SELFISH ROUTING IN THE PRESENCE OF NETWORK UNCERTAINTY*

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ABSTRACT

We study the problem of selfish routing in the presence of incomplete network information. Our model consists of a number of users who wish to route their traffic on a network of m parallel links with the objective of minimizing their latency. However, in doing so, they face the challenge of lack of precise information on the capacity of the network links. This uncertainty is modeled via a set of probability distributions over all the possibilities, one for each user. The resulting model is an amalgamation of the KP-model of [14] and the congestion games with user-specific functions of [22]. We embark on a study of *Nash equilibria* and the *price of anarchy* in this new model. In particular, we propose polynomial-time algorithms (w.r.t. our model's parameters) for computing some special cases of pure Nash equilibria and we show that negative results of [22], for the non-existence of pure Nash equilibria in the case of three users, do not apply to our model. Consequently, we propose an interesting open problem, that of the existence of pure Nash equilibria in the general case of our model. Furthermore, we consider appropriate notions for the social cost and the price of anarchy and obtain upper bounds for the latter. With respect to fully mixed Nash equilibria, we show that when they exist, they are unique. Finally, we prove that the fully mixed Nash equilibrium is the worst equilibrium.

Keywords: Selfish routing, Incomplete information, User-specific cost, Nash Equilibrium.

1. Introduction

In their pioneering work, Koutsoupias and Papadimitriou [14] introduce a non-cooperative *weighted congestion game* (named in the literature as the KP-model) where n selfish users wish to route their unsplittable traffic onto m parallel links from a source to a destination. In this class of games, each link has a certain capacity representing the rate at which the link processes traffic, and users have complete knowledge of the system's parameters such as the link capacities and the traffic induced by other users. Furthermore, users choose how to route their traffic based on a common payoff function, which essentially captures the delay to be experienced on each link. However, modern non-cooperative systems, such as computer networks and the Internet, which have motivated the study of games such as that of [14], present incomplete information on various aspects of their behavior. For example, it is often the case that network users have incomplete information regarding the link capacities. Such uncertainty may be caused if the network links are complex paths

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created by routers which are constructed differently on separate occasions according to the presence of congestion or link failures. In mobile wireless networks, uncertainty may also arise due to changes to the network connectivity caused by user mobility. Several techniques have been proposed in the literature for capturing probabilistically this network uncertainty via analysis of the mobility models and by profiling of mobile users [16, 27] leading to the development of probabilistic routing algorithms [15, 18].

In this paper we introduce an extension of the KP-model that captures these more realistic network scenarios. We consider a model where the network links may present a number of different capacities and each user's uncertainty about the capacity of the links is modeled via a probability distribution over all the possibilities. We assume that users may have different sources of information regarding the network and, therefore, take their probability distributions to be distinct from one another. This gives rise to a model with user-specific payoff functions. We may see that our model subsumes the KP-model since, in the case of users assigning probability one to the same capacity for each link, the two models coincide. Moreover, our model turns out to be an instance of *weighted congestion games with user-specific functions* studied by Milchtaich in [22].

We are interested in algorithmic problems related to *Nash equilibria* for the routing game we consider, that is, steady states in the game where no user has an incentive to unilaterally change its strategy. For example, we are interested in deciding whether and when Nash equilibria exist in our model, and, if so, determine efficiently the users' strategies that give rise to these equilibria. Furthermore, we study a notion for the *social cost* of the game, defined as the sum of the latencies experienced by each of the users, and an associated notion for the *price of anarchy* or *coordination ratio* [14] which captures the performance degradation in the game due to the lack of coordination among the users.

Prior and Related Work. Congestion games were first introduced by Rosenthal [26] and studied extensively thereafter. Rosenthal showed that these games admit pure Nash equilibria by using the notion of *potential functions*. Subsequent related work (e.g. [23]) characterized games that admit potential functions as *potential games*. The problem of computing pure Nash equilibria was studied for congestion games in [4] and for weighted congestion games in [1]. The *KP-Model* [14] and its Nash equilibria were studied extensively in the last years; see, for example, [3, 7, 13, 19, 21] and [5] for a survey. Fully mixed Nash equilibria for the KP-model were first studied in [21]. The fully mixed Nash equilibrium conjecture, stating that the fully mixed Nash equilibrium has the worst social cost among all Nash equilibria, was first formulated in [8] and it was verified in [17] for a social cost defined as the sum of the users latencies. Fischer and Berthold in [6] disproved the conjecture for a social cost defined as the expected maximum over the user latencies.

The notion of the price of anarchy was first introduced and studied in the KP-Model [14]. Subsequently, tight bounds were proposed for it in [3, 13] for identical links, in [3] for related links, and in [2] for congestion games with linear latency functions.

Gairing et al. [9] were the first to consider an extension of the KP-model with incomplete information. Their model considers a game of parallel links with incomplete information on the traffics of the users, which makes it complementary to our work (as we consider incomplete information on network capacities). The payoff functions employed by the users, which are universal and not user specific, take into account probabilistic information on the user traffics. Based on the seminal work by Harsanyi [12], the authors show that their model always admits a pure Nash equilibrium and propose an algorithm for computing such equilibria for some special cases. Also they show that the fully mixed Nash equilibrium maximizes the social cost for special cases of their model and that, in the general case, more than one fully mixed Nash equilibrium may exist. Finally, they show asymptotically-tight upper bounds on the coordination ratio.

Milchtaich [22] studied congestion games in which the payoff function associated with each user is not universal but user-specific. He shows that these games do not admit a pure Nash equilibrium in the general case, but are guaranteed to exhibit such equilibria in special cases, such as the case of unweighted users. Our work is closely related to [22] since our game is an instance of that model. Thus we inherit the positive results obtained therein. However, we show that the negative results of [22] do not necessarily apply for our model.

Recently, Gairing et al. [10] studied unweighted and weighted network congestion games with player-specific linear latency functions for both splittable and unsplittable traffic. Of specific interest to us is a result that establishes (via a counter-example with 3 players and 11 links) that weighted congestion games on parallel links with player-specific linear latency functions do not possess the *finite improvement property* [23], and thus, are not *potential games* [23]. This result also applies to our model. In a subsequent work, Mavronicolas et al. [20] introduced and studied weighted congestion games with *player-specific constants*, where each player-specific latency function is composed (by means of an abelian group operation) of a resource-specific delay function and a player-specific constant. In that respect, our model can be viewed as a congestion game with player-specific multiplicative constants and identity delay function.

Contributions. The contributions of our work are summarized as follows:

• We present an interesting new model that captures the idea of the uncertainty of the network state. Furthermore, we show that our routing game with incomplete information can be transformed into a complete information routing game with user-specific latency functions (Section 2).

• We propose polynomial-time algorithms (w.r.t. our model's parameters) for computing some special cases of pure Nash equilibria (Section 3). For one of our algorithms we introduce the notion of *tolerance* that measures the traffic a user can tolerate on a certain link before it decides to deviate from the specific link. We believe that this notion can be generalized and used in other related work dealing with selfish routing. We also demonstrate that the counter-example presented in [22], showing that pure Nash equilibria do not exist in the general case, does not apply in our model. Thus, we identify an interesting open problem in this area, that of existence of pure Nash equilibria in our model.

• We identify and employ an expression for the social cost (the sum of the users' latencies) and the associated notion for the price of anarchy. We obtain upper bounds for the latter in the general case and a special case (Section 4).

