Practically-Self-Stabilizing Virtual Synchrony*

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Abstract

The virtual synchrony abstraction was proven to be extremely useful for asynchronous, large-scale, message-passing distributed systems. Self-stabilizing systems can automatically regain consistency after the occurrence of transient faults.

We present the first practically-self-stabilizing virtual synchrony algorithm that uses a new counter algorithm that establishes an efficient practically unbounded counter, which in turn can be directly used for emulating a self-stabilizing Multiple-Writer Multiple-Reader (MWMR). Other self-stabilizing services include membership, multicast, and replicated state machine (RSM) emulation. As we base the latter on virtual synchrony, rather than consensus, the system can progress in more extreme asynchronous executions than consensus-based RSM emulations.

1 Introduction

Virtual Synchrony (VS) is an important property provided by several Group Communication Systems (GCSs) that has proved to be valuable in the scope of fault-tolerant distributed systems where communicating processors are organized in process groups with changing membership [5]. During the computation, groups change allowing an outside observer to track the history (and order) of the groups, as well as the messages exchanged within each group. The VS property guarantees that any two processors that both participate in two consecutive such groups, should deliver the same messages in their respective group. Systems that support the VS abstraction are designed to operate in the presence of fail-stop failures of a minority of the participants. Such a design fits large computer clusters, data-centers and cloud computing, where at any given time some of the processing units are non-operational. Systems that cannot tolerate such failures degrade their functionality and availability to the degree of useless systems.

Group communication systems that realize the VS abstraction provide services, such as group membership and reliable group multicast. The group membership service is responsible for providing the current group view of the recently live and connected group members, i.e., a processor set and a unique view identifier, which is a sequence number of the view installation. The reliable group multicast allows the service clients to exchange messages with the group members as if it was a single communication endpoint with a single network address and to which messages are delivered in an atomic fashion, thus any message is either delivered to all recently live and connected group members prior to the next message, or is not delivered to any member. The challenges related to VS consist of the need to maintain atomic message delivery in the presence of asynchrony and crash failures. VS facilitates the implementation of a replicated state machine [5] that is more efficient than classical consensus-based implementations that start every multicast round with an agreement on the set of recently live and connected processors. It is also usually easier to implement [5].

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**Transient faults** Transient violations of design assumptions can lead a system to an arbitrary state. For example, the assumption that error detection ensures the arrival of correct messages and the discarding of corrupted messages, might be violated since error detection is a probabilistic mechanism that may not detect a corrupt message. As a result, the message can be regarded as legitimate, driving the system to an arbitrary state after which, availability and functionality may be damaged forever, requiring human intervention. In the presence of transient faults, large multicomputer systems providing VS-based services can prove hard to manage and control. One key problem, not restricted to virtually synchronous systems, is catering for counters (such as view identifiers) reaching an arbitrary value. How can we deal with the fact that transient faults may force counters to wrap around to the zero value and violate important system assumptions and correctness invariants, such as the ordering of events? A self-stabilizing algorithm [13] can automatically recover from such unexpected failures, possibly as part of after-disaster recovery or even after benign temporal violations of the assumptions made in the design of the system. To the best of our knowledge, no *stabilizing virtual synchrony* solution exists. We tackle this issue in our work.

**Practically-self-stabilization** A relatively new self-stabilization paradigm is *practically-self-stabilization* [1, 7, 16]. Consider an asynchronous system with bounded memory and data link capacity in which corrupt pieces of data (stale information) exist due to a transient fault. (Recall that transient faults can result in the appearance of corrupted information, which the system tends to spread and thus reach an arbitrary state.) These corrupted data may appear *unexpectedly* at any processor as they lie in communication links, or may (indefinitely) remain “hidden” in some processor’s local memory until they are added to the communication links as a response to some other processor’s input. Whilst these pieces of corrupted data are bounded in number due to the boundedness of the links and local memory, they can eventually force the system to lose its safety guarantees. Such corrupt information may repeatedly drive the system to an undesired state of non-functionality. This is true for all systems and self-stabilizing systems are required to eradicate the presence of all corrupted information. In fact, whenever they appear, the self-stabilizing system is required to regain consistency and in some sense stabilize. One can consider this as an adversary with a limited number of chances to interrupt the system, but only itself knows *when* it will do this.

In this perspective, self-stabilization, as it was proposed by Dijkstra [12], is not the best design criteria for asynchronous systems for which we cannot specifically define *when* stabilization is expected to finish (in some metric like asynchronous cycles, for example). The newer criterion of practically-stabilizing systems is closely related to pseudo-self-stabilizing systems [9], as we explain next. Burns, Gouda and Miller [9] deal with the above challenge by proposing the design criteria of *pseudo-self-stabilization*, which merely bounds the number of possible safety violations. Namely, their approach is to abandon Dijkstra’s seminal proposal [12] to bound the period in which such violations occur (using some metric like asynchronous cycles). We consider a variation on the design criteria for pseudo-self-stabilization systems that can address additional challenges that appear when implementing a decentralized shared counter that uses a constant number of bits.

Self-stabilizing systems can face an additional challenge due to the fact that a single transient fault can cause the counter to store its maximum possible value and still (it is often the case that) the system needs to be able to increment the counter for an unbounded number of times. The challenge becomes greater when there is no elegant way to show that the system can always maintain an order among the different values of the counter by, say, wrapping to zero in such integer overflow events. Arora, Kulkarni and Demirbas [2] overcome the challenge of integer overflow by using non-blocking resets in the absence of faults described [2]. In case faults occur, the system recovery requires a blocking operation, which performs a distributed global reset. This work considers a design criteria for message passing systems that perform in a wait-free manner also when recovering from transient faults.

Note that, from the theoretical point of view, systems that take an extraordinary large number of steps (that accedes the counter maximum value, or even an infinite number of steps) are bound to violate any ordering constraints. This is because of the asynchronous nature of the studied system, which could arbitrarily delay a node from taking steps or defer the arrival of a message until such violations occur
after, say, a counter wraps around to zero. Having practical systems in mind, we consider systems for which the number of sequential steps that they can take throughout their lifetime is not greater than an integer that can be represented using a constant number of bits. For example, Dolev, Kat and Schiller [16] assume that counting from zero to $2^{64} - 1$ using sequential steps is not possible in any practical system and thus consider only a practically infinite period, of $2^{64}$ sequential steps, that the system takes when demonstrating that safety is not violated. The design criteria of practically-self-stabilizing systems [1, 7, 22] requires that there is a bounded number of possible safety violations during any practically infinite period of the system execution. For such (message passing) systems, we provide a decentralized shared counter algorithm that performs in a wait-free manner also when recovering from transient faults.

Contributions We present the first practically-self-stabilizing (or practically-stabilizing) virtual synchrony solution. Specifically:

• We provide a practically-self-stabilizing counter algorithm using bounded memory and communication bandwidth, where many writers can increment the counter for an unbounded number of times in the presence of processor crashes and unbounded communication delays. Our counter algorithm is modular with a simple interface for increasing and reading the counter, as well as providing the identifier of the processor that has incremented it.

• At the heart of our counter algorithm is the underlying labeling algorithm that extends the label scheme of Alon et al. [1] to support multiple writers, whilst the algorithm specifies how the processors exchange their label information in the asynchronous system and how they maintain proper label bookkeeping so as to “discover” the greatest label and discard all obsolete ones.

• An immediate application of our counter algorithm is a practically-self-stabilizing MWMR register emulation.

• The practically-self-stabilizing counter algorithm, together with implementations of a practically-self-stabilizing reliable multicast service and membership service that we propose, are composed to yield a practically-self-stabilizing coordinator-based Virtual Synchrony solution.

• Our Virtual Synchrony solution yields a practically-self-stabilizing State Machine Replication (SMR) implementation. As this implementation is based on virtual synchrony rather than consensus, the system can progress in more extreme asynchronous executions than consensus-based SMR implementations.

Related Work Leslie Lamport was the first to introduce SMR, presenting it as an example in [20]. Schneider [23] gave a more generalized approach to the design and implementation of SMR protocols. Group communication services can implement SMR by providing reliable multicast that guarantees VS [4]. Birman et al. were the first to present VS and a series of improvements in the efficiency of ordering protocols [6]. Birman gives a concise account of the evolution of the VS model for SMR in [5].

Research during the last recent decades resulted in an extensive literature on ways to implement VS and SMR, as well as industrial construction of such systems. A recent research line on stabilizing versions of replicated state machines [1, 7, 16, 17] obtains self-stabilizing replicated state machines in shared memory as well as in synchronous and asynchronous message passing systems.

The bounded labeling scheme and the use of practically unbounded sequence numbers proposed in [1], allow the creation of practically-stabilizing bounded-size solutions to the never-exhausted counter problem in the restricted case of a single writer. In [7] a practically-self-stabilizing version of Paxos was developed, which led to a practically-self-stabilizing consensus-based SMR implementation. To this end, they extended the labeling scheme of [1] to allow for multiple counter writers, since unbounded counters are required for ballot numbers. Extracting this scheme for other uses does not seem intuitive. We present a simpler and significantly more communication efficient practically infinite counter that also supports many writers, where only a pair of labels rather than a vector of labels needs to be communicated. Our solution is highly modular and can be easily used in any similar setting requiring such counters. We also note that with [1]’s single writer atomic register emulation, a quorum read of the value could return
without a value if the writer did not before perform a write to establish a maximal tag. An emulation based on our multiple-writer version guarantees that reads may always terminate with a value, since our labeling algorithm continuously maintains a maximal tag.

In what follows, Section 2 presents the system settings and the necessary definitions. Section 3 details the practically-self-stabilizing Labeling Scheme and Increment Counter algorithms. In Section 4 we present the practically-self-stabilizing Virtual Synchrony algorithm and the resulting replicate state machine emulation. We conclude with Section 5.

2 System Settings and Definitions

We consider an asynchronous message-passing system. The system includes a set $P$ of $n$ communicating processors; we refer to the processor with identifier $i$, as $p_i$. At most $n/2 - 1$ processors may fail by crashing and these may sometimes be referred to as inactive in contrast to active processors that are not crashed. We assume that the system runs on top of a stabilizing data-link layer that provides reliable FIFO communication over unreliable bounded capacity channels as the ones of [14, 15]. The network topology is of a fully connected graph where every two processors exchange (low-level messages called) packets to enable a reliable delivery of (high level) messages. When no confusion is possible we use the term messages for packets.

Communication and data link implementation The communication links have bounded capacity, so that the number of messages in every given instance is bounded by a constant $cap$, which is known to the processors. When processor $p_i$ sends a packet, $\pi$, to processor $p_j$, the operation send inserts a copy of $\pi$ to the FIFO queue that represents the communication channel from $p_i$ to $p_j$, while respecting an upper bound on the number of packets in the channel, possibly omitting the new packet or one of the already sent packets. When $p_j$ receives $\pi$ from $p_j$, $\pi$ is dequeued from the queue representing the channel. We assume that packets can be spontaneously omitted (lost) from the channel, however, a packet that is sent infinitely often is received infinitely often.

One version of a self-stabilizing FIFO data link implementation that we can use, is based on the fact that communication links have bounded capacity. Packets are retransmitted until more than the total capacity acknowledgments arrive; while acknowledgments are sent only when a packet arrives (not spontaneously) [14, 15]. Over this data-link, the two connected processors can constantly exchange a “token”. Specifically, the sender (possibly the processor with the highest identifier among the two) constantly sends packet $\pi_1$ until it receives enough acknowledgments (more than the capacity). Then, it constantly sends packet $\pi_2$, and so on and so forth. This assures that the receiver has received packet $\pi_1$ before the sender starts sending packet $\pi_2$. This can be viewed as a token exchange. We use the abstraction of the token carrying messages back and forth between any two communication entities. a heartbeat to (imperfectly) detect whether a processor is active or not; when a processor in no longer active, the token will not be returned back to the other processor.

Definitions and complexity measures Every processor, $p_i$, executes a program that is a sequence of (atomic) steps, where a step starts with local computations and ends with a single communication operation, which is either send or receive of a packet. For ease of description, we assume the interleaving model, where steps are executed atomically, a single step at any given time. An input event can be either the receipt of a packet or a periodic timer triggering $p_i$ to (re)send. Note that the system is asynchronous and the rate of the timer is totally unknown.

The state, $s_i$, of a node $p_i$ consists of the value of all the variables of the node including the set of all incoming communication channels. The execution of an algorithm step can change the node’s state. The term (system) configuration is used for a tuple of the form $(s_1, s_2, \cdots, s_n)$, where each $s_i$ is the state of node $p_i$ (including messages in transit to $p_i$). We define an execution (or run) $R = c_0, a_0, c_1, a_1, \ldots$ as an alternating sequence of system configurations $c_x$ and steps $a_x$, such that each configuration $c_{x+1}$, except
Figure 1: An execution satisfying the VS property. The grey boxes indicate a new view installation, and the example shows four views. View $v_1$ with membership $\{p_1, p_4, p_5\}$. The reliable multicast reaches all members of the group. Two new processors $p_2$ and $p_3$ join the group, forming view $v_2$. In this view, $p_5$ crashes before completing its multicast which is ignored (dashed lines). The new view $v_3$ is formed to exclude $p_5$, and in it, $p_1$ manages a successful multicast before crashing. The multicast of $p_3$ is reliable and guaranteed to be delivered to all non-crashed within the view, that is excluding $p_1$ which might or might not have received it (dotted line). A new view is then formed to encapture the failure of $p_1$.

the initial configuration $c_0$, is obtained from the preceding configuration $c_x$ by the execution of the step $a_x$.

An execution $R$ is practically infinite execution if it contains a chain of steps ordered according to Lamport’s happened-before relation [20] that are longer than $2^\tau$ ($\tau$ being, for example, 64), namely they are practically infinite for any given system [16]. Similar to an infinite execution, a processor that fails by crashing stops taking steps, and any processor that does not crash eventually takes a practically infinite number of steps. The code of self-stabilizing algorithms reflects the requirement for non-termination in that it usually consists of a do–forever loop that contains communication operations with the neighbors and validation that the system is in a consistent state as part of the transition decision. An iteration of an algorithm formed as a do–forever loop is a complete run of the algorithm starting in the loop’s first line and ending at the last line, regardless of whether it enters branches. (Note that an iteration may contain many steps).

We define the system’s task by a set of executions called legal executions (LE) in which the task’s requirements hold, we use the term safe configuration for any configuration in any execution in LE. As defined by Dijkstra in [12], an algorithm is self-stabilizing with relation to the task LE when every (unbounded) execution of the algorithm a safe configuration with relation to the algorithm and the task. We define the system’s abstract task $T$ by a set of variables (of the processor states) and constraints, which we call the system requirements, in a way that implies the desired system behavior [13]. Note that an execution $R$ can satisfy the abstract task and still not belong to $LE$, because $R$ considers only a subset of variables, whereas the configurations of executions that are in LE consider every variable in the processor states and message fields. An algorithm is practically-self-stabilizing (or just practically-stabilizing) with relation to the task $T$ if in any practically infinite execution has a bounded number of deviations $T$ [22].

