A New Model for Selfish Routing[*]

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Abstract. In this work, we introduce and study a new model for *selfish
routing* over *non-cooperative networks* that combines features from the
two such best studied models, namely the *KP model* and the *Wardrop
model* in an interesting way.

We consider a set of *n users,* each using a *mixed strategy* to ship its
unsplittable traffic over a *network* consisting of m parallel *links.* In a
Nash equilibrium, no *user* can increase its *Individual Cost* by unilaterally
deviating from its strategy. To evaluate the performance of such Nash
equilibria, we introduce *Quadratic Social Cost* as a certain *sum* of In-
dividual Costs – namely, the sum of the expectations of the squares of
the incurred *link latencies*. This definition is unlike the KP model, where
Maximum Social Cost has been defined as the *maximum* of Individual
Costs.

We analyse the impact of our modeling assumptions on the computation
of *Quadratic Social Cost*, on the structure of worst-case Nash equilibria,
and on bounds on the *Quadratic Coordination Ratio*.

1 Introduction

1.1 Motivation and Framework

Nash Equilibria and Outline. *Nash equilibrium* [23,24] is arguably the most
robust equilibrium concept in (non-cooperative) Game Theory.[1] At a Nash equi-
librium, no player of a strategic game can unilaterally improve its objective by
switching to a different *strategy*. In a *pure* Nash equilibrium, each player chooses
exactly one strategy (with probability one); in a *mixed* Nash equilibrium, the
choices of each player are modeled by a probability distribution over strategies.
Of special interest to our work is the *fully mixed Nash equilibrium* [22], where

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[1] See [25] for a concise introduction to contemporary Game Theory.

each user chooses each strategy with non-zero probability. Nash equilibria have some very nice properties; most notably, for finite games, there always exists a mixed Nash equilibrium [24].

In this work, we embark on a systematic study, within a new model for *selfish routing* over *non-cooperative networks* that we introduce, of some interesting algorithmic and mathematical properties of Nash equilibria for some specific *routing game* formulated in this context. Our new model for selfish routing is an interesting hybridization of the two most famous models for selfish routing that were studied in the literature before; these are the so called *KP model* [19] and *Wardrop model* [4,30].

The KP Model and the Wardrop Model. The KP and the Wardrop models differ with respect to the assumptions they are making about: 1. the structure of the *routing network*; 2. the *splittability* or *unsplittability* of the users' *traffics*; 3. the definition of *Individual Cost* for a user they use for defining Nash equilibria; 4. the type of Nash equilibria (pure or mixed) they consider; 5. the specific definitions they employ for *Social Cost,* a performance measure for Nash equilibria, and for *Social Optimum,* an optimality measure for traffic assignments (not necessarily equilibria). The definitions for Social Cost usually relate to Individual Costs. In either model, these two definitions give rise to *Coordination Ratio,* the maximum value of the ratio of Social Cost over Social Optimum; a *worst-case* Nash equilibrium is one that maximizes its particular Social Cost.

In the KP model, a collection of n *users* is assumed; each user employs a *mixed strategy,* which is a probability distribution over m parallel *links,* to control the shipping of its own assigned *traffic*. In the KP model, traffics are *unsplittable*. For each link, a *capacity* specifies the rate at which the link processes traffic. Allowing link capacities to vary arbitrarily gives rise to the standard model of *related links.* A special case of the model of related links is the model of *identical links,* where all link capacities are identical. Reciprocally, in the model of *identical users,* all user traffics are equal; they may vary arbitrarily in the model of *arbitrary users.* In a Nash equilibrium, each user selfishly routes its traffic on those links that minimize its *Individual Cost*: its *expected latency cost* on that link, given the network congestion caused by the other users. In the KP model, the Social Cost of a Nash Equilibrium, henceforth called *Maximum Social Cost,* is the expectation, over all random choices of the users, of the maximum, over all links, *latency* through a link; the Social Optimum, henceforth called the *Maximum Social Optimum,* is the *least possible* maximum, over all links, latency through a link that could be attained had global regulation been available; correspondingly, the Coordination Ratio in the KP model will henceforth be called the *Maximum Coordination Ratio*. It follows that the Maximum Social Cost in the KP model is the *maximum* of Individual Costs.