• We compute the fully mixed Nash equilibrium and show that when it exists, it is unique.

Also we show that, for certain instances of the game, a fully mixed Nash equilibrium always exists and it assigns all users to all links equiprobably. Finally, we verify the fully-mixed NE conjecture in our model, by proving that the fully mixed Nash equilibrium minimizes the social welfare (Section 4).

2. Model and Definitions

In this section we present the model and definitions we use throughout the paper. For all $k \in \mathbb{N}$, denote $[k] = \{1, 2, \ldots, k\}$. We consider a *network* consisting of a set of m *parallel links* $1, 2, \ldots, m$, or simply *links*, from a source to a destination, and n *network users* $1, 2, \ldots, n$, or simply *users*, who wish to route their traffic along a single link from the source to the destination. We assume that $n > 1$ and $m > 1$. (Throughout, we will be using subscripts for users and superscripts for links.) We denote by $w_i > 0$ the traffic of user $i \in [n]$. We define $\mathbf{w}$ as the $n \times 1$ vector containing the traffics of all users.

In our model, we assume that there exists uncertainty regarding the capacity of the network links. Thus, we define a *state* to be an $m \times 1$ vector, $\langle c^1, c^2, \ldots, c^m \rangle$ where, for all $\ell \in [m]$, $c^\ell > 0$ represents the capacity of link ℓ. The *state space* of the network, denoted by Φ, is defined as the set containing all the possible states the network may realize. We let ϕ range over Φ and we write c^ℓ_ϕ for the capacity of link ℓ according to state ϕ.

Each user, based on some private knowledge, may have a different *belief* regarding the capacity of the network links. We assume that this knowledge is *probabilistic* and it has the form of a probability distribution function over the set of all states. In general, we write $b \in \Delta(\Phi)$ to denote a belief probability distribution over all states, and b_i for the belief of user $i \in [n]$. Furthermore, we write $b(\phi)$ for the probability assigned to state ϕ by belief b. We define the *belief profile* $\mathbf{B}(\Phi)$ to be the $n \times 1$ vector $\langle b_1, b_2, \ldots, b_n \rangle$ containing the beliefs of all users. From this point onwards, we fix Φ and we write $\mathbf{B}$ instead of $\mathbf{B}(\Phi)$.

We consider the routing game $\mathbb{G} = (n, m, \mathbf{w}, \mathbf{B})$ where n is the number of users, m is the number of links, $\mathbf{w}$ is a traffic vector and $\mathbf{B}$ a belief profile. The KP-model [14] is a special instance of this model if we set for some $\phi \in \Phi$, $b_i(\phi) = 1$ for all $i \in [n]$.

For the remainder of the section let us fix a game $\mathbb{G} = (n, m, \mathbf{w}, \mathbf{B})$. A *pure strategy* for a user $i \in [n]$ is the selection of some link $\ell \in [m]$. A *pure strategies profile* is an n-tuple $\langle \ell_1, \ell_2, \ldots, \ell_n \rangle \in [m]^n$ of pure actions, one for each user, where ℓ_i is the selection of user $i \in [n]$. A *mixed strategy* for a user $i \in [n]$ is a probability distribution $\Delta([m])$ over pure strategies, that is, over the set of links. We denote the probability assigned by user $i \in [n]$ to link $\ell \in [m]$ by p^ℓ_i. A *mixed strategies profile* is an $n \times m$ probability matrix $\mathbf{P}$, where $\mathbf{P}[i, \ell] = p^\ell_i$ is the probability that user i chooses link ℓ and $\sum_{\ell=1}^m p^\ell_i = 1$, for all $i \in [n]$. A mixed strategies profile is said to be *fully mixed* if $p^\ell_i > 0$, for all $i \in [n]$ and $\ell \in [m]$.

For a pure strategies profile $\boldsymbol{\sigma} = \langle \ell_1, \ell_2, \ldots, \ell_n \rangle$, the *latency cost* $\lambda_{i,\phi}(\boldsymbol{\sigma})$ of user $i \in [n]$ in state ϕ is $\dfrac{\sum_{k:\ell_k=\ell_i} w_k}{c^{\ell_i}_\phi}$. On the other hand, the *expected latency cost* $\lambda_{i,b_i}(\boldsymbol{\sigma})$ over all states of user $i \in [n]$ with belief b_i is $\displaystyle\sum_{\phi \in \Phi} b_i(\phi) \cdot \lambda_{i,\phi}(\boldsymbol{\sigma})$.

For a mixed strategies profile $\mathbf{P}$, denote W^ℓ the *expected traffic* on link $\ell \in [m]$, $W^\ell =$

$\sum_{i=1}^{n} p_i^\ell \cdot w_i$. Denote **W** as the $m \times 1$ matrix containing the expected traffics on each link. Furthermore, the *expected latency cost* for user $i \in [n]$ with belief b_i on link $\ell \in [m]$, denoted by $\lambda_{i,b_i}^\ell(\mathbf{P})$, is the expectation over all possible states and over all random choices of the remaining users, of the latency cost for user i when its traffic is assigned to ℓ. Thus,

$$\lambda_{i,b_i}^\ell(\mathbf{P}) = \sum_{\phi\in\Phi} b_i(\phi)\cdot\frac{w_i + \sum_{k=1,k\neq i}^{n} p_k^\ell w_k}{c_\phi^\ell} = \sum_{\phi\in\Phi} b_i(\phi)\cdot\frac{(1-p_i^\ell)w_i + W^\ell}{c_\phi^\ell}\ .$$

For user $i \in [n]$, with belief b_i, the *minimum expected latency cost* $\lambda_{i,b_i}(\mathbf{P})$ is the minimum, over all links $\ell \in [m]$, of the expected latency cost for user i of belief b_i on ℓ:

$$\lambda_{i,b_i}(\mathbf{P}) = \min_{\ell\in[m]} \lambda_{i,b_i}^\ell(\mathbf{P}) \tag{1}$$

When **P** is inferred from the context we simply write λ_{i,b_i}^ℓ, λ_{i,b_i}.

By setting $c_i^\ell = \frac{1}{\sum_{\phi\in\Phi}\frac{b_i(\phi)}{c_\phi^\ell}}$ the expected latency cost of user i can be written as

$$\lambda_{i,b_i}^\ell = \frac{(1-p_i^\ell)w_i + W^\ell}{c_i^\ell}. \tag{2}$$

Observe that the resulting payoff function does not display the bayesian nature of our game. In fact, by embedding this information in the parameter c_i^ℓ, our game is *recasted* as a complete information routing game with *user-specific* latency functions. To implement this transformation, one needs to compute the pseudo-capacities c_i^ℓ for every user $i \in [n]$ and every link $\ell \in [m]$. Therefore, the transformation cost depends on the size of set Φ. If $|\Phi| \in$ O(1) then the transformation can be achieved in time O(n). On the other hand, if we assume that each link i may assume values from a set C_i of constant size, then $|\Phi| = |C_1|\cdot\ldots\cdot|C_m| \in$ O(2^m). Although we do not take a specific design decision on this issue, we point out that the size of set Φ is important in determining the time complexity of our algorithms for computing pure Nash equilibria presented in the next section.