This defines a measure for complexity. The asynchrony of the system makes it hard, if not impossible to infer anything on stabilization time, since we cannot predict when an element from the corrupt state of the system will reach a processor, (cf. self-stabilizing solutions that give time complexity in asynchronous rounds). Based on the above definition of practically-stabilizing algorithms, a bounded number of corrupt elements that might force the system to deviate from its task even if these may or may not (due to asynchrony) appear. Whenever a deviation happens, a number of algorithmic operations are required to satisfy $T$ once again. As a complexity measure, we bound the total of these operations throughout an execution. These operations differ by algorithm, i.e., it is label creations in the labeling scheme, counter increments for the counter increment algorithm and view creations in the virtual synchrony algorithm.

The virtual synchrony task uses the notion of a view, a group of processors that perform multicast
within the group and is uniquely identified, to ensure that any two processors that belong to two views that are consecutive according to their identifier, deliver identical message sets in these views. The legal execution of virtual synchrony is defined in terms of the input and output sequences of the system with the environment. When a majority of processors are continuously active every external input (and only the external inputs) should be atomically accepted and processed by the majority of the active processors. The system works in the primary component, i.e., it does not deal with partitions and requires that a view contains a majority of the system’s processors, i.e., its membership size is always greater than \( n/2 \). Therefore, there is no delivery and processing guarantee in executions in which there is no majority, still in these executions any delivery and processing is due to a received environment input. Figure 1 is an example of a virtually synchronous execution.

**Notation.** Throughout the paper we use the following notation. Let \( y \) and \( y' \) be two objects that both include the field \( x \). We denote \((y_x = y'_x) \equiv (y.x = y'.x)\).

### 3 Practically-Self-Stabilizing Labeling Scheme and Counter Algorithm

Many system like the ones performing replication (e.g. GCSs requiring group identifiers, of Paxos implementations requiring ballot numbers) assume access to an infinite (unbounded) counter. We proceed to give a practically-stabilizing, practically infinite counter based on a bounded labeling scheme. Note that by a *practically infinite* (or unbounded) counter we imply that a \( t \)-bit counter (e.g., 64-bit) is not truly infinite (since this is anyway not implementable on hardware), but it is large enough to provide counters for the lifetime of most conceivable systems when started at 0. We refer the reader to the example provided by Blanchard et al. [8], where a 64-bit counter initialized at 0 and incremented per nanosecond is calculated to last for around 500 years, essentially an infinity for most of today’s running systems.

The task of a practically-self-stabilizing labeling scheme is for every processor that takes an infinite number of steps to reach to a label that is maximal for all active processors in the system. The task of maintaining a practically infinite counter, is for every processor that takes an infinite yet bounded number of steps, to eventually be able to monotonically increment the counter from 0 to \( 2^t \). The latter task depends on the former to provide the maximal label in the system to be used as a sequence number epoch, so that within the same epoch, the integer sequence number is incremented as a practically infinite counter. It is implicit that the tasks are performed in the presence of corrupt information that might exist due to transient faults.

Our solutions are *practically infinite*, in the following way. A bounded amount of stale information from the corrupt initial state, may unpredictably corrupt the counter. In such cases, processors are forced to change their labels and restart their counters. A processor cannot predict whether a corrupt piece of information exists, or when will it make its appearance as this is essentially the work of asynchrony. Our solutions guarantee that only a bounded number of labels will need to change, or that only a bounded number of counter increments will need to take place before we reach to one that is eligible to last its full \( 2^7 \) length, less the fact that this maximal value is practically unattainable.

We first present and prove the correctness of a practically-stabilizing labeling algorithm, and then explain how this can be extended to implement practically stabilizing, practically unbounded counters in Section 3.3.

#### 3.1 Labeling Algorithm for Concurrent Label Creations

##### 3.1.1 Preliminaries

**Bounded labeling scheme** The bounded labeling scheme of Alon et al. [1] implements an SWMR register emulation in a message-passing system. The labels (also called epochs) allow the system to stabilize, since once a label is established, the integer counter related to this label is considered to be practically infinite, as a 64-bit integer is practically infinite and sufficient for the lifespan of any reasonable
system. We extend the labeling scheme of [1] to support multiple writers, by including the epoch creator (writer) identity to break symmetry, and decide which epoch is the most recent one, even when two or more creators concurrently create a new label.

Formally defined, we consider the set of integers $D = [1, k^2 + 1]$ such that $k \in \mathbb{N}$ a known constant to the processors, which we determine in Corollary 3.2. A label (or epoch) is a triple $(lCreator, sting, Antistings)$, where $lCreator$ is the identity of the processor that established (created) the label, $Antistings \subset D$ with $|Antistings| = k$, and $sting \in D$. Given two labels $\ell_i, \ell_j$, we define the relation $\ell_i \prec_{lb} \ell_j \equiv (\ell_i.lCreator < \ell_j.lCreator) \lor (\ell_i.lCreator = \ell_j.lCreator \land (\ell_i.sting \in \ell_j.Antistings) \land (\ell_j.sting \notin \ell_i.Antistings))$; we use $=_{lb}$ to say that the labels are identical. Note that the relation $\prec_{lb}$ does not define a total order. For example, when $\ell_i =_{lb} \ell_j$ and $(\ell_i.sting \notin \ell_j.Antistings)$ and $(\ell_j.sting \notin \ell_i.Antistings)$ these labels are incomparable.

As an example, consider the situation with $k = 3$, and $D = \{1, 2, \ldots, 10\}$. Assume the existence of three labels $\ell_1 = (i, 2, (3, 5, 9))$, $\ell_2 = (i, 1, (2, 9, 10))$, and $\ell_3 = (i + 1, 1, (3, 5, 9))$. In this case, $\ell_1 \prec_{lb} \ell_3$ and $\ell_2 \prec_{lb} \ell_3$, since the creator of $\ell_3$ has a greater identity than the creator of $\ell_1$ and $\ell_2$. We can also see that $\ell_1 \prec_{lb} \ell_2$, since the sting of $\ell_1$, namely 2, belongs to the antistings set of $\ell_2$ (which is $\{2, 9, 10\}$) while the opposite is not true for the sting of $\ell_2$. This makes $\ell_2$ “immune” to the sting of $\ell_1$.

As in [1], we demonstrate that one can still use this labeling scheme as long as it is ensured that eventually a label greater than all other labels in the system is introduced. We say that a label $\ell$ cancels another label $\ell'$, either if they are incomparable or they have the same lCreator but $\ell$ is greater than $\ell'$ (with respect to sting and Antistings). A label with creator $p_i$ is said to belong to $p_i$’s domain.

Creating a largest label Function $nextLabel()$, Algorithm 1, gets a set of at most $k$ labels as input and returns a new label that is greater than all of the labels of the input, given that all the input labels have the same creator i.e., the same lCreator. This last condition is imposed by the labeling algorithm that calls $nextLabel()$, as we will see further down with a set of labels from the same processor. It has the same functionality as the function called $NextLabel()$ in [1], but it additionally appends the label creator to the output. The function essentially composes a new Antistings set from the stings of all the labels that it receives as input, and chooses a sting that is in none of the Antistings of the input labels. In this way it ensures that the new label is greater than any of the input. Note that the function takes $k$ Antistings of $k$ labels that are not necessarily distinct, implying at most $k^2$ distinct integers and thus the choice of $|D| = k^2 + 1$ allows to always obtain a greater integer as the sting. For the needs of our labeling scheme, $k = 4(n^3cap + 2n^2 - 2n) + 1$ (Corollary 3.2).

Scheme idea and challenges When all processors are active, the scheme can be viewed as a simple extension of the one of [1]. Informally speaking, the scheme ensures that each processor $p_i$ eventually “cleans up” the system from obsolete labels of which $p_i$ appears to be the creator (for example, such labels could be present in the system’s initial arbitrary state). Specifically, $p_i$ maintains a bounded FIFO history of such labels that it has recently learned, while communicating with the other processors, and creates a label greater than all that are in its history; call this $p_i$’s local maximal label. In addition, each processor seeks to learn the globally maximal label, that is, the label in the system that is the greatest among the local maximal ones.
We note here that compared to Alon et al. [1], which only had a single writer upon the failure of whom there would be no progress thus stabilization would not be the main concern, we have multiple label creators. If these creators were not allowed to crash then the extension of the scheme would be a simple exercise. Nevertheless, when some processors can crash the problem becomes incrementally more difficult as we now explain. The problem lies in cleaning the system of these crashed processors’ labels since they will not “clean up” their local labels. Each active processor needs to do this itself, indirectly, without knowing which processor is inactive, i.e., we do not employ any form of failure detection for this algorithm. To overcome this problem, each processor maintains bounded FIFO histories on labels appearing to have been created by other processors. These histories eventually accumulate the obsolete labels of the inactive processors. The reader may already see that maintaining these histories, also creates another source of possible corrupt labels. We show that even in the presence of (a minority of) inactive processors, starting from an arbitrary state, the system eventually converges to use a global maximal label.

Let us explain why obsolete labels from inactive processors can create a problem when no one ever cleans (cancels) them up. Consider a system starting in a state that includes a cycle of labels $\ell_1 \prec \ell_2 \prec \ell_3 \prec \ell_1$, all of the same creator, say $p_x$, where $\prec$ is a relation between labels. If $p_x$ is active, it will eventually learn about these labels and introduce a label greater than them all. But if $p_x$ is inactive, the system’s asynchronous nature may cause a repeated cyclic label adoption, especially when $p_x$ has the greatest processor identifier, as these identifiers are used to break symmetry. Say that an active processor learns and adopts $\ell_1$ as its global maximal label. Then, it learns about $\ell_2$ and hence adopts it, while forgetting about $\ell_1$. Then, learning of $\ell_3$ it adopts it. Lastly, it learns about $\ell_1$, and as it is greater than $\ell_3$, it adopts $\ell_1$ once more, as the greatest in the system; this can continue indefinitely. By using the bounded FIFO histories, such labels will be accumulated in the histories and hence will not be adopted again, ending this vicious cycle. We now formally present the algorithm.

3.1.2 The Labeling Algorithm

The labeling algorithm (Algorithm 2) specifies how the processors exchange their label information in the asynchronous system and how they maintain proper label bookkeeping so as to “discover” their greatest label and cancel all obsolete ones. Specifically, we define the abstract task of the algorithm as one that lets every node to maintain a variable that holds the local maximal label. We require that, after the recovery period and as long as there are no calls to nextLabel() (Algorithm 1), these local maximal label actually refer to the same global maximal label.

As we will be using pairs of labels with the same label creator, for the ease of presentation, we will be referring to these two variables as the (label) pair. The first label in a pair is called $ml$. The second label is called $cl$ and it is either $\perp$, or equal to a label that cancels $ml$ (i.e., $cl$ indicates whether $ml$ is an obsolete label or not).

**The processor state** Each processor stores an array of label pairs, $max_i[n]$, where $max_i[i]$ refers to $p_i$’s maximal label pair and $max_i[j]$ considers the most recent value that $p_i$ knows about $p_j$’s pair. Processor $p_i$ also stores the pairs of the most-recently-used labels in the array of queues $storedLabels_i[n]$. The $j$-th entry refers to the queue with pairs from $p_j$’s domain, i.e., that were created by $p_j$. The algorithm makes sure that $storedLabels_i[j]$ includes only label pairs with unique $ml$ from $p_j$’s domain and that at most one of them is legitimate, i.e., not canceled. Queues $storedLabels_i[j]$ for $i \neq j$, have size $n + m$ whilst $storedLabels_i[j]$ has size $2(mn + 2n^2 - 2n)$ where $m$ is the system’s total link capacity in labels. We later show (c.f. Lemmas 3.3 and 3.4) that these queue sizes are sufficient to prevent overflows of useful labels.

**High level description** Each pair of processors periodically exchange their maximal label pairs and the maximal label pair that they know of the recipient. Upon receipt of such a label pair couple, the receiving processor starts by checking the integrity of its data structures and upon finding a corruption it flushes its label history queues. It then moves to see whether the two labels that it received can cancel any of its non-canceled labels and if the received labels themselves can be canceled by labels that it has
Algorithm 2: Practically-Self-Stabilizing Labeling Algorithm; code for \( p_i \)

1. **Variables:**
   - \( \text{maxLabels}[n] \) of \( m \) label pairs, \( \text{max}[j] \) refers to \( p_i \)’s label pair (canceled when \( \text{max}[j], \text{cl} \neq \bot \)).
   - \( \text{storedLabels}[n] \) as an array of queues of the most recently-used label pairs, where \( \text{storedLabels}[j] \) holds the labels created by \( p_j \) in \( P \). For \( p_j \in (P \setminus \{p_i\}) \), \( \text{storedLabels}[j] \)’s queue size is limited to \( (n + m) \) w.r.t. label pairs, where \( n = |P| \) is the number of processors in the system and \( m \) is the maximum number of label pairs that can be in transit in the system. The \( \text{storedLabels}[j] \)’s queue size is limited to \( (n^2 + m) \) pairs. The operator \( \text{add}() \) adds \( p_j \) to the front of the queue, and \( \text{emptyAllQueues()} \) clears all \( \text{storedLabels}[j] \) queues. We use \( \text{lp.remove()} \) for removing the record \( lp \in \text{storedLabels}[j] \).

2. In the case that stale information exists, the algorithm as done in line 20. Processor \( p_j \) in \( P \) has\( \text{maximal label, sentMax} \), and continues to receive new label pair couples from other processors.

3. Information exchange between processors
   - Whenever a processor \( p_j \in (P \setminus \{p_i\}) \) receives a label pair, and this satisfies \( \text{max}[j] \neq \bot \) \&\& \( \text{max}[j] \neq \bot \).
   - \( p_j \) sends a pair \( \langle \text{sentMax}, \text{lastSent} \rangle \) to \( p_i \), this processor stores the value of the arriving \( \text{sentMax} \) field in \( \text{max}[j] \) (line 19). However, \( p_j \) may have local knowledge of a label from \( p_i \)’s domain that cancels \( p_j \)’s maximal label, \( ml \), of the last received \( \text{sentMax} \) from \( p_j \) to \( p_i \) that was stored in \( \text{max}[j] \). Then \( p_i \) needs to communicate this canceling label in its next communication to \( p_j \). To this end, \( p_j \) assigns this canceling label to \( \text{max}[j], \text{cl} \) which stops being \( \bot \). Then \( p_j \) transmits \( \text{max}[j] \) to \( p_i \) as a \( \text{lastSent} \) label pair, and this satisfies \( \text{lastSent.cl} \neq \bot \) \&\& \( \text{lastSent.ml} \), i.e., \( \text{lastSent.cl} \) is either greater or incomparable to \( \text{lastSent.ml} \). This makes \( \text{lastSent} \) illegitimate and in case this still refers to \( p_i \)’s current maximal label, \( p_i \) must cancel \( \text{max}[i] \) by assigning it with \( \text{lastSent} \) (and thus \( \text{max}[i], \text{cl} \) = \( \text{lastSent.cl} \)) as done in line 20. Processor \( p_i \) then processes the two pairs received (lines 21 to 28).

