Facets of the Fully Mixed Nash Equilibrium Conjecture

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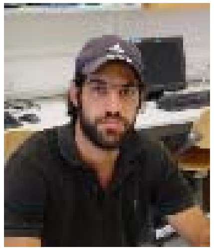

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Structure of the Talk

- Introduction
- Mathematical Tools
- Framework
- Contribution
- Conclusions

Motivation

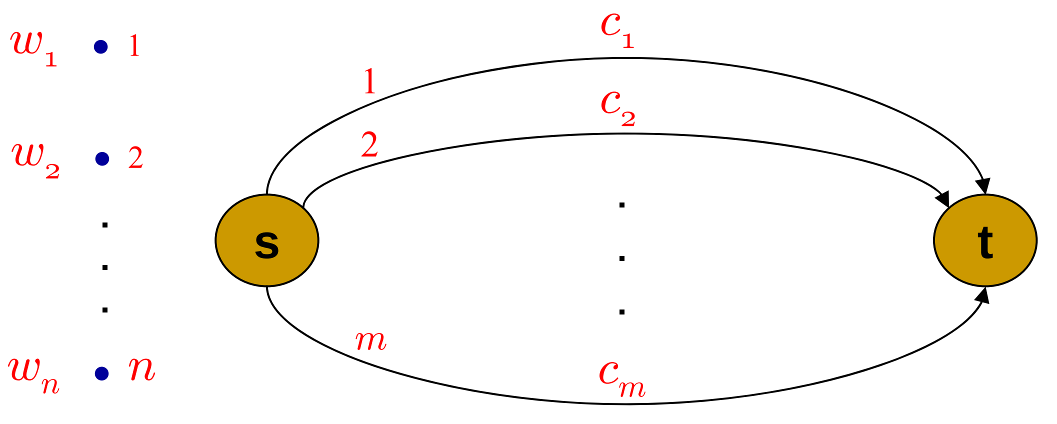
- Selfish routing in computer networks
  - Users are selfish and non-cooperative
- Cast as a non-cooperative game:
  - Users $\leftrightarrow$ Players
  - Links $\leftrightarrow$ Strategies
- Nash Equilibrium (NE):

  Stable state of the game where no user may profit by unitarily changing her strategy

  $\Rightarrow$ Identify the *worst-case* NE (with respect to Social Cost)

KP Model

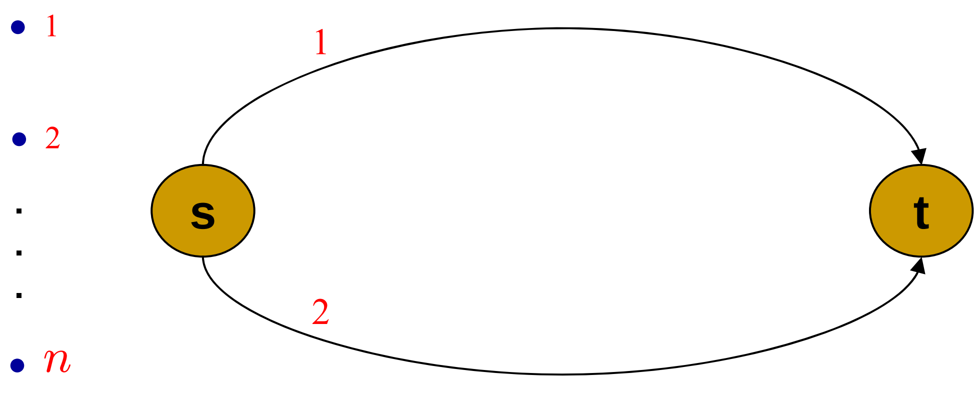
- A model of selfish routing:
  - A non-cooperative network:
    m parallel *related* links with arbitrary capacities
  - Players: n users with arbitrary weights
    - Strategy: deterministic or randomized choice of link(s)



[Koutsoupias and Papadimitriou, STACS 99]

KP Model (cont.)