Finally, note that by solving for p_i^ℓ, in the above equation for λ_{i,b_i}^ℓ, we have that for every user $i \in [n]$ and link $\ell \in [m]$

$$p_i^\ell = \frac{W^\ell + w_i - c_i^\ell\lambda_{i,b_i}^\ell}{w_i}. \tag{3}$$

The notion of a Nash equilibrium [24, 25] is defined for our model in the usual way. Specifically, a probability matrix $\mathbf{P}$ is a *Nash equilibrium* (often abbreviated as *NE*), if, for all users $i \in [n]$ and for all links $\ell \in [m]$:

$$\lambda_{i,b_i}^\ell \begin{cases} = & \lambda_{i,b_i}, \quad \text{if } p_i^\ell > 0 \\ \geq & \lambda_{i,b_i}, \quad \text{if } p_i^\ell = 0 \end{cases} \tag{4}$$

Thus, each user assigns its traffic with positive probability only to links for which its expected latency cost is minimized. This implies that there is no incentive for a user to unilaterally deviate from its strategy to improve its expected latency cost. We refer to probabilities in a Nash equilibrium as *Nash probabilities*.

Associated with a routing game $\mathbb{G}$ and a pure Nash equilibrium **P** is the *Social Cost* denoted by SC $(\mathbb{G}, \mathbf{P})$. Since every user's belief for the capacities of the network differs, there is no objective value for the latency of a link or for the exact congestion of the network. Thus, we are forced to *depart from the standard definition of the social cost* employed in the literature (the expected maximum congestion), and we consider a social cost definition that takes into account the subjective user beliefs; the sum of their individual cost:

$$\mathrm{SC}\left(\mathbb{G}, \mathbf{P}\right) = \sum_{i=1}^{n} \lambda_{i,b_i}\left(\mathbf{P}\right).$$

Analogously, we define the *Social Optimum*, or simply the *Optimum*, associated with a routing game $\mathbb{G} = (n, m, \mathbf{w}, \mathbf{B})$, denoted by $\mathrm{OPT}\left(\mathbb{G}\right)$, as the minimum over all pure assignments of the sum of their individual cost: $\mathrm{OPT}\left(\mathbb{G}\right) = \min_{\sigma \in [m]^n} \sum_{i=1}^{n} \lambda_{i,b_i}(\boldsymbol{\sigma})$.

The definition of *Coordination Ratio* (Price of Anarchy) for our model follows naturally:

$$\mathrm{CR} = \max_{\mathbb{G},\mathbf{P}} \frac{\mathrm{SC}(\mathbb{G}, \mathbf{P})}{\mathrm{OPT}\left(\mathbb{G}\right)}.$$

3. Pure Nash Equilibria

In this section we consider the existence of pure Nash equilibria for our model. It is well known ([26, 4]) that any unweighted congestion game has a pure Nash equilibium and, in the case of the KP-model, they can be efficiently computed [7]. In contrast, in [22] it is shown that weighted congestion games with user-specific functions do not always possess a pure NE by a counter-example using three users and three resources (links). Further, in [10], pure NE are shown to exist for unweighted games with user-specific linear latency functions using potential function arguments. For our model, a special case of the games of [22] and an extension of [14], we inherit the positive results of [22]. However, the counter-example of [22] is not valid: we show that for games with three users pure NE always exist. In addition, we present polynomial-time algorithms for computing pure NE for a number of special cases and we conjecture that pure NE exist in the general case.

3.1. *Polynomial-Time Algorithms for Special Cases*

3.1.1. *The case of $m = 2$ links*

First we consider the case of an arbitrary number of users and 2 links. Our algorithm solves the more general problem of finding a pure NE for games where the links have some initial traffic $\langle t^1, t^2\rangle$, where t^j is the initial traffic of link $j \in [2]$. We begin with a useful definition.

Definition 3.1. Consider a game $\mathbb{G} = (n, 2, \mathbf{w}, \mathbf{B})$ with initial traffic $\mathbf{t} = \langle t^1, t^2\rangle$. We define the *tolerance of user i for link j* as the value α_i^j which satisfies

$$\frac{t^j + \alpha_i^j}{c_i^j} = \frac{t^{\diamond j} + W - \alpha_i^j + w_i}{c_i^{\diamond j}}$$

where $W = \sum_{i\in[n]} w_i$ and $\diamond z = 3 - z$.

Thus, given a two-link game with an associated load W to be assigned on the two links, the tolerance of user i for a link j, α_i^j, is the maximum fragment of the load W the user can tolerate on link j while routing its traffic on it. This implies that, for a pure strategy profile $\langle \ell_1, \ldots, \ell_n\rangle$, if $\ell_i = j$ and link j has load $\leq \alpha_i^j$ (and consequently link $\diamond j$ has load $\geq W - \alpha_i^j$), user i has no incentive to change its strategy. We proof this as follows:

Lemma 3.1. *Consider a pure strategy profile $\langle \ell_1, \ldots, \ell_n\rangle$ in the game $\mathbb{G} = (n, 2, \boldsymbol{w}, \boldsymbol{B})$ with initial traffic $\mathbf{t} = \langle t^1, t^2\rangle$ and suppose $\ell_1 = 1$. Then, user 1 satisfies the NE condition* $\dfrac{t^1 + \sum_{\ell_i=1} w_i}{c_1^1} \leq \dfrac{t^2 + \sum_{\ell_i=2} w_i + w_1}{c_1^2}$ *if and only if* $\sum_{\ell_i=1} w_i \leq \alpha_1^1$.

Proof. First suppose that $\sum_{\ell_i=1} w_i \leq \alpha_1^1$. We have

$$\frac{t^1+\sum_{\ell_i=1} w_i}{c_1^1} \leq \frac{t^1+\alpha_1^1}{c_1^1} = \frac{t^2+\sum_{i\in[n]} w_i - \alpha_1^1 + w_1}{c_1^2}$$
$$\leq \frac{t^2+\sum_{i\in[n]} w_i - \sum_{\ell_i=1} w_i + w_1}{c_1^2} = \frac{t^2+\sum_{\ell_i=2} w_i + w_1}{c_1^2}$$

as required. To prove the other way round, suppose $\sum_{\ell_i=1} w_i > \alpha_1^1$. We have

$$\frac{t^1+\sum_{\ell_i=1} w_i}{c_1^1} > \frac{t^1+\alpha_1^1}{c_1^1} = \frac{t^2+\sum_{i\in[n]} w_i - \alpha_1^1 + w_1}{c_1^2}$$
$$> \frac{t^2+\sum_{i\in[n]} w_i - \sum_{\ell_i=1} w_i + w_1}{c_1^2} = \frac{t^2+\sum_{\ell_i=2} w_i + w_1}{c_1^2}$$

which completes the proof. □

Algorithm $A_{\mathrm{twolinks}}(\mathbb{G}, \mathbf{t})$

INPUT: A game $\mathbb{G} = (n, 2, \mathbf{w}, \mathbf{B})$ and a traffic vector **t** = $\langle t^1, t^2\rangle$
OUTPUT: A pure NE $\boldsymbol{\sigma}$

(1) Let $\boldsymbol{\sigma} = \langle 0, \ldots, 0\rangle$, $U = [n]$, and $W = \sum_{k\in[n]} w_k$

(2) Compute c_i^ℓ for every user $i \in [n]$ and every link $\ell \in [2]$

(3) while $U \neq \emptyset$ do:

   (a) For every user $u \in U$ and every link $j \in [2]$:
   Compute $\alpha_u^j = \frac{c_u^1 c_u^2}{c_u^1 + c_u^2}\left(\frac{t^{\diamond j}+W+w_u}{c_u^{\diamond j}} - \frac{t^j}{c_u^j}\right)$

   (b) Choose $k \in U$ and $\ell_k \in [2]$ such that $\alpha_k^{\ell_k} = \max_{u\in U, j\in[2]} \alpha_u^j$

   (c) Set $\boldsymbol{\sigma}[k] = \ell_k$, $U = U - \{k\}$, $W = W - w_k$ and $t^{\ell_k} = t^{\ell_k} + w_k$

(4) Return $\boldsymbol{\sigma}$

Fig. 1. Algorithm A_{twolinks}

Figure 1 presents our algorithm which, given a game and an initial traffic vector, computes a pure NE $\boldsymbol{\sigma}$. It begins by initializing $\boldsymbol{\sigma}$ to $\langle 0, \ldots, 0\rangle$ signifying that no user has been assigned to a link as yet, and it behaves greedily by selecting the user, k, with the highest tolerance over the two links, and assigning k on the specific link, ℓ_k. It then proceeds to construct an assignment for the remaining users in the same network, but where the initial load on link ℓ_k is increased by w_k. Thus, the algorithm decides the strategies of the players sequentially and irrevocably. We proceed to prove the correctness of the algorithm.

Theorem 1. For any game $\mathbb{G} = (n, 2, \mathbf{w}, \mathbf{B})$, the algorithm A_{twolinks} computes a pure Nash equilibrium in time $O(n^2 + n \cdot |\Phi|)$.

Proof. We prove the correctness of the algorithm by induction on n. Clearly, for $n = 1$ the claim holds. Assume that the claim holds for any game with $n = \nu$. Consider the execution of the algorithm with input a game $\mathbb{G} = (n, 2, \mathbf{w}, \mathbf{B})$ with $n = \nu + 1$, and an initial traffic vector $\mathbf{t}$. The algorithm initially selects the user k with the highest tolerance over both links, i.e. with $\alpha_k^{\ell_k} = \max_{i\in[n],j\in[2]} \alpha_i^j$. Without loss of generality, let us assume that $\ell_k = 1$. Consequently, the algorithm proceeds to compute a strategy profile $\boldsymbol{\sigma}'$ for the ν-user game $\mathbb{G}'$ of the remaining users with initial vector $\mathbf{t}' = \langle t^1 + w_k, t^2\rangle$ and return the strategy profile $\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}[i] = \ell_k$ for $i = k$ and $\boldsymbol{\sigma}[i] = \boldsymbol{\sigma}'[i]$ otherwise. By the induction hypothesis $\boldsymbol{\sigma}'$ is a NE for the ν-user game $\mathbb{G}'$ and initial traffic vector $\mathbf{t}'$. We have to show that $\boldsymbol{\sigma}$ is also a pure NE for the initial game $\mathbb{G}$ and initial vector $\mathbf{t}$. First note that all users $i \neq k$ satisfy the NE condition for game $\mathbb{G}$ and initial vector $\mathbf{t}$. It remains to show that this assignment is also acceptable for user k. Let N^1, N^2 be the sets of users on links 1 and 2, respectively, in strategy profile $\boldsymbol{\sigma}'$. Two cases exist. If $N^1 = \emptyset$, then the claim follows trivially by observing that the tolerance of user k for link 1 within game $\mathbb{G}$ must be such that $a_k^1 \geq w_k$, and by Lemma 3.1. On the other hand, if $N^1 \neq \emptyset$ and $j \in N^1$

$$\frac{t^1 + w_k + \sum_{u\in N^1} w_u}{c_j^1} \leq \frac{t^2 + \sum_{u\in N^2} w_u + w_j}{c_j^2},$$

which implies, by Lemma 3.1, that the tolerance of user j for link 1 within game $\mathbb{G}$ satisfies $\alpha_j^1 \geq \sum_{u\in N^1} w_u + w_k$. Since $\alpha_k^1 \geq \alpha_j^1$, by Lemma 3.1,

$$\frac{t^1 + w_k + \sum_{u\in N^1} w_u}{c_k^1} \leq \frac{t^2 + \sum_{u\in N^2} w_u + w_k}{c_k^2}$$

which completes the proof that $\boldsymbol{\sigma}$ is a pure Nash equilibrium.

The complexity of the algorithm is $O(n^2 + n\cdot|\Phi|)$, where $O(n^2)$ is the time required to for steps (1) and (3) and $O(n\cdot|\Phi|)$ the time required to compute the values c_i^ℓ (step (2)). □

3.1.2. *The case of symmetric users*

In this section we consider the case of *symmetric users*, that is, the case where all users have identical traffics, and we provide an $O(nm(n^2 + |\Phi|))$ algorithm for finding a pure NE for the model. Our algorithm, $A_{\text{symmetric}}$, shown in Figure 2, follows along the lines of the constructive proof of [22] for the same problem, for user-specific congestion games.

Algorithm $A_{\text{symmetric}}$

INPUT: A game $\mathbb{G} = (n, m, \langle w, \dots, w\rangle, \mathbf{B})$
OUTPUT: A pure NE $\boldsymbol{\sigma}$

(1) Let $|N^\ell| = 0$ for all $\ell \in [m]$ and $\boldsymbol{\sigma} = \langle 0, \dots, 0\rangle$

(2) Compute c_i^ℓ for every user $i \in [n]$ and every link $\ell \in [m]$

(3) For every user $i \in [n]$ do:

  (a) Let $\ell \in [m]$ be a link such that $\frac{|N^\ell|+1}{c_i^\ell} \leq \frac{|N^j|+1}{c_i^j}$, $\forall j \neq \ell$
  (b) Set $\boldsymbol{\sigma}[i] = \ell$ and $|N^\ell| = |N^\ell| + 1$
  (c) while there exists user k with $\ell_k = \ell$ and
    $\ell' \in [m]$ such that $(\frac{|N^\ell|}{c_k^\ell} > \frac{|N^{\ell'}|+1}{c_k^{\ell'}})$ do:
      Set $\boldsymbol{\sigma}[k] = \ell'$, $|N^\ell| = |N^\ell| - 1$,$|N^{\ell'}| = |N^{\ell'}| + 1$ and $\ell = \ell'$

Fig. 2. Algorithm $A_{\text{symmetric}}$

The contribution of our work is a simplification in the correctness proof. We will be using the following definitions and notations.

- Given a strategy profile $\boldsymbol{\sigma} = \langle \ell_1, \ldots, \ell_n \rangle$, $\ell_i \in [m]$, we define the *state induced by the strategy profile* as $\mathbf{s} = \langle N^1, \ldots, N^m \rangle$, where $N^j = \{i \in [n] \mid \ell_i = j\}$ is the set of users assigned to link j by $\boldsymbol{\sigma}$.
- A user is a *defecting user* in a state $\mathbf{s}$ if he does not satisfy the NE property in $\mathbf{s}$.
- We define the *game graph* of a game as the graph whose nodes are all possible states of the game and there exists an edge between states $\mathbf{s}$ and $\mathbf{s}'$ if $\mathbf{s} = \langle N^1, \ldots, N^m \rangle$ and $\mathbf{s}' = \langle N^1, \ldots, N^i - \{u\}, \ldots N^j \cup \{u\}, \ldots, N^m \rangle$, where u is a defecting user in $\mathbf{s}$ but not in $\mathbf{s}'$. We write $\mathbf{s} \xrightarrow{u} \mathbf{s}'$.