4. **Notation:** Let \( y \) and \( y’ \) be two records that include the field \( x \). We denote \( y = x \equiv y.x = y’.x \).

5. **Macros:**
   - Let \( \text{max} \) refer to \( p_i \)’s largest label pair, \( \text{max}[j] \) refers to \( p_i \)’s label pair (canceled when \( \text{max}[j], \text{cl} \neq \bot \)).
   - \( \text{storedLabels}[n] \) as an array of queues of the most recently-used label pairs, where \( \text{storedLabels}[j] \) holds the labels created by \( p_j \) in \( P \). For \( p_j \in (P \setminus \{p_i\}) \), \( \text{storedLabels}[j] \)’s queue size is limited to \( (n + m) \) w.r.t. label pairs, where \( n = |P| \) is the number of processors in the system and \( m \) is the maximum number of label pairs that can be in transit in the system. The \( \text{storedLabels}[j] \)’s queue size is limited to \( (n^2 + m) \) pairs. The operator \( \text{add}() \) adds \( p_j \) to the front of the queue, and \( \text{emptyAllQueues()} \) clears all \( \text{storedLabels}[j] \) queues. We use \( \text{lp.remove()} \) for removing the record \( lp \in \text{storedLabels}[j] \).

6. **Variables:**
   - \( \text{legitLabels}(\text{ml}) \) a set of maximal labels.
   - \( \text{lastSent}(\text{cl}) \) a label pair, and this satisfies \( \text{max}[j], \text{cl} \neq \bot \).
cluded in the label storage \(\text{storedLabels[]}\), otherwise it includes them (line 22). The algorithm continues to see whether, based on the new pairs added to the label storage, it is possible to cancel a non-canceled label pair (which may well be the newly added pair). In this case, the algorithm updates the canceling field of any label pair \(lp\) (line 23) with the canceling label of a label pair \(lp'\) such that \(lp'.ml \not\leq_{lb} lp.ml\) (line 23). It is implied that since the two pairs belong to the same storage queue, they have the same processor as creator. The algorithm then checks whether any pair of the \(\text{max}_i[]\) array can cause canceling to a record in the label storage (line 24), and also line 25 removes any canceled records that share the same creator identifier. The test also considers the case in which the above update may cancel any arriving label in \(\text{max}[j]\) and updates this entry accordingly based on stored pairs (line 26).

After this series of tests and updates, the algorithm is ready to decide upon a maximal label based on its local information. This is the \(\leq_{ub}\)-greatest legiti label pair among all the ones in \(\text{max}_i[]\) with respect to their \(ml\) label (line 27). When no such legit label exists, \(p_i\) requests a legit label in its own label storage, \(\text{storedLabels}_i[i]\), and if one does not exist, will create a new one if needed (line 28). This is done by passing the labels in the \(\text{storedLabels}_i[i]\) queue to the \(\text{nextLabel}()\) function. Note that the returned label is coupled with a \(\perp\) as the \(cl\) and the resulting label pair is added to both \(\text{max}_i[i]\) and \(\text{storedLabel}_i[i]\).

### 3.2 Correctness proof

We are now ready to show the correctness of the algorithm. We begin with a proof overview.

**Proof overview** The proof considers a execution \(R\) of Algorithm 2 that may initiate in an arbitrary configuration (and include a processor that takes practically infinite number of steps). It starts by showing some basic facts, such as: (1) stale information is removed, i.e., \(\text{storedLabels}_i[j]\) includes only unique copies of \(p_j\)'s labels, and at most one legitimate such label (Corollary 3.1), and (2) \(p_i\) either adopts or creates the \(\leq_{ub}\)-greatest legitimate local label (Lemma 3.2). The proof then presents bounds on the number adoption steps (Lemmas 3.3 and 3.4), that define the required queue sizes to avoid label overflows.

The proof continues to show that active processors can eventually stop adopting or creating labels, by tackling individual cases where canceled or incomparable label pairs may cause a change of the local maximal label. We show that such labels eventually disappear from the system (Lemma 3.5) and thus no new labels are being adopted or created (Lemma 3.6), which then implies the existence of a global maximal label (Lemma 3.7). Namely, there is a legitimate label \(\ell_{\text{max}}\) such that for any processor \(p_i \in P\) (that takes a practically infinite number of steps in \(R\)), it holds that \(\text{max}_i[i] = \ell_{\text{max}}\). Moreover, for any processor \(p_i \in P\) that is active throughout the execution, it holds that \(p_i\)'s local maximal (legit) label pair \(\text{max}_i[i] = \ell_{\text{max}}\) is the \(\leq_{ub}\)-greatest of all the label pairs in \(\text{max}_i[]\) and there is no label pair in \(\text{storedLabels}_i[j]\) that cancels \(\ell_{\text{max}}\); i.e., \(((\text{max}_i[j].ml \leq_{lb} \ell_{\text{max}}.ml) \land (\forall \ell \in \text{storedLabels}_i[j] : \text{legit}(\ell) \Rightarrow (\ell.ml \not\leq_{lb} \ell_{\text{max}}.ml)))\). We then demonstrate that, when starting from an initial arbitrary configuration, the system eventually reaches a configuration in which there is a global maximal label (Theorem 3.3).

Before we present the proof in detail, we provide some helpful definitions and notation.

**Definitions** We define \(\mathcal{H}\) to be the set of all label pairs that can be in transit in the system, with \(|\mathcal{H}| = m\). So in an arbitrary configuration, there can be up to \(m\) corrupted label pairs in the system’s links. We also denote \(\mathcal{H}_{i,j}\) as the set of label pairs that are in transit from processor \(p_i\) to processor \(p_j\). The number of label pairs in \(\mathcal{H}_{i,j}\) obeys the link capacity bound. Recall that the data structures used (e.g., \(\text{max}_i[]\), \(\text{storedLabels}_i[]\), etc) store label pairs. For convenience of presentation and when clear from the context, we may refer to the \(ml\) part of the label pair as “the label”. Note that in this algorithm, we consider an iteration as the execution of lines 17–28, i.e., the receive action.

#### 3.2.1 No stale information

Lemma 3.1 says that the predicate \(\text{staleInfo}()\) (line 9) can only hold during the first execution of the \(\text{receive}()\) event (line 17).
Lemma 3.1 Let $p_i \in P$ be a processor for which $\neg \text{staleInfo}_i()$ (line 9) does not hold during the $k$-th step in $R$ that includes the complete execution of the receive() event (from line 17 to 28). Then $k = 1$.

Proof. Since $R$ starts in an arbitrary configuration, there could be a queue in $\text{storedLabels}_i[]$ that holds two label records from the same creator, a label that is not stored according to its creator identifier, or more than one legitimate label. Therefore, $\text{staleInfo}_0()$ might hold during the first execution of the receive() event. When this is the case, the $\text{storedLabels}_i[]$ structure is emptied (line 21). During that receive() event execution (and any event execution after this), $p_i$ adds records to a queue in $\text{storedLabels}_i[]$ (according to the creator identifier) only after checking whether $\text{recordDoesntExist}()$ holds (line 22).

Any other access to $\text{storedLabels}_i[]$ merely updates cancelations or removes duplicates. Namely, canceling labels that are not the $\leq_{lb}$-greatest among the ones that share the same creating processors (line 23) and canceling records that were canceled by other processors (line 24), as well as removing legitimate records that share the same ml (line 25). It is, therefore, clear that in any subsequent iteration of receive() (after the first), $\text{staleInfo}()$ cannot hold. Lemma 3.1 along with the lines 9 and 26 of the Algorithm, imply Corollary 3.1.

Corollary 3.1 Consider a suffix $R'$ of execution $R$ that starts after the execution of a receive() event. Then the following hold throughout $R'$: (i) $\forall p_i, p_j \in P$, the state of $p_i$ encodes at most one legitimate label, $\ell_j = \text{lcCreator } j$ and (ii) $\ell_j$ can only appear in $\text{storedLabels}_i[j]$ and $\max_i[]$ but not in $\text{storedLabels}_i[k]$ : $k \neq j$.

3.2.2 Local $\leq_{lb}$-greatest legitimate local label

Lemma 3.2 considers processors for which $\text{staleInfo}()$ (line 9) does not hold. Note that $\neg \text{staleInfo}()$ holds at any time after the first step that includes the receive() event (Lemma 3.1). Lemma 3.2 shows that $p_i$ either adopts or creates the $\leq_{lb}$-greatest legitimate local label pair and stores it in $\max_i[]$.

Lemma 3.2 Let $p_i \in P$ be a processor such that $\neg \text{staleInfo}_i()$ (line 9), and $L_{\text{pre}(i)} = \{\max_i[j].ml : \exists p_j \in P \land \text{legit(max}_i[j]) \land (\exists(\max_i[j].ml,x) \in (\text{labels}(\max_i[j]) \setminus \{\max_i[j]\})) \Rightarrow (x = \bot))\}$ be the set of $\max_i[]$’s labels that, before $p_i$ executes lines 21 to 28, are legitimate both in $\max_i[]$ and in $\text{storedLabels}_i[]$’s queues. Let $L_{\text{post}(i)} = \{\max_i[j].ml : \exists p_j \in P \land \text{legit(max}_i[j])\}$ and $⟨\ell, \bot⟩$ be the value of $\max_i[]$ immediately after $p_i$ executes lines 21 to 28. The label $⟨\ell, \bot⟩$ is the $\leq_{lb}$-greatest legitimate label in $L_{\text{post}(i)}$. Moreover, suppose that $L_{\text{pre}(i)}$ has a $\leq_{lb}$-greatest legitimate label pair, then that label pair is $⟨\ell, \bot⟩$.

Proof. $⟨\ell, \bot⟩$ is the $\leq_{lb}$-greatest legitimate label pair in $L_{\text{post}(i)}$. Suppose that immediately before line 27, we have that $\text{legitLabels}_i[] \neq \emptyset$, where $\text{legitLabels}_i[] = \{\max_i[j].ml : \exists p_j \in P \land \text{legit}(\max_i[j])\}$ (line 14). Note that in this case $L_{\text{post}(i)} = \text{legitLabels}_i[]$. By the definition of $\leq_{lb}$-greatest legitimate label pair and line 27, $\max_i[i] = ⟨\ell, \bot⟩$ is the $\leq_{lb}$-greatest legitimate label pair in $L_{\text{post}(i)}$. Suppose that $\text{legitLabels}_i[] = \emptyset$ immediately before line 27, i.e., there are no legitimate labels in $\{\max_i[j] : \exists p_j \in P\}$. By the definition of $\leq_{lb}$-greatest legitimate label pair and line 15, $\max_i[i] = ⟨\ell, \bot⟩$ is the $\leq_{lb}$-greatest legitimate label pair in $L_{\text{post}(i)}$.

Suppose that $\text{rec} = ⟨\ell', \bot⟩$ is a $\leq_{lb}$-greatest legitimate label pair in $L_{\text{pre}(i)}$, then $\ell = \ell'$. We show that the record $\text{rec}$ is not modified in $\max_i[]$ until the end of the execution of lines 21 to 28. Moreover, the records that are modified in $\max_i[]$, are not included in $L_{\text{pre}(i)}$ (it is canceled in $\text{storedLabels}_i[]$) and no records in $\max_i[]$ become legitimate. Therefore, $\text{rec}$ is also the $\leq_{lb}$-greatest legitimate label pair in $L_{\text{post}(i)}$, and thus, $\ell = \ell'$.

Since we assume that $\text{staleInfo}_0()$ does not hold, line 21 does not modify $\text{rec}$. Lines 22, 23 and 25 might add, modify, and respectively, remove $\text{storedLabels}_i[]’s$ records, but it does not modify $\max_i[]$. Since $\text{rec}$ is not canceled in $\text{storedLabels}_i[]$ and the $\leq_{lb}$-greatest legitimate label pair in $\max_i[]$, the predicate $(\text{legit}(\max_i[j]) \land \text{notgeq}(j))$ does not hold and line 23 does not modify $\text{rec}$. Moreover, the records in $\max_i[]$, for which that predicate holds, become illegitimate. 

\[\square\]
3.2.3 Bounding the number of labels

Lemmas 3.3 and 3.4 present bounds on the number of adoption steps. These are \( n + m \) for labels by labels that become inactive in any point in \( R \) and \((mn + 2n^2 - 2n)\) for any active processor. Following the above, choosing the queue sizes as \( n + m \) for \( \text{storedLabels}_i[j] \) if \( i \neq j \), and \( 2(nm + 2n^2 - 2n) + 1 \) for \( \text{storedLabels}_i[i] \) is sufficient to prevent overflows given that \( m \) is the system’s total link capacity in labels.

**Maximum number of label adoptions in the absence of creations** Suppose that there exists a processor, \( p_j \), that has stopped adding labels to the system (the else part of line 28), say, because it became inactive (crashed), or it names a maximal label that is the \( \geq_{lb} \)-greatest label pair among all the ones in \( \text{storedLabels}_i \) (line 27). Otherwise, \( p \) max

Lemma 3.3 Let \( p_i, p_j \in P \), be two processors. Suppose that \( p_j \) has stopped adding labels to the system configuration (the else part of line 28), and sending (line 16) these labels during \( R \). Processor \( p_i \) adopts (line 27) at most \((n+m)\) labels, \( \ell_j : (\ell_j = \text{lCreator} j) \), from \( p_j \)’s unknown domain \( (\ell_j \notin \text{labels}_i(\ell_j)) \) where \( m \) is the maximum number of label pairs that can be in transit in the system.

**Proof.** Let \( p_k \in P \). At any time (after the first step in \( R \)) processor \( p_k \)’s state encodes at most one legitimate label, \( \ell_j \), for which \( \ell_j = \text{lCreator} j \) (Corollary 3.1). Whenever \( p_i \) adopts a new label \( \ell_j \) from \( p_j \)’s domain (line 27) such that \( \ell_j : (\ell_j = \text{lCreator} j) \), this implies that \( \ell_j \) is the only legitimate label pair in \( \text{storedLabels}_i[j] \). Since \( \ell_j \) was not transmitted by \( p_j \) before it was adopted, \( \ell_j \) must come from \( p_k \)’s state delivered by a transmit event (line 16) or delivered via the network as part of the set of labels that existed in the initial arbitrary state. The bound holds since there are \( n \) processors, such as \( p_k \), and \( m \) bounds the number of labels in transit. Moreover, no other processor can create label pairs from the domain of \( p_j \).