In the Wardrop model [11,30], there have been considered *arbitrary* networks with *latency functions* for edges. Moreover, the traffics are *splittable* into arbitrary pieces. Here, unregulated traffic is modeled as a *network flow. Equilibrium flows* have been classified as flows with all flow paths used between a given pair of a *source* and a *destination* having the same latency. Equilibrium flows are optimal solutions to a convex program, in case the edge latency functions are convex. An equilibrium in this model can be interpreted as a Nash equilibrium

in a game with infinitely many users, each carrying an infinitesimal amount of traffic from a *source* to a *destination*. Thus, the Wardrop model restricts to pure Nash equilibria. The *Individual Cost* of a user is defined as the sum of the edge *latencies* on a path from the user's source to its destination. The *Social Cost* of a Nash equilibrium is the sum of all Individual Costs. The *Social Optimum* is the least possible, over all network flows, sum of Individual Costs.

The New Model for Selfish Routing. Our new model for selfish routing over non-cooperative networks is a hybridization of the KP model [19] and the Wardrop model [11,30]. More specifically, we follow the KP model to consider the simple parallel links network (which, however, is also a special case for the Wardrop model). We also follow the KP model to consider unsplittable traffics and mixed Nash equilibria. The Individual Cost we adopt is also identical to that adopted in the KP model – the expected latency cost on a link. However, we follow the Wardrop model to model Social Cost as a certain *sum* of Individual Costs, which we will later describe. In some sense, our new model is the Wardrop model restricted to the simple parallel links network but modified to allow for unsplittable traffics and for mixed strategies; these two features were borrowed from the KP model. Our work is the *first* step toward accommodating unsplittable traffics within the Wardrop model.

For any link, consider the *square* of the traffic through the link divided by the link capacity; taking the expectation of this and adding up over all links yields the Social Cost for our model. Call it *Quadratic Social Cost*. In correspondence with Quadratic Social Cost, we also define and study in our new model *Quadratic Optimum* and *Quadratic Coordination Ratio*. Naturally, the former is the *least possible* sum of the squares of total traffic through a link divided by the link capacity; the latter is the maximum value of the ratio of Quadratic Social Cost over Quadratic Social Optimum. Since Nash equilibria are defined with respect to Individual Costs (but are independent of Social Cost), the Nash equilibria in our new model coincide with those in the KP-model since the two adopt the same Individual Costs.

Note that the commutativity between expectation and sum in the definition of Quadratic Social Cost has been unavailable (between expectation and maximum) in the definition of Maximum Social Cost for the KP model. So, this commutativity allows hopes for some more tractable analysis of several problems regarding some interesting algorithmic, combinatorial, structural and optimality properties of Nash equilibria in the new model.

1.2   Contribution and Significance

We partition our results into three major groups.

Combinatorial Expressions for Quadratic Social Cost. In the most general model of arbitrary users and arbitrary links, we obtain an elegant, recursive combinatorial formula for Quadratic Social Cost, implying a dynamic programming algorithm to compute Quadratic Social Cost. Furthermore, we derive simple, combinatorial expressions for the Quadratic Social Cost of the fully mixed Nash equilibrium in case of arbitrary users and identical links, and identical users and arbitrary links, respectively.

The Worst-case Nash Equilibrium. A natural problem that arises in the context of Quadratic Social Cost is to identify the *worst-case* Nash equilibrium – the one that maximizes, for each specific choice of user traffics and link capacities, the Quadratic Social Cost. We address this problem in the particular setting of the model of identical users and identical links, where the fully mixed Nash equilibrium always exists. We prove that, in this particular setting, the worst-case Nash equilibrium is the fully mixed Nash equilibrium.

Bounds on Quadratic Coordination Ratio. For the model of arbitrary users and identical links we prove that the Quadratic Coordination Ratio for pure Nash equilibria is precisely $\frac{9}{8}$. In case of identical users and related links, we discover that the Quadratic Coordination Ratio for pure Nash equilibria increases slightly to $\frac{4}{3}$. We next turn to the model of arbitrary users and identical links. Here, we restrict ourselves to the fully mixed Nash equilibrium. For this setting, we prove an upper bound of $2 - \frac{1}{m}$ on Quadratic Coordination Ratio. For identical users the Quadratic Social Cost of the fully mixed Nash equilibrium slightly drops to $1 + \min\{\frac{m-1}{n}, \frac{n-1}{m}\}$ times the optimal Quadratic Social Cost. Since in this setting the fully mixed Nash equilibrium is the worst-case Nash equilibrium, this bound holds for the Quadratic Coordination Ratio.

