Network Game with Attacker and Protector Entities

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A Network Security Problem

- Information network with
  - nodes *insecure* and vulnerable to *infection* by attackers e.g., viruses, Trojan horses, eavesdroppers, and
  - a *system security software* or a defender of limited power, e.g. able to clean a part of the network.
- In particular, we consider
  - a graph *G* with
    - ν attackers each of them locating on a node of *G* and
    - a defender, able to clean a single edge of the graph.

A Network Security Game: *Edge Model*

- We modeled the problem as a Game

$$\Pi_M(G) = \langle \mathcal{N}, \{S_i\}_{i\in\mathcal{N}}, \{\mathsf{IP}\}_{i\in\mathcal{N}} \rangle$$

  - on a graph *G(V, E)* with two kinds of players (set $\mathcal{N}$):
  - ν attackers (set $\mathcal{N}_{vp}$) or vertex players (vps) vp_i, each of them with *action set,* $S_{vp_i} = V$,
  - a defender or the edge player *ep*, with *action set,* $S_{ep} = E$,

and *Individual Profits* in a profile $\mathbf{s} = \langle s_1, \ldots, s_{|\mathcal{N}_{vp}|}, s_{ep} \rangle \in \mathcal{S}$

  - vertex player vp_i: $\mathsf{IP}_i(\mathbf{s}) = 0 \ if \ s_i \in s_{ep} \ \ or \ \ 1 \ otherwise$
    i.e., 1 if it is not caught by the edge player, and 0 otherwise.
  - Edge player ep: $IC_{ep}(\mathbf{s}) = |\{s_i : s_i \in s_{ep}\}|$,
    i.e. gains the number of vps incident to its selected edge s_{ep}.

Nash Equilibria in the Edge Model

- We consider pure and mixed strategy profiles.
- Study associated Nash equilibria (NE), where no player can unilaterally improve its Individual Cost by switching to another configuration.

Notation

- $P_s(ep, e)$: probability *ep* chooses edge *e* in *s*
- $P_s(vp_i, \upsilon)$: probability vp_i chooses vertex υ in *s*
- $P_s(vp, \upsilon) = \sum_{i \in N_{vp}} P_s(vp_i, v)$: # vps located on vertex υ in *s*
- $D_s(i)$: the support (actions assigned positive probability) of player $i \in \mathcal{N}$ in s.
- $ENeigh_s(\upsilon) = \{(u,v) \in E : (u,v) \in D_{\mathbf{s}}(ep)\}$
- $P_s(Hit(\upsilon)) = \sum_{e \in ENeigh(v)} P_{\mathbf{s}}(ep, e)$: the hitting probability of υ
- $m_s(v) = \sum_{i \in N_{vp}} P_{\mathbf{s}}(vp_i, v)$: expected # of vps choosing υ
- $m_s(e) = m_s(u) + m_s(v)$
- $Neigh_G(X) = \{u \notin X : (u,v) \in E(G)\}$

Expected Individual Costs

- vertex players vp_i:

$$\mathsf{IP}_i(\mathbf{s}) = \sum_{v \in V} P_{\mathbf{s}}(vp_i, v) \cdot (1 - P_{\mathbf{s}}(Hit(v)) \quad (1)$$

- edge player ep:

$$\mathsf{IP}_{ep}(\mathbf{s}) = \sum_{e=(u,v) \in E} P_{\mathbf{s}}(ep, e) \cdot (m_{\mathbf{s}}(u) + m_{\mathbf{s}}(v)) \quad (2)$$

Summary of Results

- No instance of the model contains a pure NE
- A graph-theoretic characterization of mixed NE
- Introduce a subclass of mixed NE:

  $\Rightarrow$ *Matching* NE

  - A characterization of graphs containing *matching* NE
  - A linear time algorithm to compute a matching NE on such graphs
  - Bipartite graphs and trees satisfy the characterization
  - Polynomial time algorithms for matching NE in bipartite graphs

Significance

- The *first* work (with an exception of ACY04) to model *network security problems* as strategic game and study its associated Nash equilibria.
- One of the few works highlighting a fruitful interaction between *Game Theory* and *Graph Theory*.
- Our results contribute towards answering the general question of Papadimitriou about the complexity of Nash equilibria for our special game.
- We believe Matching Nash equilibria (and/or extensions of them) will find further applications in other network games.

Pure Nash Equilibria

Theorem 1. *If G contains more than one edges, then $\Pi(G)$ has no pure Nash Equilibrium.*

Proof.

- Let $e=(u,v)$ the edge selected by the *ep* in s.
- $|E| > 1 \Rightarrow$ there exists an edge $(u',v') = e' \neq e$, such that $u \neq u'$.
- If there is a vp_i located on e,
  - vp_i will prefer to switch to u and gain more

  $\Rightarrow$ Not a NE.
- *Otherwise*, no vertex player is located on *e*.
  - Thus, $IC_{ep}(s)=0$,
  - *ep* can gain more by by selecting any edge containing at least one vertex player.