**Maximum number of label creations** Lemma 3.4 shows a bound on the number of adoption steps that does not depend on whether the labels are from the domain of an active or (eventually) inactive processor.

Lemma 3.4 Let \( p_i \in P \) and \( L_i = \ell_{i_0}, \ell_{i_1}, \ldots \) be the sequence of legitimate labels, \( \ell_{ik} = \text{lCreator} i \), from \( p_i \)’s domain, which \( p_i \) stores in \( \text{max}_i[i] \) through the reception (line 17) or creation of labels (line 28), where \( k \in N \). It holds that \( |L_i| \leq n(n^2 + m) \).

**Proof.** Let \( L_{i,j} = \ell_{i_0,j}, \ell_{i_1,j}, \ldots \) be the sequence of legitimate labels that \( p_i \) stores in \( \text{max}_i[j] \) during \( R \) and \( C_{i,j} = \ell_{c_{i_0,j}}, \ell_{c_{i_1,j}}, \ldots \) be the sequence of legitimate labels that \( p_i \) receives from processor \( p_j \)’s domain. We consider the following cases in which \( p_k \) stores \( L \)'s values in \( \text{max}_i[i] \).

1. **When \( \ell_{ik} = \ell_{k,j',} \) where \( p_j, p_k \in P \) and \( k \in N \).** This case considers the situation in which \( \text{max}_i[i] \) stores a label that appeared in \( \text{max}_j[j'] \) at the (arbitrary) starting configuration, (i.e. \( \ell_{j_0,j'} \in L_{j,j'} \)). There are at most \( n(n-1) \) such legitimate label values from \( p_i \)'s domain, namely \( n-1 \) arrays \( \text{max}_i[k] \) of size \( n \).

2. **When \( \ell_{ik} = \ell_{k',j} = \ell_{c_{k,j}}, \) where \( p_j, p_k \in P, k, k' \in N \) and \( \ell_{ik}, j \neq \ell_{ik}, j' \).** This case considers the situation in which \( \text{max}_i[i] \) stores a label that appeared in the communication channel between \( p_j \) and \( p_{j'} \) at the (arbitrary) starting configuration, (i.e. \( \ell_{c_{j_0,j'}, j} \in C_{j,j'} \)) and appeared in \( \text{max}_j[j'] \) before \( p_j \) communicated this to \( p_k \). There are at most \( m \) such values, i.e., as many as the capacity of the communication links in labels, namely \( |H| \).

3. **When \( \ell_{ik} \) is the return value of nextLabel() (the else part of line 28).** Processor \( p_i \) aims at adopting the \( \geq_{lb} \)-greatest legitimate label pair that is stored in \( \text{max}_i[i] \), whenever such exists (line 27). Otherwise, \( p_i \) uses a label from its domain; either one that is the \( \geq_{lb} \)-greatest legit label pair among the ones in \( \text{storedLabels}_i[i] \), whenever such exists, or the returned value of nextLabel() (line 28).

The latter case (the else part of line 28) refers to labels, \( \ell_{ik} \), that \( p_i \) stores in \( \text{max}_i[i] \) only after checking that there are no legitimate labels stored in \( \text{max}_i[i] \) or \( \text{storedLabels}_i[i] \). Note that every time \( p_i \),
executes the else part of line 28, \( p_i \) stores the returned label, \( \ell_{ik} \), in \( \text{storedLabels}_{i}[i] \). After that, there are only three events for \( \ell_{ik} \) not to be stored as a legitimate label in \( \text{storedLabels}_{i}[i] \):

(i) execution of line 21, (ii) the network delivers to \( p_i \) a label, \( \ell' \), that either cancels \( \ell_{ik} \) (\( \ell'.ml \not\leq_{lb} \ell_{ik}.ml \)), or for which \( \ell'.ml \not\leq_{lb} \ell_{ik}.ml \), and (iii) \( \ell_{ik} \) overflows from \( \text{storedLabels}_{i}[i] \) after exceeding the \((n(n^2 + m) + n)\) limit which is the size of the queue.

Note that Lemma 3.1 says that event (i) can occur only once (during \( p_i \)'s first step). Moreover, only \( p_i \) can generate labels that are associated with its domain (in the else part of line 28). Each such label is \( \not\preceq_{lb} \)-greater-equal than all the ones in \( \text{storedLabels}_{i}[i] \) (by the definition of \( \text{nextLabel}() \) in Algorithm 1).

Event (ii) cannot occur after \( p_i \) has learned all the labels \( \ell \in \text{remoteLabels}_{i} \) for which \( \ell \not\in \text{storedLabels}_{i}[i] \), where \( \text{remoteLabels}_{i} = \big((\bigcup_{j \in P} \text{localLabels}_{i,j}) \cup \mathcal{H}\big) \setminus \text{storedLabels}_{i}[i] \) and \( \text{localLabels}_{i,j} = \{ \ell' : \ell' =_{i:\text{Creator}} j, \exists p_j \in P : ((\ell' \in \text{storedLabels}[i]) \lor (\exists p_k \in P : \ell' = \text{max}[k].ml)) \} \).

During this learning process, \( p_{i} \) cancels or updates the cancellation labels in \( \text{storedLabels}_{i}[i] \) before adding a new legitimate label. Thus, this process can be seen as moving labels from \( \text{remoteLabels}_{i} \) to \( \text{storedLabels}_{i}[i] \) and then keeping at most one legitimate label available in \( \text{storedLabels}_{i}[i] \). Every time \( \text{storedLabels}_{i}[i] \) accumulates a label \( \ell \) that was unknown to \( p_i \), the use of \( \text{nextLabel}() \) allows it to create a label \( \ell_{ik} \), that is \( \not\preceq_{lb} \)-greater than any label pair in \( \text{storedLabels}_{i}[i] \) and eventually from all the ones in \( \text{remoteLabels}_{i} \).

Note that \( \text{remoteLabels}_{i} \)'s labels must come from the (arbitrary) start of the system, because \( p_i \) is the only one that can add a label to the system from its domain and therefore this set cannot increase in size. These labels include those that are in transit in the system and all those that are unknown to \( p_{i} \) but exist in the \( \text{max} \) or \( \text{storedLabels}_{i}[i] \) structures of some other processor \( p_j \). By Lemma 3.3 we know that \( |\text{storedLabels}_{i}[i]| \leq n + m \) for \( i \neq j \). From the three cases of \( L_i \) labels that we detailed at the beginning of this proof, if \( \ell \not\preceq_{lb} \exists \text{remoteLabels}_{i} \), we can bound the size of \( \text{remoteLabels}_{i} \) as follows: for \( p_j \in P : j \neq i \) we have that \( |\text{remoteLabels}_{i}| \leq (n - 1)(\text{max}[] + |\text{storedLabels}_{i}[i]|) + |\mathcal{H}| = (n - 1)(n + (n + m)) + m = mn + 2n^2 - 2n. \)

Since \( p_j \) may respond to each of these labels with a call to \( \text{nextLabel}() \), we require that \( \text{storedLabels}_{i}[i] \) has size \( 2|\text{remoteLabels}_{i}| + 1 \) label pairs in order to be able to accommodate all the labels from \( |\text{remoteLabels}_{i}| \) and the ones created in response to these, plus the current greatest. Thus, what is suggested by event (ii) of \( p_i \), i.e., receiving labels from \( \text{remoteLabels}_{i} \), stops happening before overflows (event (iii)) occurs, since \( \text{storedLabels}_{i}[i] \) has been chosen to have a size that can accommodate all the labels from \( \text{remoteLabels}_{i} \) and those created by \( p_j \) as a response to these. This size is \( 2(mn + 2n^2 - 2n) + 1 = 2(n^3 \text{cap} + 2n^2 - 2n) + 1 \) (since \( m = n^2 \text{cap} \)) which is \( O(n^3) \).

From the end of the proof of Lemma 3.4, we get Corollary 3.2.

**Corollary 3.2** The number \( k \) of antistings needed by Algorithm 1 is \( 2 \cdot (2(n^3 \text{cap} + 2n^2 - 2n) + 1) \) (twice the queue size).
and they have the same creator \( p_i \). Algorithm 2 suggests only two possible routes for some label pair \( \ell' \) to find its way in the system through \( p_j \). Either by \( p_j \) adopting \( \ell' \) (line 27), or by creating it as a new label (the else part of line 28). Note, however, that \( p_j \) is not allowed to create a label in the name of \( p_i \) and since \( \ell' = \text{lcreator} i \), the only way for \( \ell' \) to disturb the system is if this is adopted by \( p_j \) as in line 27. We use the following definitions for estimating whether there are such label pairs as \( \ell \) and \( \ell' \) in the system.

There is a risk for two label pairs from \( p_i \)’s domain, \( \ell_j \) and \( \ell_k \), to cause such a disturbance when either they cancel one another or when it can be found that one is not greater than the other. Thus, we use the predicate \( \text{risk}_{i,j,k}(\ell_j, \ell_k) = (\ell_j = i, \ell_k) \wedge \text{legit}(\ell_j) \wedge (\text{notGreater}(\ell_j, \ell_k) \lor \text{canceled}(\ell_j, \ell_k)) \) to estimate whether \( p_j \)’s state encodes a label pair, \( \ell_j = \text{lcreator} i \), from \( p_i \)’s domain that may disturb the system due to another label, \( \ell_k \), from \( p_j \)’s domain that \( p_k \)’s state encodes, where \( \text{canceled}(\ell_j, \ell_k) = (\text{legit}(\ell_j) \land \neg \text{legit}(\ell_k) \land \ell_j = ml \ell_k) \) refers to a case in which label \( \ell_j \) is canceled by label \( \ell_k \), \( \text{notGreater}(\ell_j, \ell_k) = (\text{legit}(\ell_j) \land \text{legit}(\ell_k) \land \ell_j.ml \geq ml \ell_j.ml) \) that refers to a case in which label \( \ell_k \) is not \( \geq \)ml-greater than \( \ell_j \) and \((\ell_j = i, \ell_k) \equiv (\ell_j = \text{lcreator} i \ell_k = \text{lcreator} i) \).

These two label pairs, \( \ell_j \) and \( \ell_k \), can be the ones that processors \( p_j \) and \( p_k \) name as their local maximal label, as in \( \text{max}_{i,j,k} = \{(\text{max}[j], \text{max}[k])\} \), or recently received from one another, as in \( \text{ack}_{i,j,k} = \{(\text{max}[j], \text{max}[k])\} \). These two cases also appear when considering the communication channel (or buffers) from \( p_k \) to \( p_j \), as in \( \text{hName}_{i,j,k} = \{(\ell_j, \ell_k) : \ell_j = \text{max}[j] \land (\exists (\ell_k, \bullet) \in \mathcal{H}_{k,j})\} \) and \( \text{hAck}_{i,j,k} = \{(\ell_j, \ell_k) : \ell_j = \text{max}[j] \land (\exists (\bullet, \ell_k) \in \mathcal{H}_{k,j})\} \). We also note the case in which \( p_j \) stores a label pair that might disturb the one that \( p_j \) names as its (local) maximal, as in \( \text{stored}_{i,j,k} = \{(\text{max}[j]) \times \text{storedLabels}_k[i]\} \). We define the union of these cases to be the set \( \text{risk} = \{(\ell_j, \ell_k) \in \text{max}_{i,j,k} \cup \text{ack}_{i,j,k} \cup \text{hName}_{i,j,k} \cup \text{hAck}_{i,j,k} \cup \text{stored}_{i,j,k} : \exists p_i, p_j, p_k \in P \land \text{stopped}_j \land \text{stopped}_k \land \text{risk}_{i,j,k}(\ell_j, \ell_k)\} \), where \( \text{stopped}_i = \text{true} \) when processor \( p_i \) is inactive (crashed) and false otherwise. The above notation can also be found in Table 1.

### Table 1: The notation used to identify the possible positions of label pairs \( \ell_j \) and \( \ell_k \) that can cause canceling as used in Lemmas 3.5 to 3.7 and in Theorem 3.3.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Definition</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h\text{Name}_{i,j,k} )</td>
<td>{(\ell_j, \ell_k) : \ell_j = \text{max}[j] \land (\exists (\ell_k, \bullet) \in \mathcal{H}_{k,j})}</td>
<td>In transit from ( p_k ) to ( p_j ) as \text{sentMax} feedback about \text{max}[k]</td>
</tr>
<tr>
<td>( h\text{Ack}_{i,j,k} )</td>
<td>{(\ell_j, \ell_k) : \ell_j = \text{max}[k] \land (\exists (\bullet, \ell_k) \in \mathcal{H}_{k,j})}</td>
<td>In transit from ( p_k ) to ( p_j ) as \text{lastSent} feedback about \text{max}[j]</td>
</tr>
<tr>
<td>( \text{max}_{i,j,k} )</td>
<td>{(\text{max}[j], \text{max}[k])}</td>
<td>Local maximal labels of ( p_j ) and ( p_k )</td>
</tr>
<tr>
<td>( \text{ack}_{i,j,k} )</td>
<td>{(\text{max}[j], \text{max}[k])}</td>
<td>( \ell_j ) is ( p_j )’s local maximal label and ( \ell_k = \text{max}[j] )</td>
</tr>
<tr>
<td>( \text{stored}_{i,j,k} )</td>
<td>{(\text{max}[j]) \times \text{storedLabels}_k[i]}</td>
<td>A label ( \ell_k ) in \text{storedLabels}_k[i] ) that can cancel ( \ell_j = \text{max}[j] )</td>
</tr>
</tbody>
</table>

#### Lemma 3.5
Suppose that there exists at least one processor, \( p_{\text{unknown}} \in P \) whose identity is unknown, that takes practically infinite number of steps in \( R \) during a period where \( p_j \) never adopts labels (line 27), \( \ell_j : (\ell_j = \text{lcreator} i) \), from \( p_i \)’s unknown domain \((\ell_j \notin \text{labels}_j(\ell_j))\). Then eventually \( \text{risk} = \emptyset \).

**Proof.** Suppose this Lemma is false, i.e., the assumptions of this Lemma hold and yet in any configuration \( c \in R \), it holds that \((\ell_j, \ell_k) \in \text{risk} \neq \emptyset \). We use \text{risk}’s definition to study the different cases. By the definition of \text{risk}, we can assume, without the loss of generality, that \( p_j \) and \( p_k \) are alive throughout \( R \).

**Claim:** If \( p_j \) and \( p_k \) are alive throughout \( R \), i.e. \( \text{stopped}_j = \text{stopped}_k = \text{False} \), then \( \text{risk} \neq \emptyset \iff \text{risk}_{i,j,k} = \text{True} \). This means that there exist two label pairs \((\ell_j, \ell_k)\) where \( \ell_k \) can force a cancellation to occur. Then the only way for this two labels to force \( \text{risk} \neq \emptyset \) is if, throughout the execution, \( \ell_k \) never reaches \( p_j \).