- Special case:
  - Two *identical* parallel links
  - n *unweighted* users



Social Cost

- Various measures for the evaluation of NE:

  - Maximum Social Cost (MSC)

    Expectation (over all random choices of the users) of the maximum latency on a link

    [Koutsoupias and Papadimitriou, STACS 99]

  - Quadratic Maximum Social Cost (QMSC)

    Expectation (over all random choices of the users) of the square of the maximum latency on a link

    [This work]

Fully Mixed NE

- Fully Mixed NE
  - Each user chooses each link with *non-zero* probability
- Which NE maximizes Social Cost?
  - Since randomization increases interaction

    ⇒ fully mixed NE favors collisions among users
  - Increased probability of collisions

    ⇒ increase to the expected maximum congestion on a link

⇒ Fully Mixed NE should maximize Social Cost!

Fully Mixed NE Conjecture

- Fully Mixed NE Conjecture:

  The Fully Mixed NE maximizes MSC

- Quadratic Fully Mixed NE Conjecture:

  The Fully Mixed NE maximizes QMSC

Such Conjectures *trivialize* the computation of the *worst-case* NE!

[Gairing, Lücking, Mavronicolas, Monien & Spirakis, TCS 05]

Main Result

Quadratic Fully Mixed NE Conjecture is valid:

For n *unweighted* users and two *identical* links

$\Rightarrow$ Fully Mixed NE maximizes QMSC

Related Work

- Fully Mixed NE Conjecture is valid under MSC for:
  - Two *unweighted* users and m *related* links

    [Lücking, Mavronicolas, Monien, Rode, Spirakis & Vrto, MFCS 03]

  - n *unweighted* users and two *identical* links

    [Lücking, Mavronicolas, Monien, Rode, Spirakis & Vrto, MFCS 03]

  - Two *weighted* users and m *identical* links

    [Fotakis, Kontogiannis, Koutsoupias, Mavronicolas & Spirakis, ICALP 02]

Related Work (cont.)

- Fully Mixed NE Conjecture is *not* valid under MSC for:
  - Three users and two *unrelated* links

    [Lücking, Mavronicolas, Monien, Rode, Spirakis & Vrto, MFCS 03]
  - n *weighted* users and m *identical* links

    [Fischer & Vöcking, TCS 07]

Related Work (cont.)

Model assumptions	SC	FMNE Conjecture?
n = 2, weighted users, identical links	MSC	$\surd$
Unweighted users, related links	MSC	Away by a factor of 25
Weighted users, identical links	MSC	Away by a factor of $2h(1 + \varepsilon)$*
n = 2, unweighted users, related links	MSC	$\surd$
m = 2, unweighted users, identical links	MSC	$\surd$
m = 2, n = 2, unrelated links	MSC	$\surd$
m = 2, n = 3, unrelated links	MSC	$\times$
Weighted users, identical links	MSC	$\times$

$$* \; h = \frac{\text{maximum weight}}{\text{average weight}}$$

Notation

- For $n \geq 2$, denote $[n] = \{1, \cdots, n\}$
- For integer n, Even(n) and Odd(n) are 1 when n is even and odd, respectively, and 0 otherwise
- $X \sim \mathbb{P}$: random variable X follows probability distribution $\mathbb{P}$
- $\mathbb{E}_{\mathbb{P}}(X)$: expectation of X, where $X \sim \mathbb{P}$

Generalized Medians

- Binomial function $\mathsf{B}_{N,k}(p) : [0,1] \to \mathbb{R}$ with

$$\mathsf{B}_{N,k}(p) = \sum_{j=0}^{k} \binom{N}{j} p^j (1-p)^{N-j}$$

- For any $\alpha \in [0,1]$, define the α-median of the binomial distribution

$$\mathsf{M}_{N,p}(\alpha) = \min\{k \in [0,N] : \mathsf{B}_{N,k}(p) \geq \alpha\}$$

$\Rightarrow$ For all $k < \mathsf{M}_{N,p}(\alpha)$, $\mathsf{B}_{N,k}(p) < \alpha$

  - Generalizes the (classical) median of (binomial) distribution (which is $\frac{1}{2}$-median)

Generalized Medians Bounds

- **Lemma 1**.