1.3   Related Work and Comparison

The KP model was first introduced in the work of Koutsoupias and Papadimitriou [19]; it was further studied in [9,10,12,14,16,17,18,21,22]. Fully mixed Nash equilibria were introduced and analyzed in [22]. Bounds on Maximum Coordination Ratio were proved in [9,14,18,22]. The works by Fotakis *et al.* [16], by Gairing *et al.* [17], and by Lücking *et al.* [21] dwelled into the combinatorial structure and the computational complexity of Nash equilibria for the KP model. In particular, the Fully Mixed Nash Equilibrium Conjecture was motivated by some results in [16], explicitly formulated in [17] and further studied in [21]. The Wardrop model was defined in [30] and further studied in [3,4,11]. Recent studies of selfish routing within the Wardrop model include [27,28,29].

Fotakis *et al.* [16, Theorem 8] proved that computing the Maximum Social Cost of an arbitrary Nash equilibrium is a $\#\mathcal{P}$-complete problem. This hardness result stands in very sharp contrast to our general, pseudopolynomial algorithm to compute Quadratic Social Cost (Theorem 1).

For the KP model, there are known bounds on Maximum Coordination Ratio of $\Theta\left(\frac{\lg m}{\lg \lg m}\right)$ for the model of arbitrary users and identical links [9,18,19,22], of $\Theta\left(\frac{\lg m}{\lg \lg \lg m}\right)$ for the model of arbitrary users and related links [9], and of $O\left(\sqrt{m}\right)$ for the model of arbitrary users and related links and for pure Nash equilibria [14], which improves the previous bound for small values of m. Some of these super-constant bounds stand in very sharp contrast to some of the constant bounds (independent of m and n) on Quadratic Coordination Ratio we prove in this work. However, for the Wardrop model, there have been shown constant bounds on Coordination Ratio [27,28,29].

Other works that have studied Coordination Ratio include [13] for a network creation game and [2] for a network design game. For a survey of recent work

on selfish routing in non-cooperative networks, see [15]. Work in the scheduling literature that has considered quadratic cost functions for *makespan* includes [1, 6,8,20]; work in the networking literature that has considered quadratic cost functions for network delay includes [7].

1.4   Road Map

The rest of this paper is organized as follows. Section 2 presents our definitions and some preliminaries. The Quadratic Social Cost of Nash equilibria is studied in Section 3. Section 4 proves that the fully mixed Nash equilibrium maximizes Quadratic Social Cost in the model of identical users and identical links. Our bounds on Quadratic Coordination Ratio are presented in Section 5. Due to lack of space the proofs are omitted. They can be found in the full version.

2   Framework

2.1   Mathematical Preliminaries and Notation

Throughout, denote for any integer $m \geq 2$, $[m] = \{1, \dots, m\}$. For a random variable X, denote $\mathcal{E}(X)$ the *expectation* of X. We continue to prove a simple combinatorial inequality.

Lemma 1. *For any $k, a, b \in \mathbb{N}$ with $0 < k \leq a \leq b$,* $\frac{1}{a^k} \binom{a}{k} \leq \frac{1}{b^k} \binom{b}{k}$.

Finally, we prove a combinatorial lemma that will be useful in a later proof.

Lemma 2. *Fix any real number a, where $0 < a < 1$, and positive integer r, and set $A = \frac{r}{a}$. Then, $\sum_{1 \leq k \leq r} \binom{r}{k} k^2 \left(\frac{1}{a}\right)^k \left(1 - \frac{1}{a}\right)^{r-k} = A + \frac{r-1}{r} A^2$.*

2.2   General

We consider a *network* consisting of a set of m parallel *links* $1, 2, \dots, m$ from a *source* node to a *destination* node. Each of n *network users* $1, 2, \dots, n$, or *users* for short, wishes to route a particular amount of traffic along a (non-fixed) link from source to destination. (Throughout, we will be using subscripts for users and superscripts for links.)