The above claim is verified by a simple observation of the algorithm. If \( \ell_k \) reaches \( p_j \) then lines 20,
24 and 26 guarantee a canceling and lines 22 and 23 ensure that these labels are kept canceled inside $\text{storedLabels}_{j[i]}$. The latter is also ensured by the bounds on the labels given in Lemmas 3.3 and 3.4 that do not allow queue overflows. Thus to include these two labels to $\text{risk}$, is to keep $\ell_k$ hidden from $p_j$ throughout $R$. We perform a case-by-case analysis to show that it is impossible for label $\ell_k$ to be “hidden” from $p_j$ for an infinite number of steps in $R$.

**The case of** $(\ell_j, \ell_k) \in h\text{Name}_{i,j,k}$. This is the case where $\ell_j = \text{max}_j[j]$ and $\ell_k$ is a label in $\mathcal{H}_{k,j}$ that appears to be $\text{max}_k[k]$. This may also contain such labels from the corrupt state. We note that $p_j$ and $p_k$ are alive throughout $R$. The stabilizing implementation of the data-link ensures that a message cannot reside in the communication channel during an infinite number of $\text{transmit}() – \text{receive}()$ events of the two ends. Thus $\ell_k$, which may well have only a single instance in the link coming from the initial corrupt state, will either eventually reach $p_j$ or it become lost. In the both cases (the first by the Claim for the second trivially) the two clashing labels are removed from $\text{risk}$ and the result follows.

**The case of** $(\ell_j, \ell_k) \in h\text{Ack}_{i,j,k}$. This is the case where $\ell_j = \text{max}_j[j]$ and $\ell_k$ is a label in $\mathcal{H}_{k,j}$ that appears to be $\text{max}_k[j]$. The proof line is exactly the same as the previous case.

This case follows by the same arguments to the case of $(\ell_j, \ell_k) \in \text{ack}_{i,j,k}$.

**The case of** $(\ell_j, \ell_k) \in \text{max}_{i,j,k}$. Here the label pairs $\ell_j$ and $\ell_k$ are named by $p_j$ and $p_k$ as their local maximal label. We note that $p_j$ and $p_k$ are alive throughout $R$. By our self-stabilizing data-links and by the assumption on the communication that a message sent infinitely often is received infinitely often, then $p_k$ transmits its $\text{max}_k[k]$ label infinitely often when executing line 16. This implies that $p_j$ receives $\ell_k$ infinitely often. By the Claim the canceling takes place, and the two labels are eventually removed from the global observer’s $\text{risk}$ set, giving a contradiction.

**The case of** $(\ell_j, \ell_k) \in \text{stored}_{i,j,k}$. This is the case where the labels $(\ell_j, \ell_k)$ belong to $\{(\text{max}_j[j], \text{max}_k[j])\}$. Since processor $p_k$ continuously transmits its label pair in $\text{max}_k[j]$ (line 16) the proof is almost identical to the previous case.

**The case of** $(\ell_j, \ell_k) \in \text{stored}_{j,i,k}$. This case’s proof, follows by similar arguments to the case of $(\ell_j, \ell_k) \in \text{max}_{i,j,k}$. Namely, $p_k$ eventually receives the label pair $\ell_j = \text{max}_j[j]$. The assumption that $\text{risk}_{i,j,k}(\ell_j, \ell_k)$ holds implies that one of the tests in lines 23 and 26 will either update $\text{storedLabels}_{k[i]}$, and respectively, $\text{max}_k[j]$ with canceling values. We note that for the latter case we argue that $p_j$ eventually received the canceled label pair in $\text{max}_k[j]$, because we assume that $p_j$ does not change the value of $\text{max}_j[j]$ throughout $R$.

By careful and exhaustive examination of all the cases, we have proved that there is no way to to keep $\ell_k$ hidden from $p_j$ throughout $R$. This is a contradiction to our initial assumption, and thus eventually $\text{risk} = \emptyset$.

These two label pairs, $\ell_j$ and $\ell_k$, can be the ones that processors $p_j$ and $p_k$ name as their local maximal label, as in $\text{max}_{j,k} = \{(\text{max}_j[j], \text{max}_k[k])\}$, or recently received from one another, as in $\text{ack}_{i,j,k} = \{(\text{max}_j[j], \text{max}_k[k])\}$. These two cases also appear when considering the communication channel (or buffers) from $p_k$ to $p_j$, as in $h\text{Name}_{i,j,k} = \{(\ell_j, \ell_k) : \ell_j = \text{max}_j[j] \land (\exists (\ell_k, \bullet) \in \mathcal{H}_{k,j})\}$ and $h\text{Ack}_{i,j,k} = \{(\ell_j, \ell_k) : \ell_j = \text{max}_j[j] \land (\exists (\bullet, \ell_k) \in \mathcal{H}_{k,j})\}$. We also note the case in which $p_k$ stores a label pair that might disturb the one that $p_j$ names as its (local) maximal, as in $\text{stored}_{i,j,k} = \{(\text{max}_j[j]) \times \text{storedLabels}_{k[i]}\}$.

**Lemma 3.6** Suppose that $\text{risk} = \emptyset$ in every configuration throughout $R$ and that there exists at least one processor, $p_{\text{unknown}} \in P$ whose identity is unknown, that takes practically infinite number of steps in $R$. Then $p_j$ never adopts labels (line 27), $\ell_j : (\ell_j = \text{Creator} i)$, from $p_i$’s unknown domain $(\ell_j \not\in \text{labels}_{j}(\ell_j))$.

**Proof.** Note that the definition of $\text{risk}$ considers almost every possible combination of two label pairs $\ell_j$ and $\ell_k$ from $p_j$’s domain that are stored by processor $p_j$, and respectively, $p_k$ (or in the channels to them). The only combination that is not considered is $(\ell_j, \ell_k) \in \text{storedLabels}_{j[i]} \times \text{storedLabels}_{k[i]}$. However, this combination can indeed reside in the system during a legal execution and it cannot lead to a disruption for the case of $\text{risk} = \emptyset$ in every configuration throughout $R$ because before that could happen, either $p_j$ or $p_k$ would have to adopt $\ell_j$, and respectively, $\ell_k$, which means a contradiction with the assumption that $\text{risk} = \emptyset$.

The only way that a label in $\text{storedLabels}[i]$ can cause a change of the local maximum label and be communicated to also disrupt the system, is to find its way to $\text{max}[i]$. Note that $p_j$ cannot create a
label under \( p_i \)'s domain (line 28) since the algorithm does not allow this, nor can it adopt a label from \( storedLabels_i[\ell] \) (by the definition of \( legitLabels() \), line 14). So there is no way for \( \ell_j \) to be added to \( max_j[\ell] \) and thus make \( risk \neq \emptyset \) through creation or adoption.

On the other hand, we note that there is only one case where \( p_k \) extracts a label from \( storedLabels_k[\ell] : i \neq k \) and adds it to \( max_k[\ell] \). This is when it finds a legit label \( \ell_j \in max_k[\ell] \) that can be canceled by some other label \( \ell_k \) in \( storedLabels_k[\ell] \), line 26. But this is the case of having the label pair \( (\ell_j, \ell_k) \) in \( stored_{i,k} \). Our assumption that \( risk = \emptyset \) implies that \( stored_{i,j,k} = \emptyset \). This is a contradiction. Thus a label \( \ell_k \) cannot reach \( max_k[\ell] \) in order for it to be communicated to \( p_j \).

In the same way we can argue for the case of two messages in transit, \( H_{j,k} \times H_{k,j} \) and that \( risk = \emptyset \) throughout \( R \).

**Lemma 3.7** Suppose that \( risk = \emptyset \) in every configuration throughout \( R \) and that there exists at least one processor, \( p_{\text{unknown}} \in P \) whose identity is unknown, that takes practically infinite number of steps in \( R \). There is a legitimate label \( \ell_{\text{max}} \), such that for any processor \( p_i \in P \) (that takes a practically infinite number of steps in \( R \), it holds that \( max_i[\ell] = \ell_{\text{max}} \). Moreover, for any processor \( p_j \in P \) (that takes a practically infinite number of steps in \( R \), it holds that \( (max_i[\ell].ml \preceq_{i,b} \ell_{\text{max}}.ml) \land ((\forall \ell \in storedLabels_i[\ell] : \text{legit}(\ell)) \Rightarrow (\ell.ml \preceq_{i,b} \ell_{\text{max}}.ml)) \).

**Proof.** We initially note that the two processors \( p_i, p_j \) that take an infinite number of steps in \( R \) will exchange their local maximal label \( max_i[i] \) and \( max_j[j] \) an infinite number of times. By the assumption that \( risk = \emptyset \), there are no two label pairs in the system that can cause canceling to each other that are unknown to \( p_i \) or \( p_j \) and are still part of \( max_i[i] \) or \( max_j[j] \). Hence, any differences in the local maximal label of the processors must be due to the labels' \textit{ICreator} difference.

Since \( max_i[i] \) and \( max_j[j] \) are continuously exchanged and received, assuming \( max_i[i].ml \prec_{i,b} max_j[j].ml \), where the labels are of different label creators, then \( p_i \) will be led to a receive() event of \( (\text{\textit{sentMax}}, \text{\textit{lastSent}}) \) where \( max_i[i].ml \prec_{i,b} \text{\textit{sentMax}}.ml \). By line 19, \( \text{\textit{sentMax}} \) is added to \( max_i[j] \) and since \( risk = \emptyset \) no action from line 20 to line 26 takes place. Line 27 will then indicate that the greatest label in \( max_i \) is that in \( max_i[j] \) which is then adopted by \( p_i \) as \( max_i[i] \), i.e., \( p_i \)'s local maximal. The above is true for every pair of processors taking an infinite number of steps in \( R \) and so we reach to the conclusion that eventually all such processors converge to the same \( \ell_{\text{max}} \) label, i.e., it holds that \( (\forall l \in storedLabels_i[l] : \text{legit}(l)) \Rightarrow (\ell.ml \preceq_{i,b} \ell_{\text{max}}.ml) \).

**3.2.5 Convergence**

Theorem 3.3 combines all the previous lemmas to demonstrate that when starting from an arbitrary starting configuration, the system eventually reaches a configuration in which there is a global maximal label.

**Theorem 3.3** Suppose that there exists at least one processor, \( p_{\text{unknown}} \in P \) whose identity is unknown, and that takes practically infinite number of steps in \( R \). Within a bounded number of steps, there is a legitimate label pair \( \ell_{\text{max}} \), such that for any processor \( p_i \in P \) (that takes a practically infinite number of steps in \( R \), it holds that \( p_i \) has \( max_i[i] = \ell_{\text{max}} \). Moreover, for any processor \( p_j \in P \) (that takes a practically infinite number of steps in \( R \), it holds that \( (max_i[j].ml \preceq_{i,b} \ell_{\text{max}}.ml) \land ((\forall l \in storedLabels_i[j] : \text{legit}(l)) \Rightarrow (\ell.ml \preceq_{i,b} \ell_{\text{max}}.ml)) \).

**Proof.** For any processor in the system, which may take any (bounded or practically infinite) number of steps in \( R \), we know that there is a bounded number of label pairs, \( L_i = \ell_{i,1}, \ell_{i,2}, \ldots, \) that processor \( p_i \in P \) adds to the system configuration (the else part of line 28), where \( \ell_{i,k} = \text{\textit{ICreator} } i \) (Lemma 3.4). Thus, by the pigeonhole principle we know that, within a bounded number of steps in \( R \), there is a period during which \( p_{\text{unknown}} \) takes a practically infinite number of steps in \( R \) whilst (all processors) \( p_i \) do not add any label pair, \( \ell_{i,k} = \text{\textit{ICreator} } i \), to the system configuration (the else part of line 28).

During this practically infinite period (with respect to \( p_{\text{unknown}} \)), in which no label pairs are added to the system configuration due to the else part of line 28, we know that for any processor \( p_j \in P \) that
Thus no label $\ell$ of steps in $p$ (Lemma 3.5) and remains so throughout $R$ that hosts the labels with the processor as a creator (Corollary 3.2). The local state also includes the required local memory of a processor comprises of a queue of size (in labels) $2\cdot\beta$ in an execution with $O(n)$ Risk $\ell$. From the pigeonhole principle and say that there is a period during which max risk $\ell$ is unknown in $R$ and processor $p_k \in P$ that adopts labels in $R$ (line 27), $\ell_j : (\ell_j = lCreator j)$, from $p_j$’s unknown domain ($\ell_j \notin storedLabels_k(j)$) it holds that $p_k$ adopts such labels (line 27) only a bounded number times in $R$ (Lemma 3.3). Therefore, we can again follow the pigeonhole principle and say that there is a period during which $p_{unknown}$ takes a practically infinite number of steps in $R$ whilst neither $p_i$ adds a label, $\ell_{ik} = lCreator i$, to the system (the else part of line 28), nor $p_k$ adopts labels (line 27), $\ell_j : (\ell_j = lCreator j)$, from $p_j$’s unknown domain ($\ell_j \notin labels_k(\ell_j)$).

We deduce that, when the above is true, then we have reached a configuration in $R$ where Risk $= 0$ (Lemma 3.5) and remains so throughout $R$ (Lemma 3.6). Lemma 3.7 concludes by proving that, whilst $p_{unknown}$ takes a practically infinite number of steps, all processors (that take practically infinite number of steps in $R$) name the same $\ell_{lb}$-greatest legitimate label pair which the theorem statement specifies. Thus no label $\ell = lCreator j$ in $max_i[\bullet]$ or in $storedLabels_k[j]$ may satisfy $\ell.ml \not<_{lb} \ell_{max.ml}.$

### 3.2.6 Algorithm Complexity

The required local memory of a processor comprises of a queue of size (in labels) $2(n^3cap + 2n^2 - 2n)$ that hosts the labels with the processor as a creator (Corollary 3.2). The local state also includes $n - 1$ queues of size $n + n^2cap$ to store labels by other processors, and a single label for the maximal label of every processor. We conclude that the space complexity is of order $O(n^3)$ in labels. Given the number of possible labels in the system by the same processor is $\beta = n^3cap + 2n^2 - 2n$, as shown in the proof of Lemma 3.4, we deduce that the size of a label in bits is $O(\beta log \beta)$.

By Theorem 3.3 we can bound the stabilization time based on the number of label creations. Namely, in an execution with $O(n \cdot \beta)$ label creations (e.g., up to $n$ processors can create $O(\beta)$ labels), there is a practically infinite execution suffix (of size $2^\beta$ iterations) where the receipt of a label which starts an iteration never changes the maximal label of any processor in the system.

### 3.3 Increment Counter Algorithm

We adjust the labeling algorithm to work with counters, so that our counter increment algorithm is a stand-alone algorithm. In this subsection, we explain how we can enhance the labeling scheme presented in the previous subsection to obtain a practically (infinite) self-stabilizing counter increment algorithm.

