 2\sqrt{n}\left((n-3)(n-r-1)^2 - n(n-r-2)(n-2r-1)\right) - 3r(n+1)(n-r-1) \\ &= 2(r-2)n^{\frac{5}{2}} - 3rn^2 - 2(r^2-3r-5)n^{\frac{3}{2}} + 3r^2n - 6(r+1)^2\sqrt{n} + 3r(r+1). \end{aligned}$$

We shall prove that $f_r(n) \geq 0$. Write

$$f_r(n) \quad = \quad ((r-2)\sqrt{n} - 3r)n^2 + ((r-2)n - 2r^2 + 6r + 10)n^{\frac{3}{2}} + 3(r^2\sqrt{n} - 2(r+1)^2)\sqrt{n} + 3r(r+1).$$

Note that

$$f_r(n) \quad > \quad (\underbrace{(r-2)\sqrt{n} - 3r}_{\mathsf{A}})n^2 + (\underbrace{(r-2)n - 2r^2 + 6r + 10}_{\mathsf{B}})n^{\frac{3}{2}} + 3(\underbrace{r^2\sqrt{n} - 2(r+1)^2}_{\mathsf{C}})\sqrt{n}.$$

Since $n \geq 81$ and $r \geq 3$, $\mathsf{A} > 9(r-2) - 3r \geq 0$. Since $n > 2r$ and $r \geq 3$, $\mathsf{B} > (r-2)n - 2r^2 + 4r = (n-2r)(r-2) > 0$. Since $n \geq 81$ and $r \geq 3$, $\mathsf{C} > 9r^2 - 2(r+1)^2 = 7r^2 - 4r - 2 > 0$. Since $f_r(n) > \mathsf{A}n^2 + \mathsf{B}n^{\frac{3}{2}} + 3\mathsf{C}\sqrt{n}$, the claim follows. ■

7 Conclusions

We have presented an extensive proof for the validity of the FMNE *Conjecture* for a special case of the selfish routing model of Koutsoupias and Papadimitriou [14] where users are unweighted and there are only two identical (related) links. We adopted a new, well-motivated kind of Social Cost, called Quadratic Maximum Social Cost. The proof required a variety of combinatorial arguments and analytical estimations.

We believe that our work contributes significantly, both conceptually and technically, to enriching our knowledge about the many facets of the FMNE *Conjecture*. Based on this improved understanding, we have extended the QMFMNE *Conjecture* formulated and proven in this work to an *Extended* QMFMNE *Conjecture* for the more general case with an arbitrary number of unweighted users, an arbitrary number of identical (related) links and Social Cost as the expectation of a polynomial with non-negative coefficients of the maximum congestion on a link. Settling this *Extended* QMFMNE *Conjecture* remains a major challenge.

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