Denote w_i the *traffic* of user $i \in [n]$. Define the $n \times 1$ *traffic vector* $\mathbf{w}$ in the natural way. Assume throughout that $m > 1$ and $n > 1$. Assume also, without loss of generality, that $w_1 \geq w_2 \geq \dots \geq w_n$. Denote $W = \sum_{i \in [n]} w_i$.

A *pure strategy* for user $i \in [n]$ is some specific link. A *mixed strategy* for user $i \in [n]$ is a probability distribution over pure strategies; thus, a mixed strategy is a probability distribution over the set of links. The *support* of the mixed strategy for user $i \in [n]$, denoted $\mathsf{sup}(i)$, is the set of those pure strategies (links) to which i assigns positive probability. A *pure strategy profile* is represented by an n-tuple $\langle \ell_1, \ell_2, \dots, \ell_n \rangle \in [m]^n$; a *mixed strategy profile* is represented by an $n \times m$ *probability matrix* $\mathbf{P}$ of nm probabilities p_i^j, $i \in [n]$ and $j \in [m]$, where p_i^j is the probability that user i chooses link j.

For a probability matrix $\mathbf{P}$, define *indicator variables* $I_i^j \in \{0,1\}$, where $i \in [n]$ and $j \in [m]$, such that $I_i^j = 1$ if and only if $p_i^j > 0$. Thus, the support of the mixed strategy for user $i \in [n]$ is the set $\{j \in [m] \mid I_i^j = 1\}$. For each link $j \in [m]$, define the *view* of link j, denoted $view(j)$, as the set of users $i \in [n]$ that potentially assign their traffics to link j; so, $view(j) = \{i \in [n] \mid I_i^j = 1\}$. For each link $j \in [m]$, denote $V^j = |view(j)|$.

A mixed strategy profile $\mathbf{P}$ is *fully mixed* [22, Section 2.2] if for all users $i \in [n]$ and links $j \in [m]$, $I_i^j = 1$. Throughout, we will cast a pure strategy profile as a special case of a mixed strategy profile in which all (mixed) strategies are pure.

2.3 System, Models, and Cost Measures

Denote $c^\ell > 0$ the *capacity* of link $\ell \in [m]$, representing the rate at which the link processes traffic. So, the *latency* for traffic w through link ℓ equals w/c^ℓ. In the model of *uniform capacities,* all link capacities are equal to c, for some constant $c > 0$; link capacities may vary arbitrarily in the model of *arbitrary capacities*. Assume throughout, without loss of generality, that $c^1 \geq c^2 \geq \ldots \geq c^m$. Denote $C = \sum_{j \in [m]} c^j$. In the model of *identical traffics,* all user traffics are equal to 1; user traffics may vary arbitrarily in the model of *arbitrary traffics*.

For a pure strategy profile $\langle \ell_1, \ell_2, \ldots, \ell_n \rangle$, the *latency cost for user* $i \in [n]$, denoted λ_i, is $(\sum_{k:\ell_k=\ell_i} w_k)/c^{\ell_i}$; that is, the latency cost for user i is the latency of the link it chooses. For a mixed strategy profile $\mathbf{P}$, denote δ^ℓ the *actual traffic* on link $\ell \in [m]$; so, δ^ℓ is a random variable. For each link $\ell \in [m]$, denote θ^ℓ the *expected traffic* on link $\ell \in [m]$; thus, $\theta^\ell = \mathcal{E}(\delta^\ell) = \sum_{i=1}^n p_i^\ell w_i$. Given $\mathbf{P}$, define the $m \times 1$ *expected traffic vector* Θ induced by $\mathbf{P}$ in the natural way. Given $\mathbf{P}$, denote Λ^ℓ the *expected latency* on link $\ell \in [m]$; clearly, $\Lambda^\ell = \frac{\theta^\ell}{c^\ell}$. Define the $m \times 1$ *expected latency vector* Λ in the natural way. For a mixed strategy profile $\mathbf{P}$, the *expected latency cost* for user $i \in [n]$ on link $\ell \in [m]$, denoted λ_i^ℓ, is the expectation, over all random choices of the remaining users, of the latency cost for user i had its traffic been assigned to link ℓ; thus,

$$\lambda_i^\ell = \frac{w_i + \sum_{k=1, k \neq i} p_k^\ell w_k}{c^\ell} = \frac{(1 - p_i^\ell)w_i + \theta^\ell}{c^\ell} \, .$$