For any $\epsilon > 0$, the following hold for $p = \frac{1}{2} - \frac{r}{2(n-r-1)}$:

(1) $\mathsf{M}_{n-r-2,p}\left(\frac{1}{2}+\epsilon\right) > \left\lceil\frac{n-3}{2}\right\rceil - r - 1$, where $1 \leq r \leq \left\lfloor\frac{n-3}{2}\right\rfloor - 4$.

(2) $\mathsf{M}_{n-r-2,p}\left(\frac{3}{7}+\epsilon\right) > \left\lceil\frac{n-3}{2}\right\rceil - r - 1$, where $n \geq 134$ is even and $r = \left\lfloor\frac{n-3}{2}\right\rfloor - 3$.

(3) $\mathsf{M}_{n-r-2,p}\left(\frac{2}{5}+\epsilon\right) > \left\lceil\frac{n-3}{2}\right\rceil - r - 1$, where $n \geq 134$ is even and $r = \left\lfloor\frac{n-3}{2}\right\rfloor - 2$.

(4) $\mathsf{M}_{n-r-2,p}\left(\frac{1}{3}+\epsilon\right) > \left\lceil\frac{n-3}{2}\right\rceil - r - 1$, where $n \geq 134$ is even and $r = \left\lfloor\frac{n-3}{2}\right\rfloor - 1$.

(5) $\mathsf{M}_{n-r-2,p}\left(\frac{1}{4}+\epsilon\right) > \left\lceil\frac{n-3}{2}\right\rceil - r - 1$, where $n \geq 134$ is even and $r = \left\lfloor\frac{n-3}{2}\right\rfloor$.

(6) $\mathsf{M}_{n-r-2,p}\left(\frac{3}{11}+\epsilon\right) > \left\lceil\frac{n-3}{2}\right\rceil - r - 1$, where $n \geq 135$ is odd and $r = \left\lfloor\frac{n-3}{2}\right\rfloor - 3$.

(7) $\mathsf{M}_{n-r-2,p}\left(\frac{2}{9}+\epsilon\right) > \left\lceil\frac{n-3}{2}\right\rceil - r - 1$, where $n \geq 135$ is odd and $r = \left\lfloor\frac{n-3}{2}\right\rfloor - 2$.

(8) $\mathsf{M}_{n-r-2,p}\left(\frac{1}{7}+\epsilon\right) > \left\lceil\frac{n-3}{2}\right\rceil - r - 1$, where $n \geq 135$ is odd and $r = \left\lfloor\frac{n-3}{2}\right\rfloor - 1$.

(9) $\mathsf{M}_{n-r-2,p}\left(\epsilon\right) > \left\lceil\frac{n-3}{2}\right\rceil - r - 1$, where $n \geq 135$ is odd and $r = \left\lfloor\frac{n-3}{2}\right\rfloor$.

The Model

- Two parallel *identical* links from source to destination
  - Each link has capacity 1
- $n \geq 2$ users wish to route from source to destination
  - Each user has weight 1

Strategies

- Pure strategy s_i for user $i \in [n]$:

  Some specific link

- Mixed strategy σ_i for user $i \in [n]$:

  Probability distribution over pure strategies

  $\Rightarrow$ Probability distribution over links

Profiles

- Pure profile $\mathsf{s} = \langle s_1, \cdots, s_n \rangle$
- Mixed profile $\sigma = \langle \sigma_1, \cdots, \sigma_n \rangle$

  $\Rightarrow$ Induces a (product) probability measure $\mathbb{P}_\sigma$ on pure profiles:

  For each pure profile s, $\mathbb{P}_\sigma(\mathsf{s}) = \prod_{k \in [n]} \sigma_k(s_k)$
- Fully Mixed profile σ :

  $\sigma_i(j) > 0$ for each $i \in [n]$ and $j \in [2]$

Costs

- Congestion on link j in pure profile s

$$\mathbf{c}(j, \mathbf{s}) = |\{i \in [n] : s_i = j\}|$$

- Expected Congestion on link j in mixed profile σ

$$\mathbf{c}(j, \sigma) = \mathbb{E}_{\mathbf{s} \sim \mathbb{P}_\sigma}(\mathbf{c}(j, \mathbf{s}))$$

- Individual Cost of user i in pure profile s

$$\mathsf{IC}_i(\mathbf{s}) = \mathbf{c}(s_i, \mathbf{s})$$

- Expected Individual Cost of user i in mixed profile σ

$$\mathsf{IC}_i(\sigma) = \mathbb{E}_{\mathbf{s} \sim \mathbb{P}_\sigma}(\mathsf{IC}_i(\mathbf{s}))$$

Costs (cont.)