To prove the correctness of the algorithm we use the following lemma:

Lemma 3.2. *Consider state $\langle N^1, \ldots, N^m \rangle$ where $i \in N^j$ and suppose that user i satisfies the NE property, that is, for all $k \neq j$: $\frac{|N^j|}{c_i^j} \leq \frac{|N^k|+1}{c_i^k}$. Then, for any state $\langle L^1, \ldots, L^m \rangle$ satisfying $|L^k| \geq |N^k|$ for $k \neq j$ and $|L^j| \leq |N^j|$, user i continues to satisfy the NE property.*

Proof. We have for any $k \neq j$: $\frac{|L^j|}{c_i^j} \leq \frac{|N^j|}{c_i^j} \leq \frac{|N^k|+1}{c_i^k} \leq \frac{|L^k|+1}{c_i^k}$, as desired. □

We now proceed to prove the correctness of the algorithm.

Theorem 2. Given a game $\mathbb{G} = (n, m, \langle w, \ldots, w \rangle, \mathbf{B})$, the algorithm $A_{\text{symmetric}}$ computes a pure Nash equilibrium in time $O(n^3 m + nm|\Phi|)$.

Proof. Without loss of generality, we assume that $w_i = 1$, for all $i \in [n]$. We will prove the correctness of the algorithm by induction on n. Clearly, for $n = 1$ the claim holds. Assume that the claim holds for $n = \nu - 1$. We will show that it holds for $n = \nu$. By the induction hypothesis, at the end of the $(\nu-1)^{th}$ iteration, $\nu - 1$ users have been assigned on links and the assignment induced for this game, say $\boldsymbol{\sigma}_{\nu-1}$, is a Nash equilibrium. In the ν^{th} iteration, user ν assigns its traffic on link j which minimizes its latency (step 3(b)). Then one or more users may wish to deviate from link j to another link. To prove the claim we will show that $\boldsymbol{\sigma}_{\nu-1}$ can be transformed into a NE in $O(\nu)$ moves (step 3(c)).

Let $\mathbf{s_0} = \langle N^1, \ldots, N^m \rangle$ be the state induced by $\boldsymbol{\sigma}_{\nu-1}$. Suppose that, given this state, user P_ν chooses to route its traffic on link 1, giving rise to state $\mathbf{s_1} = \langle N^1 \cup \{\nu\}, N^2, \ldots, N^m \rangle$. Suppose that this placement results in a sequence of moves $\mathbf{s_1} \xrightarrow{u_2} \mathbf{s_2} \xrightarrow{u_3} \mathbf{s_3} \xrightarrow{u_4} \ldots$ where $\mathbf{s_i} = \langle N_i^1, N_i^2, \ldots, N_i^m \rangle$. We may prove that for all i, there exists j_i such that

(1) $|N_i^{j_i}| = |N^{j_i}| + 1$, and $|N_i^j| = |N^j|$, for $j \neq j_i$,
(2) the defecting user $u_{i+1} \in N_i^{j_i}$, and
(3) all $u \in N_i^j$, $j \neq j_i$, satisfy the NE criterion.

We prove this by induction on i. For the base case consider $i = 1$. Clearly, $\mathbf{s_1}$ satisfies property (1), with $j_1 = 1$. While all users in $\{\nu\} \cup N^2 \cup \ldots \cup N^m$ can be seen to continue to

Algorithm $A_{\text{uniform}}(\mathbb{G}, \mathbf{t})$

INPUT: A game $\mathbb{G} = (n, m, \mathbf{w}, \mathbf{B})$ and a traffic vector **t** = $\langle t^1, \dots, t^m \rangle$
OUTPUT: A pure NE $\boldsymbol{\sigma}$

(1) Compute $c_i = c_i^1$ ($= c_i^\ell$, for all ℓ) for every user $i \in [n]$

(2) Let $\boldsymbol{\sigma} = \langle 0, \dots, 0 \rangle$

(3) Sort the users in decreasing order of traffics

(4) For every user $i \in [n]$ do:

  (a) Let ℓ_k be a link such that $\frac{w_k + t^{\ell_k}}{c_k} \leq \frac{w_k + t^j}{c_k}, \forall j \neq \ell_k$
  (b) Set $\boldsymbol{\sigma}[k] = \ell_k$ and let $t^{\ell_k} = t^{\ell_k} + w_k$.

Fig. 3. Algorithm A_{uniform}

satisfy the NE criterion in this new state, it is possible that some user $u \in N_1^1$ is no longer satisfied, that is the defecting user, if one exists, is some $u \in N_1^1$.

Suppose now that the claim holds for $i = k$ and consider $i = k + 1$. We observe, by the induction hypothesis, that, if we pick j_{k+1} to be the new strategy of the defecting player u_{k+1}, $\mathbf{s_{k+1}}$ satisfies property (1). In addition, user u_{k+1} is satisfied in $\mathbf{s_{k+1}}$, and so are all users in N_{k+1}^q, $q \neq j_{k+1}$, which completes the induction.

Now consider users $u_1 = \nu, u_2, \dots$, in the execution above. We may see that, since user u_i satisfies the NE criterion in state $\mathbf{s_i}$, by Lemma 3.2, he will continue to satisfy it in every subsequent step. Thus, user u_i, will not defect in any of the moves following state $\mathbf{s_i}$, which implies that any user may defect at most once. In other words, the execution is finite and will converge to a NE in at most ν steps. This completes the proof that $\boldsymbol{\sigma}_\nu$ is a pure NE.

We may see that the complexity of the algorithm is in $O(n^3 m + nm|\Phi|)$. In particular, we note that in the i^{th} iteration of the algorithm there may be at most $i - 1$ defecting users, which amounts to a total of $O(n^2)$ defecting steps. The candidate users for defection may be identified in a single pass over all users, proceeding Step 3(c), in time $O(nm)$. As before the term $nm|\Phi|$ is obtained due to the calculation of the values c_i^ℓ. □

3.1.3. *The case of uniform user beliefs*

We now turn to the *model of uniform user beliefs*, that is, games where each user believes all links to have equal capacity ($\forall i \in [n],\ \exists c_i : \forall \ell \in [m]\ c_i^\ell = c_i$). We present an algorithm that computes a pure NE for the model in the case where the links have some initial traffic $\mathbf{t} = \langle t^1, \dots, t^m \rangle$, where t^j is the initial traffic of link $j \in [m]$. The algorithm, A_{uniform}, presented in Figure 3, is essentially the algorithm of [7] (which in turn can be viewed as a variant of Graham's Longest Processing Time (LPT) algorithm [11]) proceeded by a step that calculates the values of the c_i^ℓ. Essentially, the algorithm constructs a pure NE in a greedy fashion by processing the users in decreasing order of their traffics, and, for each user k, it assigns the user on its preferred link ℓ_k and proceeds with the remaining users in the network where the initial load of link ℓ_k has increased by w_k.

We may prove the following for our algorithm.

Theorem 3. Given a game $\mathbb{G} = (n, m, \mathbf{w}, \mathbf{B})$ under the model of uniform user beliefs, the

Algorithm A_{uniform} computes a pure Nash equilibrium in time $O(n(\log n + m + |\Phi|))$.