For each user $i \in [n]$, the *minimum expected latency cost,* denoted λ_i, is the minimum, over all links $\ell \in [m]$, of the expected latency cost for user i on link ℓ; thus, $\lambda_i = \min_{\ell \in [m]} \lambda_i^\ell$. For a probability matrix $\mathbf{P}$, define the $n \times 1$ *minimum expected latency cost vector* λ induced by $\mathbf{P}$ in the natural way.

Associated with a traffic vector $\mathbf{w}$, a capacity vector $\mathbf{c}$ and a mixed strategy profile $\mathbf{P}$ is the *Quadratic Social Cost,* denoted $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})$, which is the expectation of the sum of squares of the incurred link latencies; thus,

$$\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) = \mathcal{E}\left(\sum_{\ell \in [m]} \frac{(\sum_{k:\ell_k=\ell} w_k)^2}{c^\ell} \right)$$

$$= \sum_{\langle \ell_1, \ell_2, \ldots, \ell_n \rangle \in [m]^n} \left(\prod_{k=1}^n p_k^{\ell_k} \cdot \sum_{\ell \in [m]} \frac{(\sum_{k:\ell_k=\ell} w_k)^2}{c^\ell} \right) .$$

Since the expectation of a sum is equal to the sum of expectations, we can write

$$\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) = \sum_{\ell \in [m]} \sum_{A \subset [n]} \left(\prod_{i \in A} p_i^\ell \right) \left(\prod_{i \notin A} (1 - p_i^\ell) \right) \frac{\left(\sum_{k: \ell_k = \ell} w_k \right)^2}{c^\ell}. \quad (1)$$

The *Quadratic Optimum* associated with a traffic vector $\mathbf{w}$ and a capacity vector $\mathbf{c}$, denoted $\mathsf{QOPT}(\mathbf{w}, \mathbf{c})$, is the *least possible* sum of squares of the incurred link latencies. Note that while $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})$ is defined in relation to a mixed strategy profile $\mathbf{P}$, $\mathsf{QOPT}(\mathbf{w}, \mathbf{c})$ refers to the *optimum* pure strategy profile.

The *Maximum Social Cost*, denoted $\mathsf{MSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})$, which is used in the original KP model, is defined as the expectation of the maximum of the incurred link latencies. Correspondingly, the *Maximum Optimum*, denoted $\mathsf{MOPT}(\mathbf{w}, \mathbf{c})$, is the minimum, over all assignments, maximum incurred link latency.

2.4   Nash Equilibria

We are interested in a special class of mixed strategies called Nash equilibria [24] that we describe below. Formally, the probability matrix $\mathbf{P}$ is a *Nash equilibrium* [19, Section 2] if for all users $i \in [n]$ and links $\ell \in [m]$, $\lambda_i^\ell = \lambda_i$ if $I_i^\ell = 1$, and $\lambda_i^\ell > \lambda_i$ if $I_i^\ell = 0$. Thus, each user assigns its traffic with positive probability only on links (possibly more than one of them) for which its expected latency cost is minimized; this implies that there is no incentive for a user to unilaterally deviate from its mixed strategy in order to avoid links on which its expected latency cost is higher than necessary.

Mavronicolas and Spirakis [22, Lemma 15] show that in the model of arbitrary users and identical links, all links are equiprobable in a fully mixed Nash equilibrium.

Lemma 3 (Mavronicolas and Spirakis [22]). *Consider the model of arbitrary users and identical links. Then, there exists a unique fully mixed Nash equilibrium with associated Nash probabilities $p_i^\ell = 1/m$, for any user $i \in [n]$ and link $\ell \in [m]$.*

2.5   Coordination Ratio and Quadratic Coordination Ratio

The *Quadratic Coordination Ratio* is the maximum value, over all traffic vectors $\mathbf{w}$, capacity vectors $\mathbf{c}$, and Nash equilibria $\mathbf{P}$ of the ratio $\frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})}$. In a corresponding way, the *Maximum Coordination Ratio* is defined in [19] as the maximum value, over all traffic vectors $\mathbf{w}$, capacity vectors $\mathbf{c}$ and Nash equilibria $\mathbf{P}$ of the ratio $\frac{\mathsf{MSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{MOPT}(\mathbf{w}, \mathbf{c})}$.