- Quadratic Maximum Social Cost for mixed profile σ:

$$\begin{aligned}\mathsf{QMSC}(\sigma) &= \mathbb{E}_{\mathbf{s}\sim\mathbb{P}_\sigma}\left(\left(\max_{j\in[2]} \mathbf{c}(j,\mathbf{s})\right)^2\right) \\ &= \sum_{\mathbf{s}} \mathbb{P}_\sigma(\mathbf{s}) \cdot \left(\max_{j\in[2]} \mathbf{c}(j,\mathbf{s})\right)^2 \\ &= \sum_{\mathbf{s}} \left(\prod_{k\in[n]} \sigma_k(s_k)\right) \cdot \left(\max_{j\in[2]} \mathbf{c}(j,\mathbf{s})\right)^2\end{aligned}$$

Nash Equilibrium (NE)

Notation:

$\sigma_{-i} \diamond \sigma'_i$: the mixed profile obtained by substituting the mixed strategy σ_i in σ with σ'_i

Mixed profile σ is a NE if for each user $i \in [n]$, for each mixed strategy σ'_i she has,

$$\mathsf{IC}_i(\sigma) \leq \mathsf{IC}_i(\sigma_{-i} \diamond \sigma'_i)$$

Recalls

- Fully mixed NE ϕ exists uniquely (for *identical* links)
  [Mavronicolas & Spirakis, Algorithmica 07]

- For n *unweighted* users and two *identical* links,
  - $\mathsf{MSC}(\phi) = \frac{n}{2} + \frac{n}{2^n}\binom{n-1}{\lceil \frac{n}{2} \rceil - 1}$
  - For an arbitrary NE σ, $\mathsf{MSC}(\phi) \geq \mathsf{MSC}(\sigma)$

  [Lücking, Mavronicolas, Monien, Rode, Spirakis & Vrto, MFCS 03]

Quadratic Fully Mixed NE Conjecture is Valid

Theorem.

For the fully mixed NE ϕ and an arbitrary NE σ,

$$\mathsf{QMSC}(\phi) \geq \mathsf{QMSC}(\sigma)$$

Proof Sketch

We first prove:

$$\mathsf{QMSC}(\phi) = \frac{n}{4} + \frac{n^2}{4} + \frac{n}{2^n}\binom{n-1}{\lceil \frac{n}{2} \rceil - 1}$$

Proof Sketch

Fix an arbitrary NE σ. We identify three sets of users:

- U_1 = pure users choosing link 1
- U_2 = pure users choosing link 2
- U_{12} = fully mixed users choosing either link 1 or link 2

Proof Sketch (cont.)

Let $u = \min \{ |U_1|, |U_2| \}$

$\Rightarrow$ there are $2u$ pure users of which u choose link 1 and the other u choose link 2

- $\widehat{\sigma}$: the mixed NE derived from σ by eliminating those $2u$ users

  $\Rightarrow$ $\widehat{\sigma}$ has simpler syntactic form

- $\widehat{\phi}$: the fully mixed NE of n-$2u$ users

Proof Sketch (cont.)

We shall now prove:

Lemma 2.

$$\mathsf{QMSC}(\phi) - \mathsf{QMSC}(\sigma) \geq \mathsf{QMSC}(\widehat{\phi}) - \mathsf{QMSC}(\widehat{\sigma})$$

Proof Sketch (cont.)