#### 3.3.1 From labels to counters and to a counter version of Algorithm 2

**Counters.** To achieve this task, we now need to work with practically unbounded counters. A counter $cnt_i$ is a triplet $(lbl, seqn, wid)$, where $lbl$ is an epoch label as defined in the previous subsection, $seqn$ is a $\tau$-bit integer sequence number and $wid$ is the identifier of the processor that last incremented the counter’s sequence number, i.e., $wid$ is the counter writer. Then, given two counters $cnt_i, cnt_j$ we define the relation $cnt_i <_{ct} cnt_j \equiv ((cnt_i.lbl <_{lb} cnt_j.lbl) \land (cnt_i.seqn < cnt_j.seqn)) \lor ((cnt_i.lbl = cnt_j.lbl) \land (cnt_i.seqn = cnt_j.seqn)) \lor (cnt_i.wid < cnt_j.wid))$. Observe that when the labels of the two counters are incomparable, the counters are also incomparable.
The relation $\prec_{ct}$ defines a total order (as required by practically unbounded counters) for counters with the same label, thus, only when processors share a globally maximal label. Conceptually, if the system stabilizes to use a global maximal label, then the pair of the sequence number and the processor identifier (of this sequence number) can be used as an unbounded counter, as used, for example, in MWMR register implementations [18, 21].

**Structures.** We convert the label structures $\text{max}[]$ and $\text{storedLabels}[]$ of the labeling algorithm into the structures $\text{maxC}[]$ and $\text{storedCnts}[]$ that hold counters rather than labels (see Figure 2). Each label can yield many different counters with different $(\text{seqn}, \text{wid})$. Therefore, in order to avoid increasing the size of the queues of $\text{storedCnts}$ (with respect to the number of elements stored), we only keep the highest sequence number observed for each label (breaking ties with the $\text{wid}$).

This is encapsulated in the definition of the $\text{add()}$ operator (Figure 2 – Operators). In particular, we define the operator $\text{add}(ctp)$ (Fig. 2) to enqueue a counter pair $(\text{ctp}, \text{ctp})$ to a queue of $\text{storedCnts}[n]$, where in case a counter with the same label already exists, the following two rules apply: (1) if at least one of the two counters is canceled we keep a canceled instance, and (2) if both counters are legitimate, we keep the greatest counter with respect to $(\text{seqn}, \text{wid})$. The counter is placed at the front of the queue. In this way we allow for labels for which the counters have not been exhausted to be reused. We denote a counter pair by $(\text{ctp}, \text{ctp})$, with this being the extension of a label pair $(\text{ml}, \text{cl})$, where $\text{ct}$ is a canceling counter for $\text{mct}$, such that either $\text{ct lbl} \not\prec_{ct} \text{mct lbl}$ (i.e., the counter is canceled), or $\text{ct lbl} = \bot$.

**Exhausted counters.** These are the ones satisfying $\text{seqn} \geq 2^\tau$, and they are treated in a way similar to the canceled labels in the labeling algorithm; an exhausted counter $\text{mct}$ in a counter pair $(\text{mct}, \text{ctl})$ is canceled, by setting $\text{mct lbl} = \text{ctl lbl}$ (i.e., the counter’s own label cancels it) and hence cannot be used as a local maximal counter in $\text{maxC}[]$. This cannot increase the number of labels that are created, since the initial set of corrupt counters remains the same as the one for labels, for which we have already produced a proof in Section 3.1.

**The enhanced labeling algorithm.** Figure 4 presents a standalone version of the labeling algorithm adjusted for counters. Each processor $p_j$ uses the token-based communication to transmit to every other processor $p_i$ its own maximal counter and the one it currently holds for $p_j$ in $\text{maxC}[]$ (line 1). Upon receipt of such an update from $p_j$, $p_i$ first performs canceling of any exhausted counters in $\text{storedCnts}[]$ (line 4), in $\text{maxC}[]$ (line 6) and in the received couple of counter pairs (line 5). Having catered for exhaustion, it then calls $\text{maintainCntrs}(\bullet, \bullet)$ with the received two counter pairs as arguments. This is essentially a counter version of Algorithm 2. Macros that require some minor adjustments to handle counters are seen in Figure 3 lines 5 to 10. We also address the need to update counters of $\text{maxC}[]$ w.r.t. $\text{seqn}$ and $\text{wid}$ based on counters from the $\text{storedCnts}[]$ structure and vice versa in lines 19 and 10.

```
// Where macros coincide with Algorithm 2 we do not restate them.
1 Macros: exhaused(ctp) = (ctp.mct.seqn ≥ 2\^{\tau})
2 cancelExh(ctp) : ctp.ct ← ctp.mct
3 cancelExhMaxC(): foreach p_j ∈ P, c ∈ maxC[j] : exhaused(c) do cancelExh(maxC[j]);
4 legit(ctp) = (ctp.ct = ⊥)
5 staleCntInfo() = staleInfo() \lor (\exists p_j \in P, x ∈ storedCnts[j] : exhaused(x) \land legit(x))
6 retCntQ(ctp) : return (storedCnts[ctp.lblCreator])
7 retMaxCnt(ctp) = return (maxC[ctp]) where ct' ∈ retCntQ(ctp) ∧ (ct = \text{ctl ct'})
8 useOwnCntInfo() = if (\exists p ∈ storedCnts[i] : legit(ip)) then maxC[i] ← cp else storedCnts[i],add(maxC[i] ←
9 (\langle nextLabel(), 0, i, \bot \rangle)) // For every cp ∈ storedCnts[i], we pass to nextLabel() both cp.mct.lbl and cp.ct.lbl.
10 getMaxSeq() : return max\_{\text{seqn}}(\{\text{ctp} : \text{ctp.lbl ∈ legitCnts()} \land \text{maxC[i] = mct.lbl ctp} \})
11 initWrite = \{\langle\text{maxC[i], responseSet, status} = \langle\text{maxC[i]}, 0, \text{MAX_WRITE} \rangle\};
12 increment() = \{\langle\text{maxC[i], mct.lbl, maxC[i], mct.seqn + 1, i} \};
13 correctResponse(A, B) = return (\langle status = \text{MAX_REQUEST} \land (A, B \notin \{\bot\}) \lor ((status = \text{MAX_WRITE}) ∧
14 ((A, B) = (\bot, \text{maxC[i]}))
```

Figure 3: Macros for Algorithm 3.
We define the operator `add(ctp)` (Fig. 2) to enqueue a counter pair `ctp` to a queue of `storedCnts[n]`, where in case a counter with the same label already exists the following two rules apply: (1) if at least one of the two counters is cancelled we keep a cancelled instance, and (2) if both counters are legitimate, we keep the greatest counter with respect to `(seqn, wid)`. The counter is placed at the front of the queue.

### 3.3.2 Counter Increment Algorithm

Algorithm 3 shows a self-stabilizing counter increment algorithm where multiple processors can increment the counter. We start with some useful definitions and proceed to describe the algorithm.

**Quorums** We define a quorum set $Q$ based on processors in $P$, as a set of processor subsets of $P$ (named quorums), that ensure a non-empty intersection of every pair of quorums. Namely, for all quorum pairs $Q_i, Q_j \in Q$ such that $Q_i, Q_j \subseteq P$, it must hold that $Q_i \cap Q_j \neq \emptyset$. This intersection property is useful to propagate information among servers and exploiting the common intersection without having to write a value $v$ to all the servers in a system, but only to a single quorum, say $Q$. If one wants to retrieve this value, then a call to *any* of the quorums (not necessarily $Q$), is expected to return $v$ because there is least one processor in every quorum that also belongs to $Q$. In the counter algorithm we exploit the intersection property to retrieve the currently greatest counter in the system, increment it, and write it back to the system, i.e., to a quorum therein. Note that majorities form a special case of a quorum system.

**Algorithm description** To increment the counter, a processor $p_i$ enters status `MAX_REQUEST` (line 2) and starts sending a request to all other processors, waiting for their maximal counter (via line 8). Processors receiving this request respond with their current maximal counter and the last sent by $p_i$, (line 12). When such a response is received (line 15), $p_i$ adds this to the local counter structures via the counter bookkeeping algorithm of Figure 4. Once a quorum of responses (line 4) have been processed, $maxC[i]$ holds the maximal counter that has come to the knowledge of $p_i$ about the system’s maximal
Algorithm 3: Increment Counter; code for \( p_i \)

```plaintext
1 interface function incrementCounter() begin
2   let (responseSet, status) ← (\( \emptyset \), MAX_REQUEST);
3   repeat
4     if status = MAX_REQUEST ∧ (∃Q ∈ Q : Q ⊆ {responseSet}) then
5       initWrite(); increment()
6     else if status = MAX_WRITE ∧ (∃Q ∈ Q : Q ⊆ {responseSet}) then
7       (status ← COMPLETE)
8     foreach \( p_j \) ∈ P do send (status, maxC[i], maxC[j]);
9   until status = COMPLETE;
10  return maxC[i]
11 upon receive of \( m = (subj, sentMax, lastSent) \) from \( p_j \) begin
12    if (m.subj = MAX_REQUEST) then send (ACK, maxC[i], maxC[j]) to \( p_j \);
13    else if (m.subj = MAX_WRITE) then
14      processCntr(sentMax, lastSent, j); send (ACK, ⊥, lastSent) to \( p_j \);
15    else if (m.subj = ACK ∧ correctResponse(sentMax, lastSent)) then
16      processCntr(sentMax, lastSent, j); responseSet ← j
```

counter. This counter is then incremented locally and \( p_i \) enters status MAX_WRITE by initiating the propagation of the incremented counter (line 5), and waiting to gather acknowledgments from a quorum (the condition of line 6). When the latter condition is satisfied, the function returns the new counter. This is, in spirit, similar to the two-phase write operation of MWMR register implementations, focusing on the sequence number rather than on an associated value.

3.3.3 Proof of correctness

Proof outline Initially we prove, by extending the proof of the labeling algorithm, that starting from an arbitrary configuration the system eventually reaches to a global maximal label (as given in Theorem 3.3), even in the presence of exhausted counters (Lemma 3.4). By using the intersection property of quorums we establish that a counter that was written is known by at least one processor in every quorum (Lemma 3.5). We then combine the two previous lemmas to prove that counters increment monotonically.

Lemma 3.4 In a bounded number of steps of Algorithm 4 every processor \( p_i \) has counter \( maxC_i[i] = ct \) with \( ct.lbl = \ell_{\text{max}} \) the globally maximal non-exhausted label.

Proof. For this lemma we refer to the enhanced labeling algorithm for counters (Figure 4). The lemma proof can be mapped on the arguments proving lemmas Lemma 3.1 to Lemma 3.4 of Algorithm 2. Specifically, consider a processor \( p_i \) that has performed a full execution of \( processCntr() \) (Fig. 4 line 3) at least once due to a receive event. This implies a call to \( maintainCntrs \) and thus to \( staleCntrInfo() \) (Fig. 4 line 11) which will empty all queues if exhausted non-canceled counters exist. Also there is a call to \( cancelExhaustedMaxC() \) which cancels all counters that are exhausted in \( maxC[] \). By observation of the code, after a single iteration, there is no local exhausted counter that is not canceled.

Since every counter that is received and is exhausted becomes canceled, and since the arbitrary counters in transit are bounded, we know that there is no differentiation between exhausted labels that may cause a counter’s label to be canceled. Namely, the size of the queues of \( storedCntrs[] \) remain the same while at the same time provide the guarantees provided by the proof of the labeling algorithm. It follows from the labeling algorithm correctness and by our cancellation policy on the exhausted counters, that Theorem 3.3 can be extended to also include the use of counters without any need to locally keep more counters than those that are labels.

We proceed to deduce that, eventually, any processor taking practically infinite number of steps in \( R \) obtains a counter with globally maximal label \( \ell_{\text{max}} \). □
Lemma 3.5 In an execution where Lemma 3.4 holds, it also holds that \( \forall Q \in \mathbb{Q}, \exists p_j \in Q : \max C_j[j] = ct \land (ct' \prec_{ct} ct) \), where \( ct' \in \{\text{storedCnts}_j[k'] \cup \max C_k[k'] : ct' = \text{ld ct}_p \cup_p \in P \setminus \{ct\} \} \), i.e., \( ct' \) is every counter in the system with identical label but less than \( ct \) w.r.t. \( \text{seqn} \) or \( \text{wid} \) and \( ct \) is the last counter increment.

Proof. Observe that upon a quorum write, the new incremented counter \( ct \) with the maximal label \( lb \) is propagated (lines 6 and 8) until a quorum of acknowledgments have been received. Upon receiving such a counter by \( p_i \), a processor \( p_j \) will first add \( ct \) to its structures via \( \text{processCntr}() \) and will then acknowledge the write. If this is the maximal counter that it has received (there could be concurrent ones) then the call to \( \text{processCntr}() \) will also have the following effects: (i) the counter’s \( \text{seqn} \) and \( \text{wid} \) will be updated in the \( \text{storedCnts}_j[] \) structure in the queue of the creator of \( lb \), (ii) \( \max C_j[j] \leftarrow ct \).

Since \( p_i \) waits for responses by a quorum before it returns, it follows that by the intersection property of the quorums, the lemma must hold when \( p_i \) reaches status COMPLETE.

Theorem 3.6 Given an execution \( R \) of the counter increment algorithm in which at least a majority of processors take a practically infinite number of steps, the algorithm ensures that counters eventually increment monotonically.

Proof. Consider a configuration \( c \in R' \) where \( R' \) is a suffix of \( R \) in which Lemma 3.4 holds, and in which \( ct_{\text{max}} \) is the counter which is maximal with respect to \( \prec_{ct} \). There are two cases that the counter may be incremented.

In the first case, a legal execution, the counter \( ct_{\text{max}} \) is only incremented by a call to \( \text{incrementCounter()} \). By Lemma 3.5 any call to \( \text{incrementCounter()} \) will return the last written maximal counter (namely \( ct_{\text{max}} \)). When this is incremented giving \( ct'_{\text{max}} \) then \( ct'_{\text{max}}.\text{seqn} = ct_{\text{max}}.\text{seqn} + 1 \) which is monotonically greater than \( ct'_{\text{max}} \) and in case of concurrent writes the \( \text{wid} \) is unique and can break symmetry enforcing the monotonicity.

The second case arises when \( ct_{\text{max}} \) comes from the arbitrary initial state, is not known by a quorum, and resides in either a local state or is in transit. When \( ct_{\text{max}} \) eventually reaches a processor, it becomes the local maximal and it is propagated either via counter maintenance or in the first stage of a counter increment when the maximal counters are requested by the writer. In this case the use of \( ct_{\text{max}} \) is also a monotonic increment, and from this point onwards any increment in \( R' \) proceeds monotonically from \( ct_{\text{max}} \), as described in the previous paragraph.