3   The Quadratic Social Cost of Nash Equilibria

In this section, we study the Quadratic Social Cost of arbitrary (mixed) Nash equilibria. We start by proving:

Theorem 1 (Quadratic Social Cost of Arbitrary Nash Equilibrium).
Fix any traffic vector $\mathbf{w}$*, any capacity vector* $\mathbf{c}$*, and any arbitrary Nash equilib-rium* $\mathbf{P}$*. Then,* $\mathsf{QSC}\,(\mathbf{w},\mathbf{c},\mathbf{P})$ *can be computed in time* $O(nmW)$*.*

We next establish that the Quadratic Social Cost takes a particularly nice form for the case of the fully mixed Nash equilibrium. We prove:

Theorem 2 (Quadratic Social Cost of Fully Mixed Nash Equilibrium).
Consider the model of arbitrary users and identical links. Then, for any traffic vector $\mathbf{w}$*,*

$$\mathsf{QSC}\,(\mathbf{w},\mathbf{F}) = W_1 + \frac{2}{m}\,W_2\,,$$

where $W_1 = \sum_{i\in[n]} w_i^2$ *and* $W_2 = \sum_{i,k\in[n],i\neq k} w_i w_k$*.*

The next Lemma is used in the proof of Proposition 1.

Lemma 4. *Let* $a,n \in \mathbb{N}$*, a even, let $p_i \in [0,1]$ for all $1 \le i \le n$. Denote* $P = (p_1,\dots,p_n)$ *and* $p = \sum_{1\le i\le n} p_i$*. Set*

$$H(P) = \sum_{A\subset[n]} |A|^a \left\{ \prod_{i\in A} p_i \right\} \left\{ \prod_{j\notin A}(1-p_j) \right\}.$$

Define $\tilde{P}$ *by* $\tilde{p}_i = \frac{1}{n}\cdot p$ *for all* $1\le i\le n$*. Then* $H(P) \le H(\tilde{P})$*.*

Proposition 1. *Consider the model of identical users and identical links. Then, for any arbitrary Nash equilibrium* $\mathbf{P}$*,*

$$\mathsf{QSC}\,(\mathbf{P}) \le \sum_{j\in[m]} \left(\theta^j + \frac{r_j-1}{r_j}\,(\theta^j)^2 \right),$$

where $\theta^j = \sum_{i\in[n]} p_i^j$ *and* $r_j = |view(j)|$*.*

We finally prove:

Theorem 3. *Consider the model of identical users and related links. Then,*

$$\mathsf{QSC}\,(\mathbf{c},\mathbf{F}) = \frac{n\,(n+m-1)}{C}\,.$$

Corollary 1 (Quadratic Social Cost of Fully Mixed Nash Equilibrium).
Consider the model of identical users and identical links. Then,

$$\mathsf{QSC}\,(\mathbf{w},\mathbf{c},\mathbf{F}) = \frac{n\,(n+m-1)}{m}\,.$$

4   The Worst-Case Nash Equilibrium

We now establish that, for the model of identical users and identical links, the worst-case Nash equilibrium is the fully mixed Nash equilibrium. We start our proof with a technical lemma which holds in the more general model of arbitrary users, and then return to the model of identical users and identical links.

Lemma 5. *Consider n arbitrary users on m identical links, and let $j, k \in [m]$.*

1. *If $view(j) = view(k) \neq \emptyset$, then $|view(j)| = 1$ or $\theta^j = \theta^k$ and $p_i^j = p_i^k$ for all $i \in [n]$.*
2. *If $view(j) \subsetneq view(k)$, then $\theta^j > \theta^k$.*

Theorem 4. *Consider the model of identical users and identical links. Then, for any arbitrary Nash equilibrium $\mathbf{P}$, $\mathsf{QSC}\,(\mathbf{w}, \mathbf{c}, \mathbf{P}) \leq \mathsf{QSC}\,(\mathbf{w}, \mathbf{c}, \mathbf{F})$.*