Proof of Lemma 2

- First compare $\mathsf{QMSC}(\sigma)$ and $\mathsf{QMSC}(\hat{\sigma})$:

$$
\begin{aligned}
& \mathsf{QMSC}(\hat{\sigma}) \\
= \;& \mathbb{E}_{\mathbb{P}_\sigma}\left((\max\{\mathsf{c}(1,\sigma),\mathsf{c}(2,\sigma)\} - u)^2\right) \\
= \;& \mathbb{E}_{\mathbb{P}_\sigma}\left((\max\{\mathsf{c}(1,\sigma),\mathsf{c}(2,\sigma)\})^2 - 2u\max\{\mathsf{c}(1,\sigma),\mathsf{c}(2,\sigma)\} + u^2\right) \\
= \;& \mathbb{E}_{\mathbb{P}_\sigma}\left((\max\{\mathsf{c}(1,\sigma),\mathsf{c}(2,\sigma)\})^2\right) - 2u\mathbb{E}_{\mathbb{P}_\sigma}\left(\max\{\mathsf{c}(1,\sigma),\mathsf{c}(2,\sigma)\}\right) + u^2 \\
= \;& \mathsf{QMSC}(\sigma) - 2u\ \mathsf{MSC}(\sigma) + u^2
\end{aligned}
$$

Proof Sketch (cont.)

Proof of Lemma 2 (cont.)

- Now compare $\mathsf{QMSC}(\phi)$ and $\mathsf{QMSC}(\widehat{\phi})$:

$$
\begin{aligned}
&\mathsf{QMSC}(\phi) - \mathsf{QMSC}(\widehat{\phi}) \\
= \;& \frac{n}{4} + \frac{n^2}{4} + \frac{n^2}{2^n}\binom{n-1}{\lceil \frac{n}{2} \rceil - 1} - \frac{n-2u}{4} - \frac{(n-2u)^2}{4} - \frac{(n-2u)^2}{2^{n-2u}}\binom{n-2u-1}{\lceil \frac{n-2u}{2} \rceil - 1} \\
= \;& -\mathsf{QMSC}(\widehat{\sigma}) - \mathsf{QMSC}(\sigma) - 2u\,\mathsf{MSC}(\sigma) \\
& + u\left(n + \frac{1}{2}\right) + \frac{n^2}{2^n}\binom{n-1}{\lceil \frac{n}{2} \rceil - 1} - \frac{(n-2u)^2}{2^{n-2u}}\binom{n-2u-1}{\lceil \frac{n-2u}{2} \rceil - 1}
\end{aligned}
$$

Proof Sketch (cont.)

Proof of Lemma 2 (cont.)

- Hence:

$$
\begin{aligned}
& \mathsf{QMSC}(\phi) - \mathsf{QMSC}(\sigma) - (\mathsf{QMSC}(\widehat{\phi}) - \mathsf{QMSC}(\widehat{\sigma})) \\
= \; & -2u\,\mathsf{MSC}(\sigma) + u\left(n + \frac{1}{2}\right) + \frac{n^2}{2^n}\binom{n-1}{\lceil \frac{n}{2} \rceil - 1} - \frac{(n-2u)^2}{2^{n-2u}}\binom{n-2u-1}{\lceil \frac{n-2u}{2} \rceil - 1} \\
\geq \; & -2u\,\mathsf{MSC}(\phi) + u\left(n + \frac{1}{2}\right) + \frac{n^2}{2^n}\binom{n-1}{\lceil \frac{n}{2} \rceil - 1} - \frac{(n-2u)^2}{2^{n-2u}}\binom{n-2u-1}{\lceil \frac{n-2u}{2} \rceil - 1} \\
= \; & \frac{u}{2} - 2u\frac{n}{2^n}\binom{n-1}{\lceil \frac{n}{2} \rceil - 1} + \frac{n^2}{2^n}\binom{n-1}{\lceil \frac{n}{2} \rceil - 1} - \frac{(n-2u)^2}{2^{n-2u}}\binom{n-2u-1}{\lceil \frac{n-2u}{2} \rceil - 1} \\
\geq \; & 0 \qquad \boxtimes
\end{aligned}
$$

Proof Sketch (cont.)