  $\Rightarrow$ Not a NE.

Characterization of Mixed NE

Theorem 2. *A mixed configuration s is a Nash equilibrium for any $\Pi(G)$ if and only if:*

1. $D_s(ep)$ *is an edge cover of G and*
2. $D_s(vp)$ *is a vertex cover of the graph obtained by* $D_s(ep)$.
3. (a) $P(Hit(v)) = P_s(Hit(u)) = min_v P_s (Hit(v))$, $8\ u,v\ 2\ D_s(vp)$,
   (b) $\sum_{e\ 2\ D_s(ep)} P_s(ep,e) = 1$
4. (a) $m_s(e_1)=m_s(e_2)=max_e\ m_s(e)$, $8\ e_1, e_2\ 2\ D_s(ep)$ and
   (b) $\sum_{v\ 2\ V(Ds(ep))} m_s(v)=\nu$.

1. (Edge cover) Proof:

If there exists a set of vertices NC ≠ ∅, Not covered by $D_s(ep)$,

⇒ $D_s(vp_i)$ µ NC, for all vp_i 2 N_{vp} ⇒ $IC_s(ep)=0$

⇒ ep can switch to an edge with at least one vp and gain more.

Matching Nash Equilibria

Definition 1. *A matching configuration s of $\Pi(G)$ satisfies:*

1. $D_s(vp)$ *is an independent set of G and*
2. *each vertex v of* $D_s(vp)$ *is incident to only one edge of* $D_s(ep)$.

Lemma 1. *For any graph G, if in $\Pi(G)$ there exists a matching configuration which additionally satisfies condition 1 of Theor. 2,*

- *then setting* $D_s(vp_i) := D_s(vp)$, 8 vp_i 2 N_{vp} *and*
- *applying the uniform probability distribution on the support of each player,*

we get a NE for $\Pi(G)$, which is called matching NE.

Characterization of Matching NE

Definition 2. The graph G is an S-expander graph if for every set X µ S µ V, |X|· |Neigh$_G$(X)|.

Marriage Theorem. *A graph G has a matching M in which set X µ V is matched into V\X in M if and only if for each subset Sµ X, |Neigh$_G$(S)| ¸ |S|.*

Theorem 3. *For any G, Π(G) contains a matching NE if and only if the vertices of G can be partitioned into two sets:*