5   Bounds on Quadratic Coordination Ratio

In this section, we present our bounds on Quadratic Coordination Ratio. We start by proving:

Theorem 5 (Quadratic Coordination Ratio for Pure Nash Equilibria). *Consider the model of arbitrary users and identical links, restricted to pure Nash equilibria. Then,*

$$\max_{\mathbf{w}, \mathbf{P}} \frac{\mathsf{QSC}\,(\mathbf{w}, \mathbf{P})}{\mathsf{QOPT}\,(\mathbf{w})} = \frac{9}{8}\,.$$

We give here only a sketch of the proof. Let there be n users and m links. If $n \leq m$, then every pure Nash equilibrium has optimal social cost. Now assume $n > m$. Let $\mathbf{P}$ be any pure Nash equilibrium. Let us first assume that $w_i \leq \frac{W}{m}$ holds for all users $i \in [n]$. Let $B = \min_{j \in [m]} \delta^j$ be the minimum traffic on any of the links. Then $B > 0$, and it has been shown in [17], that on every link the load is bounded by $2B$.
We use some iterative procedure to compute an upper bound for $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})$. When the algorithm terminates, then we know that there exists some $k \in [m]$, such that

- $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) \leq \sum_{j \in [m]} x_j^2$,
- $x_j = 2B$ for k links,
- $x_j = B + x$, $0 \leq x \leq B$, for one link, and
- $x_j = B$ for $m - k - 1$ links.

Note, that $\mathsf{QOPT}(\mathbf{w}, \mathbf{c}) \geq \frac{W2}{m}$, and therefore

$$\frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})} \leq \frac{((3k + m - 1)B2 + (B + x)2)m}{(mB + kB + x)2} = f(k)\,.$$

Maximizing the function $f(k)$ shows the upper bound for the case that $w_i \leq \frac{W}{m}$ for all $i \in [n]$.

In case that $w_i > \frac{W}{m}$ holds for some user $i \in [n]$, such a user is alone on its link in every Nash equilibrium $\mathbf{P}$, and both the user and the link can be omitted, increasing the coordination ratio. To show tightness, we give an instance.

We continue by a similar result for the reciprocal case of identical users and related links:

Theorem 6 (Quadratic Coordination Ratio for Pure Nash Equilibria).
Consider the model of identical users and related links, restricted to pure Nash equilibria. Then,

$$\max_{\mathbf{w},\mathbf{c},\mathbf{P}} \frac{\mathsf{QSC}\,(\mathbf{w},\mathbf{c},\mathbf{P})}{\mathsf{QOPT}\,(\mathbf{w},\mathbf{c})} = \frac{4}{3}\,.$$

We give here only a sketch of the full proof. First, we show that no instance with traffic vector $\mathbf{w} = \{1\}^n$, capacity vector $\mathbf{c}$ and pure Nash equilibrium $\mathbf{P}$ exists with Quadratic Coordination Ratio greater than $\frac{4}{3}$. Therefore, we assume, by way of contradiction, that such an instance exists, and fix the minimal (in the number of links) such counterexample, its worst case Nash equilibrium $\mathbf{P}$ and an optimal assignment $\mathbf{Q}$. We denote the traffic of each link j by $\delta^j(\mathbf{P})$ when referring to $\mathbf{P}$, and by $\delta^j(\mathbf{Q})$ when referring to the optimum assignment. Lemma 6 shows, that $\delta^j(\mathbf{P})$ is at most by one smaller than $\delta^j(\mathbf{Q})$ for any link j. Lemma 7 shows, that for the instance under consideration, only for exactly one link k, $\delta^k(\mathbf{P})$ is greater than $\delta^k(\mathbf{Q})$, and that no link has the same traffic according to $\mathbf{P}$ and $\mathbf{Q}$. This implies $\delta^k(\mathbf{P}) = \delta^k(\mathbf{Q}) + m - 1$, because all remaining links must have $\delta^j(\mathbf{P}) = \delta^j(\mathbf{Q}) - 1$. Lemma 8 shows, that, if not all links except for k have the same capacity c^j and the same traffic, then we can create a new instance with $m - 1$ identical links, having the same traffic, and one additional link, which has at least the same Quadratic Coordination Ratio. Hence, we can consider this new instance in order to bound the Quadratic Coordination Ratio of the original instance from above. To do so, we write down an optimization problem which overestimates the Quadratic Coordination Ratio of the new instance and includes, as constraints, the Nash equilibrium property and optimality criterion from Lemma 9. The optimization problem evaluates to $\frac{4}{3}$, which contradicts the initial assumption.