By Lemma 2 $\Rightarrow$ it suffices to show:

Lemma 3.

$$\mathsf{QMSC}(\widehat{\phi}) - \mathsf{QMSC}(\widehat{\sigma}) \geq 0$$

Proof Sketch (cont.)

Rename the variables so that both $\widehat{\phi}$ and $\widehat{\sigma}$ refer to n users

- All n users in $\widehat{\phi}$ are fully mixed
- Assume that in $\widehat{\sigma}$:
  - $r \geq 1$ pure users choose link 1 with probability 1
  - n-r mixed users choose both links with *non-zero* probability

Proof Sketch (cont.)

Lemma 4.

For the NE $\widehat{\sigma}$, for each mixed user $i \in U_{12}$, $\widehat{\sigma}_i(1) = \frac{1}{2} - \frac{r}{2(n-r-1)}$ and $\widehat{\sigma}_i(2) = 1 - \widehat{\sigma}_i(1)$. Furthermore, $r \leq \lfloor \frac{n-3}{2} \rfloor$.

[Lücking, Mavronicolas, Monien, Rode, Spirakis & Vrto, MFCS 03]

Denote $p = \widehat{\sigma}_i(1)$ and $q = \widehat{\sigma}_i(2)$

Proof Sketch (cont.)

- Calculate QMSC of $\widehat{\sigma}$:

$$
\begin{aligned}
&\mathsf{QMSC}(\widehat{\sigma}) \\
=\ &\mathsf{Even}(n)\cdot\frac{n^2}{4}\binom{n-r}{\frac{n}{2}-r}p^{\frac{n}{2}-r}q^{\frac{n}{2}}+\sum_{i=\lfloor\frac{n}{2}\rfloor+1}^{n} i^2\binom{n-r}{i-r}p^{i-r}q^{n-i} \\
&+\sum_{i=\lfloor\frac{n}{2}\rfloor+1}^{n-r} i^2\binom{n-r}{i}p^{n-r-i}q^{i}
\end{aligned}
$$

Proof Sketch (cont.)

So, observe:

$$\begin{aligned}&\mathsf{QMSC}(\widehat{\phi}) - \mathsf{QMSC}(\widehat{\sigma})\\ \geq\ & \frac{n}{4} + \frac{n^2}{4} + \frac{n^2}{2^n}\binom{n-1}{\lceil \frac{n}{2} \rceil} - q(n-r) - q^2(n-r)(n-r-1)\\ &+(q^2-p^2)(n-r)(n-r-1)\mathsf{Q} + \mathsf{D},\ \text{where}\end{aligned}$$

$$\mathsf{Q} = \sum_{i=\lceil \frac{n+1}{2} \rceil - r}^{n-r} \binom{n-r-2}{i-2} p^{i-2} q^{n-r-i} = 1 - \mathsf{B}_{n-r-2, \lceil \frac{n-3}{2} \rceil - r - 1}(p)$$

$$\begin{aligned}\mathsf{D}\ =\ & -q^2(n-r)(n-r-1)\binom{n-r-2}{\lceil \frac{n-2}{2} \rceil - r} p^{\lceil \frac{n-2}{2} \rceil - r} q^{\lfloor \frac{n-2}{2} \rfloor} + (pq-p^2)(n-r)\binom{n-r-2}{\lceil \frac{n-3}{2} \rceil - r} p^{\lceil \frac{n-3}{2} \rceil - r} q^{\lfloor \frac{n-1}{2} \rfloor}\\ & -\mathsf{Odd}(n) \cdot q(n-r)\left(\binom{n-r-1}{\frac{n-1}{2} - r} p^{\frac{n-1}{2} - r} q^{\frac{n-1}{2}} + q(n-r-1)\binom{n-r-2}{\frac{n-3}{2} - r} p^{\frac{n-3}{2} - r} q^{\frac{n-1}{2}}\right)\\ & -\mathsf{Even}(n) \cdot \frac{n^2}{4}\binom{n-r}{\frac{n}{2} - r} p^{\frac{n}{2} - r} q^{\frac{n}{2}}\end{aligned}$$

Proof Sketch (cont.)