3.3.4 Algorithm Complexity

The local memory of a processor implementing the counter increment is not different in order to the labeling algorithm’s, since converting to the counter structures only adds an integer (the sequence number). Hence the space complexity of the algorithm is \( O(n^3) \) in counters. The upper bound on stabilization time in the number of counter increments that are required to reach a period of practically infinite counter increments can be deduced by Theorem 3.6. For some \( t \) such that \( 0 \leq t \leq 2^7 \) in an execution with \( O(n \cdot \beta \cdot t) \) counter increments (recall that \( \beta = n^3 \text{cap} + 2n^2 - 2n \)), there is a practically infinite period of \( (2^7) \) monotonically increasing counter increments in which the label does not change.

3.3.5 MWMR Register Emulation

Having a practically-self-stabilizing counter increment algorithm, it is not hard to implement a practically-self-stabilizing MWMR register emulation. Each counter is associated with a value and the counter increment procedure essentially becomes a write operation: once the maximal counter is found, it is increased and associated with the new value to be written, which is then communicated to a majority of processors. The read operation is similar: a processor first queries all processors about the maximum counter they are aware of. It collects responses from a majority and if there is no maximal counter,
it returns ⊥ so the processor needs to attempt to read again (i.e., the system hasn’t converged to a maximal label yet). If a maximal counter exists, it sends this together with the associated value to all the processors, and once it collects a majority of responses, it returns the counter with the associated value (the second phase is a required to preserve the consistency of the register (c.f. [3, 18]).

4 Virtually Synchronous Stabilizing Replicated State Machine

Group communication systems (GCSs) that guarantee the virtual synchrony property, essentially suggest that processes that remain together in consecutive groups (called views) will deliver the same messages in the desired order [5]. This is particularly suited to maintain a replicated state machine service, where replicas need to remain consistent, by applying the same changes suggested by the environment’s requests. A key advantage of multicast services (with virtual synchrony) is the ability to reuse the same view during many multicast rounds, and which allows every automaton step to require just a single multicast round, as compared to other more expensive solutions.

GCSs provide the VS property by implementing two main services: a reliable multicast service, and a membership service to provide the membership set of the view, whilst they also assume access to unbounded counters to use as unique view identifiers. We combine existing self-stabilizing versions of the two services (with adaptations where needed), and we use the counter from the previous section to build the first (to our knowledge) practically self-stabilizing virtually synchronous state machine replication. While the ideas appear simple, combining the services is not always intuitive, so we first proceed to a high-level description of the algorithm and the services, and then follow the algorithm with a more technical description and the correctness proof.

4.1 Preliminaries

The algorithm progresses in state replication by performing multicast rounds, when a view, a tuple composed of a members set taken from P, and of a unique identifier (ID) that is a counter as defined in the previous section, is installed. This view must include a primary component (defined formally in Definition 4.1), namely it must contain a majority of the processors in P, i.e., \( n/2 + 1 \). In our version, a processor, the coordinator, is responsible: (1) to progress the multicast service which we detail later, (2) to change the view when its failure detector suggests changes to the composition of the view membership. Therefore, the output of the coordinator’s failure detector defines the set of view members; this helps to maintain a consistent membership among the group members, despite inaccuracies between the various failure detectors.

On the other hand, the counter increment algorithm that runs in the background allows the coordinator to draw a counter for use as a view identifier and in this case, the counter’s writer identifier (wid) is that of the view’s coordinator. This defines a simple interface with the counter algorithm, which provides an identical output. Pairing the coordinator’s member set with a counter as view identity we obtain a view. Of course as we will describe later, reaching to a unique coordinator may require issuing several such view proposals, of which one will prevail. We first suggest a possible implementation of a failure detector (to provide membership) and of a reliable multicast service over the self-stabilizing FIFO data link given in Section 2, and then proceed to an algorithm overview.

**Definition 4.1** We say that the output of the (local) failure detectors in execution \( R \) includes a primary partition when it includes a supporting majority of processors \( P_{maj} : P_{maj} \subseteq P \), that (mutually) never suspect at least one processor, i.e., \( \exists p_e \in P \) for which \( |P_{maj}| > \lfloor n/2 \rfloor \) and \( (p_i \in (P_{maj} \cap FD_i)) \iff (p_e \in (P_{maj} \cap FD_e)) \) in every \( c \in R \), where \( FD_x \) returns the set of processors that according to \( p_x \)’s failure detector are active.

**Failure detector** We employ the self-stabilizing failure detector of [7] which is implemented as follows. Every processor \( p \) uses the token-based mechanism to implement a heartbeat (see Section 2) with every
other processor, and maintain a heartbeat integer counter for every other processor $q$ in the system. Whenever processor $p$ receives the token from processor $q$ over their data link, processor $p$ resets the counter’s value to zero and increments all the integer counters associated with the other processors by one, up to a predefined threshold value $W$. Once the heartbeat counter value of a processor $q$ reaches $W$, the failure detector of processor $p$ considers $q$ as inactive. In other words, the failure detector at processor $p$ considers processor $q$ to be active, if and only if the heartbeat associated with $q$ is strictly less than $W$.

As an example, consider a processor $p$ which holds an array of heartbeat counters for processors $p_i, p_j, p_k$ such that their corresponding values are $(2, 5, W - 1)$. If $p_j$ sends its heartbeat, then $p$’s array will be changed to $(3, 0, W)$. In this case, $p_k$ will be suspected as crashed, and the failure detector reading will return the set $p_i, p_j$ as the set of processors considered correct by $p$.

Note that our virtual synchrony algorithm, employs the same implementation but has weaker requirements than [7] that solve consensus, and thus they resort to a failure detector at least as strong as $\Omega$ [10]. Specifically, in Definition 4.1 we pose the assumption that just a majority of the processors do not suspect at least one processor of $P$ for sufficiently long time, in order to be able to obtain a long-lived coordinator. This is different, as we said before, to an eventually perfect failure detector that ensures that after a certain time, no active processor suspects any other active processor.

Our requirements, on the other hand, are stronger than the weakest failure detector required to implement atomic registers (when more than a majority of failures are assumed), namely the $\Sigma$ failure detector [11], since virtual synchrony is a more difficult task. In particular, whilst the $\Sigma$ failure detector eventually outputs a set of only correct processors to correct processors, we require that this set in at least half of the processors, will contain at least one common processor. In this perspective our failure detector seems to implement a self-stabilizing version of a slightly stronger failure detector than $\Sigma$. It would certainly be of interest for someone to study what is the weakest failure detector required to achieve practically-self-stabilizing virtually synchronous state replication, and whether this coincides with our suggestion.

### Reliable multicast implementation

The coordinator of the view controls the exchange of messages (by multicasting) within the view. The coordinator requests, collects and combines input from the group members, and then it multicasts the updated information. Specifically, when the coordinator decides to collect inputs, it waits for the token (see Section 2) to arrive from each group participant. Whenever a token arrives from a participant, the coordinator uses the token to send the request for input to that participant, and waits the token to return with some input (possibly $\bot$, when the participant does not have a new input). Once the coordinator receives an input from a certain participant with respect to this multicast invocation, the corresponding token will not carry any new requests to receive input from the same participant; of course, the tokens continue to move back and forth to sustain the heartbeat-based failure detector. When all inputs are received, the processor combines them and again uses the token to carry the updated information. Once this is done, the coordinator can proceed to the next input collection.

We provide the pseudocode for the practically-stabilizing replicate state machine implementation as Algorithm 4, a high level description, and proceed to a line-by-line description and correctness analysis.

### 4.2 Virtual Synchrony Algorithm

#### 4.2.1 High-level algorithm description

Each participant maintains a replica of the state machine and the last processed (composite) message. Note that we bound the memory used to store the history of the replicated state machine by deciding to have the (encapsulated influence of the history represented by the) current state of the replicated state machine. In addition, each participant maintains the last delivered (composite) message to ensure common reliable multicast, as the coordinator may stop being active prior to ensuring that all members received a copy of the last multicast message. Whenever a new coordinator is installed, the coordinator
Algorithm 4: A practically-self-stabilizing automaton replication using virtual synchrony, code for processor $p_i$

1. **Constants:** $PCE$ (periodic consistency enforcement) number of rounds between global state check;
2. **Interfaces:** `fetch()` next multicast message, `apply(state, msg)` applies the step `msg` to `state` (while producing side effects), `synchState(replica)` returns a replica consolidated state, `synchMsgs(replica)` returns a consolidated array of last delivered messages, `failureDetector()` returns a vector of processor pairs $(pid, crdID)$, `inc()` returns a counter from the increment counter algorithm;
3. **Variables:** `rep[$i]$` = `(view = (ID, set), status ∈ {Multicast, Propose, Install}, (multicast round number) rnd, (replica) state, (last delivered messages) msg[$i]$)` to the state machine; `(last fetched)` input to the state machine, `propV = (ID, set), (no coordinator alive) noCrd, (recently live and connected component) FD)` an array of state replica of the state machine, where `rep[$i]$` refers to the one that processor $p_i$ maintains. A local variable `FDin` stores the `failureDetector()` output. `FD` is an alias for `{FD.in.pid, FD.in.pid = j}`, i.e. the set of processors that the failure detector considers as active. Let crd$(j) = \{FD.in.pid : FD.in.pid = j\}$, i.e. the id of $p_i$’s local coordinator, or ⊥ if none.
4. **Do forever begin**
5. let `FDin = failureDetector();`
6. let `seemCrd = \{p_1 = rep[i],propV.IID, pid ∈ FD : ((rep[i].propV.set) > \lfloor n/2 \rfloor) ∧ ((rep[i].FD) > \lfloor n/2 \rfloor) ∧ (p_2 ∈ rep[i].propV.set) ∧ (p_1 ∈ rep[i].propV.set) ∧ rep[i] ∈ rep[k].FD) ∧ ((rep[i].status = Multicast) → (rep[i].view = propV) ∧ crd(i) = \ell) ∧ (rep[i].status = Install) → crd(i) = \ell});`
7. let `valCrd = \{p_1 ∈ seemCrd : (\forall \ell ∈ propV.IID ≤ rep[i].propV.IID);`
8. noCrd ← (\exists valCrd) \neq 1; crdID ← valCrd;
9. if `(\{\ell \in FD : p_1 ∈ rep[k].FD ∧ rep[k].noCrd\}) > \lfloor n/2 \rfloor)) = \ell)` then `(valCrd = \{p_j\}) ∧ (FD \neq propV.set) ∧ ((\{p_k ∈ FD : rep[k].propV = propV\} > \lfloor n/2 \rfloor)) then (status, propV) ← (Propose, \\{\ell\});`
10. else if `(valCrd = \{p_j\}) ∧ (\forall p_j ∈ view.set : rep[j].view, status, rnd) = (view, status, rnd) \lor (\forall \\{\ell \in propV.set : rep[j].view, status, propV\} = (propV, Propose)) then`
11. if `status = Multicast then`
12. apply(state, msg); input ← fetch();
13. foreach $p_j ∈ P$ do if `p_j ∈ view.set then msg[j] ← rep[j].input; else msg[j] ← ⊥;`
14. else if `status = Propose then (state, status, msg) = (synchState(rep), Install, synchMsgs(rep));`
15. else if `status = Install then (view, status, rnd) = (propV, Multicast, 0);`
16. else if `valCrd = \{p_j\} ∧ \ell \neq i \lor (\{rep[i],rnd \lor \neq \ell \in rep[l],rnd \land \neq rep[i],\ell \neq \ell);`
17. if `rep[i],status = Multicast then`
18. if `rep[i],state = ⊥ then rep[i].state ← state /* PCE optimization, line 25 */;`
19. if `rep[i],msg = propV.apply(state, rep[i],msg); /* for the sake of side-effects */ input ← fetch();`
20. else if `rep[i].status = Propose then (status,propV) ← rep[i].status,propV);`
21. else if `rep[i],status = Install then rep[i] ← rep[i];`
22. else if `rep[i].status = Propose then (status,propV) ← rep[i].status,propV);`
23. let `m = rep[i] / * sending messages: all to coordinator and coordinator to all */ ;`
24. if `status = Multicast ∧ rnd(mod PCE) = 0 then m.state ← ⊥ / * PCE optimization, line 19 */ ;`
25. let `sendSet = (seemCrd ∪ \{p_1 ∈ propV.set : valCrd = \{p_j\} ∪ \{p_k ∈ FD : noCrd ∨ (status = Propose))
26. foreach `p_j ∈ sendSet do send(m);`
27. Upon message arrival `m from `p_i` do `rep[j] ← m;`
inquires all members (forming a majority) for the most updated state and delivered message. Since at least one of the members, say $p_i$, participated in the view in which the last completed state machine transition took place, $p_i$’s information will be recognized as associated with the largest counter, adopted by the coordinator that will in turn assign the most updated state and available delivered message to all the current group members, in essence satisfying the virtual synchrony property.

After this, the coordinator, as part of the multicast procedure, will collect inputs received from the environment before ensuring that all group members apply these inputs to the replica state machine. Note that the received multicast message consists of input (possibly $\perp$) from each of the processors, thus, the processors need to apply one input at a time, the processors may apply them in an agreed upon sequential order, say from the input of the first processor to the last. Alternatively, the coordinator may request one input at a time in a round-robin fashion and multicast it. Finally, to ensure that the system stabilizes when started in an arbitrary configuration, every so often, the coordinator assigns the state of its replica to the other members.

If the system reaches a configuration with no coordinator, e.g., due to an arbitrary configuration that the system starts in, or due to the coordinator’s crash. Each participant detecting the absence if a coordinator, seeks for potential candidates based on the exchanged information. A processor $p$ regards a processor $q$ as a candidate, if $q$ is active according to $p$’s failure detector, and there is a majority of processors that also think so (all these are based on $p$’s knowledge, which due to asynchrony might not be up to date). When there is more than one such candidate, processor $p$ checks whether there is a candidate that has proposed a view with a highest identifier (i.e., counter) among the candidates. If there is one, then $p$ considers this to be the coordinator and waits to hear from it (or learn that it is not active).

If there is none such, and if based on its local knowledge there is a majority of processors that also do not have a coordinator, then processor $p$ acquires a counter from the counter increment algorithm and proposes a new view, with view ID, the counter, and group membership, the set of processors that appear active according to its failure detector. As we show, if $p$ receives an “accept” message from all the processors in the view, then it proceeds to install the view, unless another processor who has obtained a higher counter does so. In a transition from one view to the next, there can be several processors attempting to become the coordinator (namely, those who according to their knowledge have a supporting majority). Still, by exploiting the intersection property of the supporting majorities we prove that each of these processors will propose a view at most once, and out of these, one view will be installed (i.e., we do not have never-ending attempts for new views to be installed).

As an aside, we note that GCSs that provide VS often leverage on the system’s ability to preserve (when possible) the coordinator during view transitions rather than venturing to install a new one, a certainly more expensive procedure. Our solution naturally follows this approach through our assumption of a supportive majority (Definition 4.1), where coordinators enjoy the support of a majority of processors by never being supported throughout a very long period. During such a period, our algorithm persists on using the same coordinator, even when views change.