Proof. The correctness of our algorithm can be proved by induction on n similarly to [11]. It is easy to see that the complexity of the algorithm is in $O(n(\log n + m + |\Phi|))$. □

3.1.4. *The case of* $n = 3$

Here we show that any game with three users has a pure NE. Recall that the game graph of a game is the graph whose nodes are all possible states of the game and there exists an edge between two states if they differ only with respect to a single defecting player. The proof employs the notion of a *best-response cycle* which is a cycle in the game graph of a game. Specifically, it has been shown in [23] that, if the game graph of a game has no best-response cycles, then the game possesses at least one pure NE. Based on this, our proof establishes in an exhaustive manner that any game with three users possesses no best-response cycles, which implies that a pure NE exists. We begin with a lemma:

Lemma 3.3. *For any game* $\mathbb{G} = (2, m, \boldsymbol{w}, \boldsymbol{B})$ *and initial traffic* $\mathbf{t}$ *a pure NE exists.*

Proof. We may construct a pure NE for the game in at most three steps as follows. Place player P_1 on his preferred link, j_1, of the initial network. Place player P_2 on his preferred link, j_2, on the network resulting by the above placement. If $j_1 \neq j_2$, $\langle j_1, j_2 \rangle$ is trivially a NE. On the other hand, if $j_1 = j_2$ and P_1 is a defecting player, the state resulting from the defecting move of P_1 is trivially a NE. □

Theorem 4. For any game $\mathbb{G} = (3, m, \mathbf{w}, \mathbf{B})$, the game graph of $\mathbb{G}$ contains no best-response cycles, and hence $\mathbb{G}$ has a pure NE.

Proof. For simplicity, in this proof, we use $\boldsymbol{\sigma} \xrightarrow{u} \boldsymbol{\sigma}'$ to denote that strategies $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ induce states $\mathbf{s}$ and $\mathbf{s}'$, respectively, where $\mathbf{s} \xrightarrow{u} \mathbf{s}'$. Consider a pure NE, $\boldsymbol{\sigma}$, for the subgame of $\mathbb{G}$, $\mathbb{G}' = (2, m, \{w_1, w_2\}, \mathbf{B})$. Two cases exist:

• Suppose that the strategies of players P_1 and P_2 in $\boldsymbol{\sigma}$ are distinct. Without loss of generality we may assume that $\boldsymbol{\sigma} = \langle 1, 2 \rangle$. Let us place player P_3 on his preferred link, j. If $j = 3$, then $\boldsymbol{\sigma}' = \langle 1, 2, 3 \rangle$ is clearly a NE and the proof follows. Otherwise, suppose $j = 1$ and that a best-response cycle exists:

$$\langle 1,2,1 \rangle \xrightarrow{1} \langle 2,2,1 \rangle \xrightarrow{2} \langle 2,1,1 \rangle \xrightarrow{3} \langle 2,1,2 \rangle \xrightarrow{1} \langle 1,1,2 \rangle \xrightarrow{2} \langle 1,2,2 \rangle \xrightarrow{3} \langle 1,2,1 \rangle$$

Note that this is the only possible best-response cycle from the initial state considered. Any valid, alternative step from the cycle's states leads to a NE. In particular, from each state there can be at most one defecting player, which is the player on whose link a move has been made, and, if this player moves to link 3, the resulting state is a NE. So let us consider the above cycle. By the moves of player P_1 we obtain the following inequalities:

$$\frac{w_1 + w_3}{c_1^1} > \frac{w_1 + w_2}{c_1^2} \qquad \text{and} \qquad \frac{w_1 + w_3}{c_1^2} > \frac{w_1 + w_2}{c_1^1}$$

By algebraic manipulation of the above we conclude that $w_3 > w_2$. Similarly, by considering the moves of players P_2 and P_3, we obtain that $w_1 > w_3$ and $w_2 > w_1$, respectively. The three inequalities give a contradiction and the claim follows.

• Suppose that the two first players play on the same link. Without loss of generality, suppose that $\boldsymbol{\sigma} = \langle 1, 1 \rangle$. Let us place player P_3 on his preferred link, j. If $j \neq 1$, then $\boldsymbol{\sigma}' = \langle 1, 1, j \rangle$ is clearly a NE and the proof follows. Otherwise, suppose $j = 1$. This implies that for any strategy $\langle j_1, j_2, 1 \rangle$ player P_3 does not wish to change his strategy. Consequently, this scenario reduces to the problem of finding a NE for the game $\mathbb{G}' = (2, m, \{w_1, w_2\}, \mathbf{B})$ with initial traffic $\mathbf{t} = \langle 0, w_3, 0, \ldots \rangle$. By Lemma 3.3 the result follows. □

We remark that Mavronicolas et al. [20, Corollary 3] extended the above result for weighted congestion games with player-specific constants and 3 players on parallel links.

3.2. *Existence of Pure Nash Equilibria (Conjecture)*

The existence of pure Nash equilibria for this model in the general case remains open. Work for answering this question has been carried out in various directions. It is easy to show that our game is not an *exact potential game* [23] and therefore it does not admit an exact potential function. Further, our game is not an *ordinal potential game*, since it has been shown by Gairing, Monien and Tiemann [10] that the state space of an instance of the game contains a cycle. Therefore, potential functions [23], a popular and powerful method for proving NE existence, cannot be used for our model. Our efforts to apply graph-theoretic methods and inductive arguments have also not been successful for proving pure NE existence in our model. The arguments end up failing mainly due to the arbitrary relation between the different user beliefs on the capacity of the network links (unlike the special cases presented before where beliefs are related or additional information is present). Naturally, and given the non-existence result on weighted congestion games with user specific payoff-functions [22], we attempted to disprove the existence of NE in our model. Typically, simple counter-examples considering a small number of resources (links) and users are used for such purposes (for example, in [22], the counter-example involves 3 users and 3 resources). This appears not to be the case in our model: we have seen that for games with three users (and arbitrary number of links) pure NE always exist; also simulations ran on numerous instances of the game (dealing with small number of users and links) suggest the existence of pure NE. Given the lack of a simple counter-example, the polynomial-time algorithms for special cases, and our intuition, we conjecture that:

Conjecture. *For any game $\mathbb{G} = (n, m, \boldsymbol{w}, \boldsymbol{B})$ there is at least one pure Nash equilibrium.*

4. Fully Mixed Nash Equilibria

4.1. *Existence and Uniqueness of Fully Mixed Nash Equilibria.*

In this section we study fully mixed Nash equilibria for our model and we compute the probabilities that yield such equilibria. Furthermore, we illustrate that if a fully-mixed NE exists in some game, it is unique and it maximizes the social cost.