We consider two cases:

- Case 1: n is even
  - Substitute p and q from Lemma 4 to get:

$$D \geq \binom{n-r-2}{\frac{n-2}{2}-r} p^{\frac{n-2}{2}-r} q^{\frac{n-2}{2}} \left(-\frac{(n-1)^2(n-r)}{4(n-r-1)} - \frac{n(n-1)(n-2r-1)(n-r)}{4(n-r-1)(n-2r)} \right)$$

  - Hence:

$$\begin{aligned} & \mathsf{QMSC}(\hat{\phi}) - \mathsf{QMSC}(\hat{\sigma}) \\ > \; & \underbrace{\frac{n^2}{2^{n+1}}\binom{n}{\frac{n}{2}} - \binom{n-r-2}{\frac{n-2}{2}-r} p^{\frac{n-2}{2}-r} q^{\frac{n-2}{2}} \left(\frac{n^2(n-r)}{2(n-r-2)} \right) + r(n-r)Q}_{G} - \frac{r(n+1)}{4(n-r-1)} \end{aligned}$$

  - Case analysis on the range of values of r to prove that

$$G - \frac{r(n+1)}{4(n-r-1)} \geq 0$$

Proof Sketch (cont.)

Case 1.1: $1 \leq r \leq \left\lfloor \frac{n-3}{2} \right\rfloor - 4$

- Lemma 1 implies that $Q \geq \frac{1}{2}$.
- By substituting p and q from Lemma 4, we get:

$$
\begin{aligned}
G \geq\ & \frac{n^2}{2^{n+1}\binom{n}{\frac{n}{2}}} \left(1 - \frac{n}{n-r-2} \frac{\prod_{i=0}^{r}(n-2r+2i)}{\prod_{i=1}^{r}(n-r+i)} \frac{(n-1)(n-2r-1)^{1-r}}{(n-r-1)^{3-r}} \right. \\
& \left. \left(\frac{n-2r-1}{n-r-1}\right)^{\frac{n-4}{2}} \left(\frac{n-1}{n-r-1}\right)^{\frac{n-4}{2}} \right) + \frac{r(n-r)}{2} \\
\geq\ & \cdots \\
\geq\ & \frac{r(n+1)}{4(n-r-1)}
\end{aligned}
$$

Proof Sketch (cont.)

Case 1.2: $\left\lfloor \frac{n-3}{2} \right\rfloor - 3 \leq r \leq \left\lfloor \frac{n-3}{2} \right\rfloor$

- By Lemma 1, get lower bounds on Q
- By substituting p and q from Lemma 4, we get:

$$\begin{aligned}
&\mathsf{QMSC}(\hat{\phi}) - \mathsf{QMSC}(\hat{\sigma}) \\
\geq\ & -\frac{n-r}{2}\left(\frac{n^2}{2(n-r-1)} - \frac{n^2}{2(n-r)} - 2r\mathsf{Q}\right. \\
&\left. + \frac{n(n-2r-1)^{\frac{n-2}{2}-r}(n^2-n-2nr+r)}{2^{\frac{n-2r-4}{2}}\left(\frac{n-2}{2}-r\right)!(n-2r)}\left(\frac{n-1}{2(n-r-1)}\right)^{\frac{n}{2}}\right) \\
\geq\ & \cdots \\
\geq\ & 0
\end{aligned}$$

Proof Sketch (cont.)

- <u>Case 2: n is odd</u>
  - Similar proof. Uses again Lemma 1.

Summary of Results

- Proved the Fully Mixed NE Conjecture for a special case of the KP model under QMSC

- Proof derived some new estimations on generalized medians of the binomial distribution

  $\Rightarrow$ independent interest!

Open Problem

Assume:

- $n \geq 2$ *unweighted* users
- $m \geq 2$ *identical* links
- Polynomial Maximum Social cost:

  Expectation of a polynomial (with non-negative coefficients) of the maximum congestion on a link

Prove:

Polynomial Fully Mixed NE Conjecture:

Fully Mixed NE maximizes Polynomial Maximum Social Cost

Thank you!