To proof that the bound is tight, we construct an instance with Quadratic Coordination Ratio $\frac{4}{3}$ for any number of links m.

Lemma 6. *Let $(\mathbf{w},\mathbf{c})$, $\mathbf{Q}$, $\mathbf{P}$, $\delta^j(\mathbf{Q})$ and $\delta^j(\mathbf{P})$ be as in the proof of Theorem 6. Then, $\delta^j(\mathbf{Q}) - \delta^j(\mathbf{P}) \leq 1$ for all $j \in [m]$.*

Lemma 7. *Let $(\mathbf{w},\mathbf{c})$, $\mathbf{Q}$, $\mathbf{P}$, $\delta^j(\mathbf{Q})$ and $\delta^j(\mathbf{P})$ be as in the proof of Theorem 6. Then, $\delta^k(\mathbf{P}) = \delta^k(\mathbf{Q}) + m - 1$ for some $k \in [m]$, and $\delta^j(\mathbf{P}) = \delta^j(\mathbf{Q}) - 1$ for all $j \in [m] \backslash \{k\}$.*

Lemma 8. *Let* $(\mathbf{w}, \mathbf{c})$, $\mathbf{Q}$, $\mathbf{P}$, $\delta^j(\mathbf{Q})$ *and* $\delta^j(\mathbf{P})$ *be as in the proof of Theorem 6. Then, there exists an instance* $(\tilde{\mathbf{w}}, \tilde{\mathbf{c}})$ *that has the same number m of links as* $(\mathbf{w}, \mathbf{c})$ *with optimal assignment* $\tilde{\mathbf{Q}}$, *Nash equilibrium assignment* $\tilde{\mathbf{P}}$, *such that* $\delta^k(\tilde{\mathbf{P}}) = \delta^k(\tilde{\mathbf{Q}}) + m - 1$ *for some link* $k \in [m]$, $\delta^i(\tilde{\mathbf{Q}}) = \delta^j(\tilde{\mathbf{Q}})$ *and* $\tilde{c}_i = \tilde{c}_j$ *for all* $i, j \in [m] \backslash \{k\}$ *and* $\frac{\mathsf{QSC}(\tilde{\mathbf{w}}, \tilde{\mathbf{c}}, \tilde{\mathbf{P}})}{\mathsf{QOPT}(\tilde{\mathbf{w}}, \tilde{\mathbf{c}})} \geq \frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})}$.

Lemma 9. *Let* $\mathbf{Q}$ *be any pure assignment for an instance* $(\mathbf{w}, \mathbf{c})$ *of the model of identical traffics and related links, let* $\mathbf{w} = (w, \ldots, w)$. *Then,* $\mathbf{Q}$ *is optimal, i.e.,* $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) = \mathsf{QOPT}(\mathbf{w}, \mathbf{c})$, *if and only if for every pair of links* $i, j \in [m]$

$$\frac{(\delta^i(\mathbf{Q}) + w)^2}{c_i} + \frac{(\delta^j(\mathbf{Q}) - w)^2}{c_j} \geq \frac{(\delta^i(\mathbf{Q}))^2}{c_i} + \frac{(\delta^j(\mathbf{Q}))^2}{c_j} .$$

We next prove:

Theorem 7. *Consider the model of arbitrary users and identical links. Then,*

$$\max_{\mathbf{w}, \mathbf{c}} \frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})} \leq 2 - \frac{1}{m} .$$

We next prove:

Theorem 8. *Consider the model of identical users and identical links. Then, for any traffic vector* $\mathbf{w}$, *capacity vector* $\mathbf{c}$ *and mixed Nash equilibrium* $\mathbf{P}$,

$$\frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})} \leq 1 + \min\left\{\frac{m-1}{n}, \frac{n-1}{m}\right\} \leq 2 - \frac{1}{m} .$$

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