By employing techniques similar to [21] we may obtain the follows lemmas:

Lemma 4.1. *Consider a game $\mathbb{G}$ and an associated fully mixed Nash equilibrium $\boldsymbol{P}$. Then, for all users $i \in [n]$ and links $\ell \in [m]$ we have*

$$p_i^\ell = 1 - \frac{(m-1)c_i^\ell}{\sum_{j=1}^m c_i^j} - \frac{1}{n-1}\frac{1}{w_i}\left[\left(1 - \sum_{k=1}^n \frac{c_k^\ell}{\sum_{j=1}^m c_k^j} + \frac{(n-1)c_i^\ell}{\sum_{j=1}^m c_i^j}\right)\sum_{k=1}^n w_k \right. \\ \left. -(m-1)\sum_{k=1}^n \frac{c_k^\ell}{\sum_{j=1}^m c_k^j} w_k\right].$$

Lemma 4.2. *If for every $i \in [n]$ and every $\ell \in [m]$*

$$p_i^\ell = 1 - \frac{(m-1)c_i^\ell}{\sum_{j=1}^m c_i^j} - \frac{1}{n-1}\frac{1}{w_i}\left[\left(1 - \sum_{k=1}^n \frac{c_k^\ell}{\sum_{j=1}^m c_k^j} + \frac{(n-1)c_i^\ell}{\sum_{j=1}^m c_i^j}\right)\sum_{k=1}^n w_k \right. \\ \left. - (m-1)\sum_{k=1}^n \frac{c_k^\ell}{\sum_{j=1}^m c_k^j} w_k\right] \in (0,1),$$

then these probabilities constitute a fully mixed Nash equilibrium.

By Lemma 4.1 and Lemma 4.2 we establish:

Theorem 5. (Existence and Uniqueness of Nash Equilibria) Consider the fully mixed case. Then for every user $i \in [n]$ and every link $\ell \in [m]$,

$$p_i^\ell = 1 - \frac{(m-1)c_i^\ell}{\sum_{j=1}^m c_i^j} - \frac{1}{n-1}\frac{1}{w_i}\left[\left(1 - \sum_{k=1}^n \frac{c_k^\ell}{\sum_{j=1}^m c_k^j} + \frac{(n-1)c_i^\ell}{\sum_{j=1}^m c_i^j}\right)\sum_{k=1}^n w_k \right. \\ \left. - (m-1)\sum_{k=1}^n \frac{c_k^\ell}{\sum_{j=1}^m c_k^j} w_k\right] \in (0,1)$$

if and only if there exist a Nash equilibrium which must be unique and the p_i^ℓ's are its associated Nash probabilities.

Theorem 5 implies the following.

Corollary 4.1. *The fully mixed Nash equilibrium when it exists can be calculated in $O(nm + nm|\Phi|) = O(nm|\Phi|)$ time, where $nm|\Phi|$ is the time needed for all users $i \in [n]$ to compute each c_i^ℓ for every $\ell \in [m]$.*

From Theorem 5, Lemma 4.1 and algebraic manipulation of the NE probabilities we get the following result for the model of uniform user beliefs ($\forall i \in [n],\ \exists c_i : \forall \ell \in [m]\ c_i^\ell = c_i$).

Theorem 6. In the model of uniform user beliefs, for any game $\mathbb{G}$ there exists a fully mixed Nash equilibrium. Additionally, for any user $i \in [n]$ and any link $\ell \in [m]$, $p_i^\ell = \frac{1}{m}$.

4.2. *Worst Case Equilibrium and Price of Anarchy*

We show that the fully mixed Nash equilibrium maximizes the social cost. Since the social cost is based on the individual costs of every user $i \in [n]$, we first extend a known relation previously shown in other related models [17, 9].

Lemma 4.3. *Take any game $\mathbb{G}$, a (mixed) Nash equilibrium* ***P*** *and the fully mixed Nash equilibrium* ***F***. *Then for any user $i \in [n]$, $\lambda_{i,b_i}(\boldsymbol{P}) \leq \lambda_{i,b_i}(\boldsymbol{F})$*

Proof. Let p_k^ℓ and f_k^ℓ for every user $k \in [n]$ and for every link $\ell \in [m]$, be the probabilities of the mixed and fully mixed Nash equilibrium respectively. Then since $\sum_{\ell=1}^m \left(\sum_{k=1,k\neq i}^n p_k^\ell w_k\right) = \sum_{k=1,k\neq i}^n \left(w_k \sum_{\ell=1}^m p_k^\ell\right) = \sum_{k=1,k\neq i}^n w_k$ and $\sum_{\ell=1}^m \left(\sum_{k=1,k\neq i}^n f_k^\ell w_k\right) = \sum_{k=1,k\neq i}^n \left(w_k \sum_{\ell=1}^m f_k^\ell\right) = \sum_{k=1,k\neq i}^n w_k$ it follows that

$$\sum_{\ell=1}^{m}\left(\sum_{k=1,k\neq i}^{n} p_k^\ell w_k\right) = \sum_{\ell=1}^{m}\left(\sum_{k=1,k\neq i}^{n} f_k^\ell w_k\right). \tag{5}$$

Therefore, there exists a link $\ell_0 \in [m]$ such that $\sum_{k=1,k\neq i}^n p_k^{\ell_0} w_k \leq \sum_{k=1,k\neq i}^n f_k^{\ell_0} w_k$. By adding w_i on both sides and by dividing with the believed capacity $c_i^{\ell_0}$ of user i for link ℓ_0 we get that $\lambda_{i,b_i}^{\ell_0}(\mathbf{P}) \leq \lambda_{i,b_i}^{\ell_0}(\mathbf{F})$. By definition of $\lambda_{i,b_i}(\mathbf{P})$ (since $\lambda_{i,b_i}(\mathbf{P})$ is the minimum of all $\lambda_{i,b_i}^\ell(\mathbf{P})$) we get that $\lambda_{i,b_i}(\mathbf{P}) \leq \lambda_{i,b_i}^{\ell_0}(\mathbf{P}) \leq \lambda_{i,b_i}^{\ell_0}(\mathbf{F}) = \lambda_{i,b_i}(\mathbf{F})$ (since **F** is a Nash equilibrium) as needed. □

The following theorem follows from Lemma 4.3 and the definition of the social cost.

Theorem 7. The fully mixed Nash equilibrium maximizes the social cost $\mathrm{SC}(\mathbb{G}, \mathbf{F})$.

Theorems 6 and 7 lead to the following two theorems (the first for the model of uniform user beliefs and the second for the general case):

Theorem 8. Take any game $\mathbb{G}$ and any Nash equilibrium **P** in the model of uniform user beliefs, then

$$\frac{\mathrm{SC}(\mathbb{G}, \mathbf{P})}{\mathrm{OPT}(\mathbb{G})} \leq \left(\frac{c_{max}}{c_{min}}\right) \frac{m+n-1}{m},$$

where $c_{max} = \max\limits_{i\in[n],\ \ell\in[m]} c_i^\ell$, and $c_{min} = \min\limits_{i\in[n],\ \ell\in[m]} c_i^\ell$.