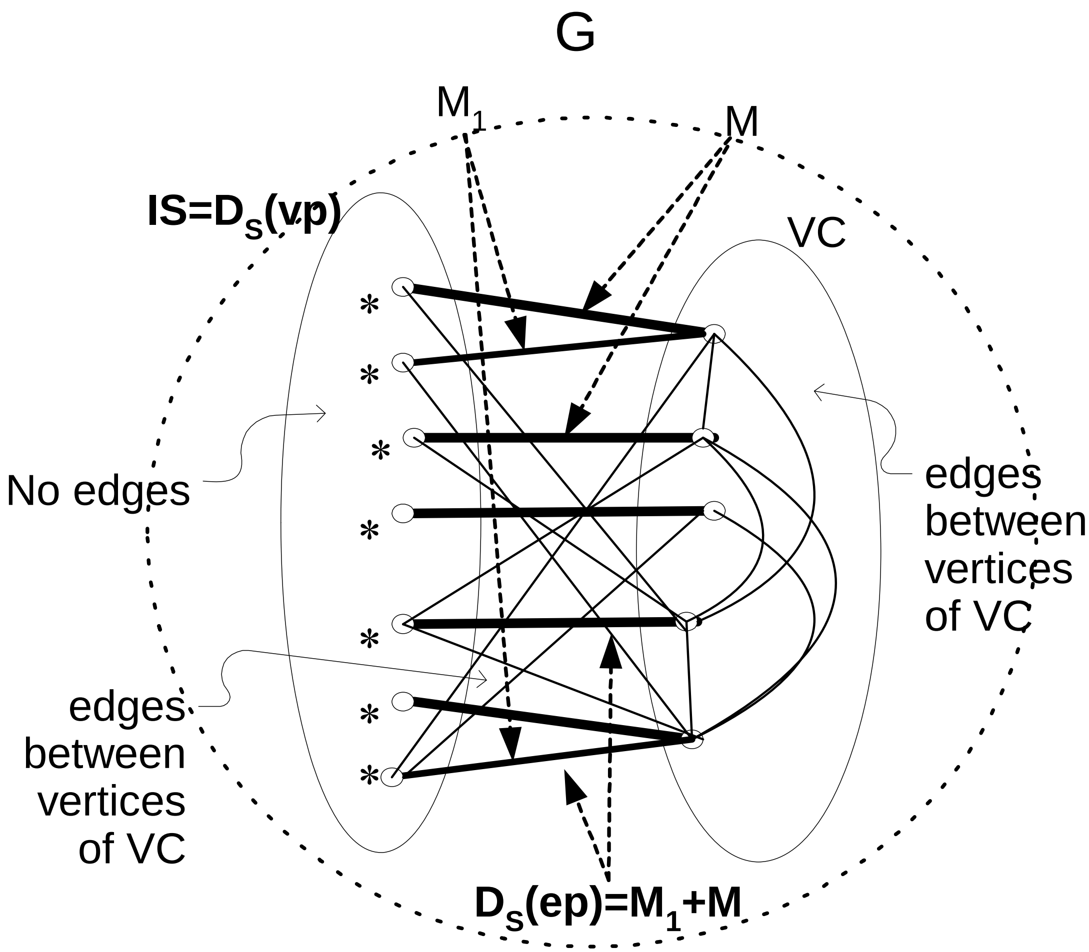
- *IS and VC= V \ IS*

such that IS is an independent set of G and G is a VC-expander graph.

Proof of Theorem 3.

- *If G contains an independent set IS and G is VC-expander then Π(G) contains a matching NE. Proof:*
- G is VC-expander ⇒ by the Marriage Theorem, *G* has a matching *M* such that each vertex *u 2 VC* is matched into *V\VC* in *M*.
- Partition IS into two sets:
  - IS_1 = {*v 2 IS* such that there exists an *e=(u,v) 2 M* and *u 2 VC*}.
  - IS_2 = the remaining vertices of *IS*.
- Define a configuration s as follows:
  - For each *v2* IS_2, add one edge *(u,v) 2 E* in set M_1.
  - Set $D_s(vp) = D_s(vp_i)_{8\ vpi\ 2\ Nvp} := IS$ and $D_s(ep) := M[\ M_1$.
  - Apply the uniform distribution for all players

Proof of Theorem 3. (An example)



- By construction, **s** is matching NE.

Proof of Theorem 3. (Cont.)

- *If Π(G) contains a matching NE then G contains an independent set IS and G is VC-expander, where VC = V \ IS. Proof:*
- Define set IS=D_s(vp)
  - IS is an independent set of G
  - for each v2 VC, there exists (u,v) 2 D_s(ep) such that v2 IS
  - for each v2 VC, add edge (u,v) 2 D_s(ep) in a set Mµ E.

  ⇒ M matches each vertex of VC into V \ VC =IS

  ⇒by the Marriage's Theorem, |Neigh(VC')|¸ |VC'|, for all VC' µ VC, i..e.

  ⇒ G is a VC-expander

A polynomial time Algorithm *A(Π(G), IS))*

Input: *Π(G),* independent set *IS*, such that G is *VC*-expander, where VC=V\IS.

Output: a matching NE of *Π(G)*

1. Compute a matching *M* covering all vertices of set *VC*.
2. Partition *IS = V\VC* into two sets:
   - IS_1 = { v 2 IS such that there exists an *e=(u,v) 2 M* and *u 2 VC* }
   - IS_2 = the remaining vertices of *IS*.
3. Compute set M_1: for each *v2* IS_2, add one edge *(u,v) 2 E* in set $M_{1.}$
4. Set $D_s(vp) = D_s(vp_i)_{8\ vpi\ 2\ Nvp}$:= *IS* and $D_s(ep)$:= *M[* M_1 and apply the uniform distribution for all players

Correctness and Time Complexity

Theorem 4. *Algorithm $A(\Pi(G), IS)$ computes a matching (mixed) Nash equilibrium for $\Pi(G)$ in time $O(m)$.*

Proof.

The algorithm follows the constructive proof of Theorem 3.

Application of Matching NE: Bipartite Graphs

Lemma 2. *In any bipartite graph G there exists a matching M and a vertex cover VC such that*

1. *every edge in M contains exactly one vertex of VC and*
2. *every vertex in VC is contained in exactly one edge of M.*

Proof Sketch.

- Consider *a minimum vertex cover* VC
- By the minimality of VC and since G is bipartite,
  - for each Sµ VC, Neigh$_G$(S)µ S

    $\Rightarrow$ by the Marriage Theorem, G has a matching M covering all vertices of VC (condition 2)
  - every edge in M contains exactly one vertex of VC (condition 1)

Application of Matching NE: Bipartite Graphs

Theorem 5. (Existence and Computation)

If G is a bipartite graph, then

- *Π(G) contains a matching mixed NE of Π(G) and*
- *one can be computed in polynomial time,*
  $\max\{O(m\sqrt{n}), O(n^{2.5}/\sqrt{\log n})\}$ *using Algorithm A.*

Proof Sketch.

- Utilizing the constructive proofs of Lemma 2 and Theorem 3,
- we compute an independent set IS such that G is VC-expander, where VC = V\IS, as required by algorithm A.
- Thus, algorithm A is applicable for *Π(G).*

Current and Future Work

- Compute other structured/unstructured Polynomial time NE
  - for specific graph families,
  - exploiting their special properties
- Existence and Complexity of Matching equilibria for general graphs
- Generalizations of the Edge model

Thank you

for your Attention !