Proof. By Theorem 6, we have that a fully mixed Nash equilibrium exists for the model, with $p_i^\ell = \frac{1}{m}$ for every user $i \in [n]$ and link $\ell \in [m]$. Furthermore, by Theorem 7 we know that $\mathrm{SC}(\mathbb{G}, \mathbf{P}) \leq \mathrm{SC}(\mathbb{G}, \mathbf{F})$ where **F** is the fully mixed Nash equilibrium. Then,

$$\mathrm{SC}(\mathbb{G}, \mathbf{P}) \leq \mathrm{SC}(\mathbb{G}, \mathbf{F}) \leq \sum_{i=1}^{n} \frac{w_i + \sum_{k=1,k\neq i}^n \frac{1}{m} w_k}{c_{min}} = \frac{1}{c_{min}}\left(\frac{m+n-1}{m}\right)\sum_{k=1}^{n} w_k.$$

Since $\mathrm{OPT}(\mathbb{G})$ is the least possible maximum over all pure strategies of the individual cost of all users,

$$\mathrm{OPT}(\mathbb{G}) \geq \sum_{i=1}^{n} \frac{\sum_{k=1:\ell_k=\ell_i}^n w_k}{c_{max}} \geq \frac{1}{c_{max}} \sum_{k=1}^{n} w_k.$$

By combining the two inequalities the desired upper bound follows. □

Theorem 9. Take any game $\mathbb{G}$ and any Nash equilibrium **P**, then

$$\frac{\mathrm{SC}\left(\mathbb{G},\mathbf{P}\right)}{\mathrm{OPT}\left(\mathbb{G}\right)} \leq \frac{(c_{max})^2}{c_{min}} \frac{1}{\sum_{j=1}^{m} c_{min}^{j}} (m+n-1),$$

where $c_{max} = \max_{i\in[n],\ \ell\in[m]} c_i^\ell$, $c_{min} = \min_{i\in[n],\ \ell\in[m]} c_i^\ell$, and $c_{min}^\ell = \min_{i\in[n]} c_i^\ell,\ \ell \in\ [m]$.

Proof. Consider the values p_i^ℓ produced by Lemma 4.1. As it is shown in the proof of Lemma 4.2, these values, even if not in the range $(0,1)$, satisfy $\sum_{i\in[n]} p_i^\ell = 1$ and, when substituted in eq. (2) for $\lambda_{i,b}^\ell$, yield $\lambda_{i,b}^\ell = \lambda_{i,b}$. If additionally we let $\mathbf{F}$ be the matrix containing these values, i.e. $\mathbf{F}[i,\ell] = p_i^\ell$, for all $i\in[n], \ell\in[m]$, we may verify that all steps of Lemma 4.3 continue to hold, thus bounding $\lambda_{i,b_i}(\mathbf{P})$ by $\lambda_{i,b_i}(\mathbf{F})$ for any $i\in[n]$ and any Nash equilibrium $\mathbf{P}$.

Furthermore, since $\lambda_{i,b_i}\left(\mathbf{F}\right) = \lambda_{i,b_i}^\ell\left(\mathbf{F}\right) \geq \lambda_{i,b_i}^\ell\left(\mathbf{P}\right) \geq 0$ for all $i\in[n]$ and all $\ell\in[m]$,

$$\begin{aligned}\mathrm{SC}\left(\mathbb{G},\mathbf{F}\right) &= \sum_{i=1}^{n} \frac{w_i + \sum_{k=1,k\neq i}^{n} p_k^\ell w_k}{c_i^\ell} \leq \sum_{i=1}^{n} \frac{w_i + \sum_{k=1,k\neq i}^{n} p_k^\ell w_k}{c_{min}^\ell} \\ &= \frac{1}{c_{min}^\ell} \sum_{i=1}^{n} \left(w_i + \sum_{k=1,k\neq i}^{n} p_k^\ell w_k \right) = \frac{1}{c_{min}^\ell} \left(\sum_{i=1}^{n} w_i + (n-1) \sum_{i=1}^{n} p_i^\ell w_i \right).\end{aligned}$$

From equations (3), (4) and algebraic manipulation we get that for any link $\ell\in[m]$

$$\begin{aligned}\mathrm{W}^\ell &= \frac{1}{n-1} \sum_{i=1}^{n} \left((m-1) \frac{c_i^\ell}{\sum_{j=1}^{m} c_i^j} + \sum_{k=1}^{n} \frac{c_k^\ell}{\sum_{j=1}^{m} c_k^j} - 1 \right) w_i \\ &= \frac{1}{n-1} \left((m-1) \sum_{i=1}^{n} \frac{c_i^\ell}{\sum_{j=1}^{m} c_i^j} w_i + \sum_{k=1}^{n} \frac{c_k^\ell}{\sum_{j=1}^{m} c_k^j} \sum_{i=1}^{n} w_i - \sum_{i=1}^{n} w_i \right). \qquad (6)\end{aligned}$$

By substituting (6) on the above upper bound of the social cost we get

$$\begin{aligned}\mathrm{SC}\left(\mathbb{G},\mathbf{P}\right) &\leq \frac{1}{c_{min}^\ell} \left(\sum_{i=1}^{n} w_i + (m-1) \sum_{i=1}^{n} \frac{c_i^\ell}{\sum_{j=1}^{m} c_i^j} w_i + \sum_{k=1}^{n} \frac{c_k^\ell}{\sum_{j=1}^{m} c_k^j} \sum_{i=1}^{n} w_i - \sum_{i=1}^{n} w_i \right) \\ &\leq \frac{1}{c_{min}} \frac{c_{max}}{\sum_{j=1}^{m} c_{min}^j} (m+n-1) \sum_{i=1}^{n} w_i.\end{aligned}$$

On the other hand $\mathrm{OPT}\left(\mathbb{G}\right)$ is the least possible maximum over all pure strategies of the individual cost of all users, thus for any user $i\in[n]$,

$$\mathrm{OPT}\left(\mathbb{G}\right) = \min_{(\ell_1,\ell_2,\ldots,\ell_n)\in[m]^n} \sum_{i=1}^{n} \frac{\sum_{k=1:\ell_k=\ell_i}^{n} w_k}{c_i^{\ell_i}} \geq \min_{(\ell_1,\ell_2,\ldots,\ell_n)\in[m]^n} \sum_{i=1}^{n} \frac{\sum_{k=1:\ell_k=\ell_i}^{n} w_k}{c_{max}^{\ell_i}} \geq \frac{1}{c_{max}} \sum_{i=1}^{n} w_i.$$

By combining the two inequalities the desired upper bound is obtained. □

Remark 4.1. The above bounds presented for the price of anarchy are loose. In particular, a trivial lower bound for the price of anarchy for the model of uniform user beliefs is $\frac{m+n-1}{m}$. This is computed by considering the following setting: $c_{max} = c_{min} = c$, $n = m$ and $w_i = 1\ \forall i\in[n]$. Then, $\lambda_{i,b_i}\left(\mathbf{F}\right) = \frac{1}{c}\left(1+\frac{1}{m}(n-1)\right) = \frac{1}{cm}(m+n-1)$, for all $i\in[n]$. Hence, $\mathrm{SC}\left(\mathbb{G},\mathbf{P}\right) = \frac{1}{cm}\sum_{i=1}^{n}(m+n-1) = \frac{m+n-1}{cm}$. From the fact that $\mathrm{OPT}\left(\mathbb{G}\right) = \frac{1}{c}$ the claimed lower bound follows. This lower bound, for the case of $m=n$ is tight within a $\frac{c_{max}}{c_{min}}$ factor for the upper bound for the uniform user belief case. However, the general upper bound presented is much looser. Improving these bounds is a challenging task, which we leave for future work.

Remark 4.2. We note that in [10] a tight bound on the price of anarchy is presented that when applied to our model in the case of symmetric users (otherwise the social cost considered in [10] is different from ours) it improves linearly (w.r.t. n) the upper bound of Theorem 9. In particular, we get $\frac{\mathrm{SC}(\mathbb{G},\mathbf{P})}{\mathrm{OPT}(\mathbb{G})} \leq \frac{\Delta+2+\sqrt{\Delta(\Delta+4)}}{2}$, where $\Delta = \frac{c_{max}}{c_{min}}$.

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