### 4.2.2 Detailed algorithm description

The existence of coordinator $p_\ell$ is in the heart of Algorithm 4. Processors that belong to and accept $p_\ell$’s view proposal are called the followers of $p_\ell$. The algorithm determines the availability of a coordinator and acts towards the election of a new one when no valid such exists (lines 5 to 9). The pseudocode details the coordinator-side (lines 10 to 16) and the follower-side (lines 17 to 22) actions. At the end of each iteration the algorithm, defines how $p_\ell$ and its followers exchange messages (lines 25 to 28).

**The processor state and interfaces** The state of each processor includes its current view, and status = \{Propose, Install, Multicast\}, which refers to usual message multicast operation when in Multicast, or view establishment rounds in which the coordinator can Propose a new view and proceed to Install it once all preparations are done (line 3). During multicast rounds, $\text{rnd}$ denotes the round number, state stores the replica, $\text{msg}[n]$ is an array that includes the last delivered messages to the state machine, which is the input fetched by each group member and then aggregated by the coordinator during the
previous multicast round. During multicast rounds, it holds that \( \text{prop}V = \text{view} \). However, whenever \( \text{prop}V \neq \text{view} \) we consider \( \text{prop}V \) as the newly proposed view and \( \text{view} \) as the last installed one. Each processor also uses \( \text{noCrd} \) and \( \text{FD} \) to indicate whether it is aware of the absence of a recently active and connected valid coordinator, and respectively, of the set of processor present in the connected component, as indicated by its local failure detector. The processors exchange their state via message passing and store the arriving messages in the replica’s array, \( \text{rep}[n] \) (line 28), where \( \text{rep}[i].(\text{view}, \ldots, \text{noCrd}) \) is an alias to the aforementioned variables and \( \text{rep}[j] \) refers to the last arriving message from processor \( p_j \) containing \( p_j \)'s \( \text{rep}[j] \). Our presentation also uses subscript \( k \) to refer to the content of a variable at processor \( p_k \), e.g., \( \text{rep}_k[j].\text{view} \), when referring to the last installed view that processor \( p_k \) last received from \( p_j \).

Algorithm 4 assumes access to the application’s message queue via \( \text{fetch}() \), which returns the next multicast message, or \( \bot \) when no such message is available (line 2). It also assumes the availability of the automaton state transition function, \( \text{apply}(\text{state}, \text{msg}) \), which applies the aggregated input array, \( \text{msg} \), to the replica’s \( \text{state} \) and produces the local side effects. The algorithm also collects the followers’ replica states and uses \( \text{synchState}(\text{replica}) \) to return the new state. The function \( \text{failureDetector}() \) provides access to \( p_i \)'s failure detector, and the function \( \text{inc}() \) (counter increment) fetches a new and unique (view) identifier, \( \text{ID} \), that can be totally ordered by \( \preceq_{ct} \), and \( \text{ID}.\text{wid} \) is the identity of the processor that incremented the counter, resulting to the counter value \( \text{ID} \) (hence view \( \text{IDs} \) are counters as defined in Section 3.3). Note that when two processors attempt to concurrently increment the counter, due to symmetry breaking, one of the two counters is the largest. Each processor will continue to propose a new view based on the counter written, but then (as described below) the one will the highest counter will succeed (line 7).

**Determining coordinator availability** The algorithm takes an agile approach to multicasting with atomic delivery guarantees. Namely, a new view is installed whenever the coordinator sees a change to its local failure detector, \( \text{failureDetector}() \), which \( p_i \) stores in \( \text{FD}_i \) (line 5). Nevertheless, we might reach a configuration without a view coordinator as a result of an arbitrary initial configuration, or of a coordinator becoming inactive. Using the failure detector heartbeat exchange, processors can detect such initially corrupted states. Each participant that detects that it has no coordinator, seeks for potential candidates based on the exchanged information.

Processor \( p_i \) can see the set of processors, \( \text{seemCrd}_i \), that each \textit{seems} to be the view coordinator, because \( p_i \) stored a message from \( p_i \in \text{FD}_i \) in which \( p_i = \text{rep}[\ell].\text{prop}V.\text{ID}.\text{wid} \). Note that \( p_i \) cannot consider \( p_i \) as a (seemly) coordinator unless the conditions in line 6 hold. Intuitively, such a processor must be active according to \( p_i \)'s failure detector, and there is a majority of processors that also think so. Note that all these are based on local knowledge, which due to asynchrony might not be up to date. The next step is for \( p_i \) to consider the processor in \( \text{seemCrd}_i \) with the \( \preceq_{ct} \)-greatest view identifier (line 7) as the valid coordinator. Here, set \( \text{valCrd}_i \) is either a singleton or empty (line 8). If \( p_i \) considers some processor \( p_t \) as a valid coordinator, it waits to hear from \( p_t \) (or learn that it is not active). We call \( p_i \) a \textit{follower} of \( p_t \). If there is no such processor, \( p_i \) will only propose a new view if its failure detector indicates that there exists a supportive majority of active processors that are also without a valid coordinator (line 9). If such a majority exists, \( p_i \) acquires a counter from the counter increment algorithm and proposes a new view, with the counter as the view \( \text{ID} \), and the set of processors that appear active according to its failure detector as the group membership.

As we show, if \( p_i \)'s view is accepted from \textit{all} the processors in the view, then it proceeds to install the view, unless another processor who has obtained a higher counter does so. In a transition from one view to the next, there can be several processors attempting to become the coordinator (namely, those who according to their knowledge have a supporting majority). Still, by exploiting the intersection property of the supporting majorities we prove that each of these processors will propose a view at most once, and out of these, one view will be installed (i.e., we do not have never-ending attempts for new views to be installed). To satisfy the VS property, no new multicast message is delivered to a new view, before the coordinator of this new view has collected all the participants’ last delivered messages (of their prior
views) and has resent the messages appearing not to have been delivered uniformly.

**The coordinator-side** Processor $p_i$ is aware of its valid coordinatorship when $(\text{valCrd}_i = \{p_i\})$ (line 10). It takes action related to its role as a coordinator when it detects the round end, based on input from other processors. During a normal Multicast round, $p_i$ observes the round end once for every view member $p_j$ it holds that $(\text{rep}_p[j].(\text{view}, \text{status}, \text{rnd}) = (\text{view}_i, \text{status}_i, \text{rnd}_i))$ in line 10. In the cases of Propose and Install rounds, the algorithm does not need to consider the round number, $\text{rnd}$.

Depending on its status, the coordinator $p_i$ proceeds once it observes the successful round conclusion. At the end of a normal Multicast round (line 11), the coordinator increments the round number (line 14) after applying the changes to its local replica (line 12) and aggregating the followers’ input (line 13). The coordinator continues from the end of a Propose round to an Install round after using the most recently received replicas to install a synchronized state of the emulated automaton (line 15). At the end of a successful Install round, the coordinator proceeds to a Multicast round after installing the proposed view and the first round number (line 16). (Note that implicitly the coordinator creates a new view if it detects that the round number is exhausted ($\text{rnd} > 2^\tau$), or if there is another member of its view that has a greater round number than the one this coordinator has. This can only be due to corruption in the initial arbitrary state which affected $\text{rnd}$ part of the state.)

**The follower-side** Processor $p_i$ is aware of its coordinator’s identity when $(\text{valCrd}_i = \{p_i\})$ and $i \neq \ell$ (line 17). Being a follower, $p_i$ only enters this block of the pseudocode when it receives a new message, i.e., the first message round when installing a new view ($\text{rep}_i[\ell].\text{rnd} = 0$), the first time a message arrives ($\text{rnd} < \text{rep}_i[\ell].\text{rnd}$) or a new view is proposed ($\text{rep}_i[\ell].(\text{view} \neq \text{propV})$).

During normal Multicast rounds (lines 18–21) the follower $p_i$ adopts the coordinator’s replica, applies the aggregated message of this round to its current automaton state so that it produces the needed side-effects, and then fetches new messages from the environment. Note that, in the case of a Propose round (line 23), the algorithm design stops $p_i$ from overwriting its round number, thus allowing the coordinator to know what was the last round number that it delivered during the last installed view. This is only overwritten on upon the installation of the new view (line 22).

**The exchanging message and PCE optimization** Each processor periodically sends its current replica (line 27) and stores the received ones (line 28). As an optimization, we propose to avoid sending the entire replica state in every Multicast round. Instead, we consider a predefined constant, PCE (periodic consistency enforcement), that determines the maximum number of Multicast rounds during which the followers do not transmit their replica state to the coordinator and the coordinator does not send its state to them (lines 19 and 25). Note that the greater the PCE’s size, the longer it takes to recover from transient faults. Therefore, one has to take this into consideration when extending the approach of periodic consistency enforcement to other elements of replica, e.g., in $\text{view}$ and $\text{propV}$, one might want to reduce the communication costs that are associated with the set field and the epoch part of the ID field.

### 4.3 Correctness Proof of Algorithm 4

The correctness proof shows that starting from an arbitrary state in an execution $R$ of Algorithm 4 and once the primary partition property (Definition 4.1) holds throughout $R$, we reach a configuration $c \in R$ in which some processor with supporting majority $p_I$ will propose a view including its supporting majority. This view is either accepted by all its member processors or in the case where $p_I$ experiences a failure detection change, it can repropose a view.

We conclude by proving that any execution suffix of $R$ that begins from such a configuration $c$ will preserve the virtual synchrony property and implement state machine replication. We begin with some definitions. Intuitively, the latter part of the proof is deduced as follows: once a processor does not have a coordinator, it stops participating in group multicasting, and prior to delivering a new multicast
message in a new view, the algorithm assures that the coordinator of this new view has collected all the participants’ last delivered messages (in their prior views) and resends the messages appearing not to have been delivered uniformly. To do so, each participant keeps the last delivered message and the view identifier that delivered this message. This, together with the intersection property of majorities, provides the virtual synchrony property. We begin with some definitions.

Once the system considers processor $p_k$ as the view coordinator (Definition 4.1) its supporting majority can extend the support throughout $R$ and thus $p_k$ continues to emulate the automaton with them. Furthermore, there is no clear guarantee for a view coordinator to continue to coordinate for an unbounded period when it does not meet the criteria of Definition 4.1 throughout $R$. Therefore, for the sake of presentation simplicity, the proof considers any execution $R$ with only definitive suspicions, i.e., once processor $p_i$ suspects processor $p_j$, it does not stop suspecting $p_j$ throughout $R$. The correctness proof implies that eventually, once all of $R$’s suspicions appear in the respective local failure detectors, the system elects a coordinator that has a supporting majority throughout $R$.

Consider a configuration $c$ in an execution $R$ of Algorithm 4 and a processor $p_i \in P$. We define the local (view) coordinator of $p_i$, say $p_j$, to be the only processor that, based on $p_i$’s local information, has a proposed view satisfying the conditions of lines 6 and 7 such that $valCrd = \{p_j\}$. $p_i$ is also considered the global (view) coordinator if for all $p_k$ in $p_i$’s proposed view ($propV_j$), it holds that $valCrd_k = \{p_j\}$. When $p_i$ has a (local) coordinator then $p_i$’s local variable $noCrd = False$, whilst when it has no local coordinator, $noCrd = True$. Moving to the proof, we consider the following useful remark on Definition 4.1 of page 22.

**Remark 4.1** Definition 4.1 suggests that we can have more than one processor that has supporting majority. In this case, it is not necessary to have the same supporting majority for all such processors. Thus for two such processors $p_i, p_j$ with respective supporting majorities $P_{maj}(i)$ and $P_{maj}(j)$ we do not require that $P_{maj}(i) = P_{maj}(j)$, but $P_{maj}(i) \cap P_{maj}(j) \neq \emptyset$ trivially holds.

**Lemma 4.1** Let $R$ be an execution with an arbitrary initial configuration, of Algorithm 4 such that Definition 4.1 holds. Consider a processor $p_i \in P_{maj}$ which has a local coordinator $p_k$, such that $p_k$ is either inactive or it does not have a supporting majority throughout $R$. There is a configuration $c \in R$, after which $p_i$ does not consider $p_k$ to be its local coordinator.

**Proof.** There are the two possibilities regarding processor $p_k$.

**Case 1:** We first consider the case where $p_k$ is inactive throughout $R$. By the design of our failure detector, $p_i$ is informed of $p_k$’s inactivity such that line 5 will return an $FD_i$ to $p_i$ where $p_k \notin FD_i$. The threshold $W$ that we set for our failure detector determines how soon $p_k$ is suspected. By the first condition of line 6 we have that $p_k \notin FD_i \Rightarrow p_k \notin seemCrd \Rightarrow p_k \notin valCrd_i$, i.e., $p_i$ stops considering $p_k$ as its local coordinator. By definitive suspicions, that $p_i$ does not stop suspecting $p_k$ throughout $R$.

We now turn to the case where $p_k$ is active, however it does not have a supporting majority throughout $R$, but $p_i$ still considers $p_k$ as its local coordinator, i.e. $valCrd_i = \{p_k\}$. Two subcases exist:

**Case 2(a):** $p_k$ considers itself to have a supporting majority, and $p_i \in propV_k$. Note that the latter assumption implies that $p_k$ is forced by lines 24 - 27 to propagate $rep_k[k]$ to $p_i$ in every iteration. By the failure detector, there exists an iteration where $p_k$ will have $|FD_k = n/2 + 1|$ and is informed that some $p_j \in propV_k$ has $p_k \notin FD_j$ and so the condition of line 6 ($FD > |n/2|$) fails for $p_k$, which stops being the coordinator of itself. If $p_k$ does not find a new coordinator, hence $noCrd_k = True$, then $p_k$ propagates its $rep_k[k]$ to $p_i$. But this implies that $p_i$ receives $rep_k[k]$ and stores it in $rep_i[k]$. Upon the next iteration of this reception, $p_i$ will remove $p_k$ from its $seemCrd$ set because $p_k$ does not satisfy the condition $|rep_i[k].FD| < |n/2|$ of line 6. We conclude that $p_i$ stops considering $p_k$ as its local coordinator if $p_k$ does not find a new coordinator. Nevertheless, $p_k$ may find a new coordinator before propagating $rep_k[k]$. If $p_k$ has a coordinator other than itself, then it only propagates $rep_k[k]$ to its coordinator and thus $p_i$ does not receive this information. We thus refer to the next case:

**Case 2(b):** $p_k$ has a different local coordinator than itself. This can occur either as described in Case 2(a) or as a result of an arbitrary initial state in which $p_i$ believes that $p_k$ is its local coordinator but $p_k$ has a different local coordinator. We note that the difficulty of this case is that $p_k$ only sends $rep_k[k]$
to its coordinator, and thus the proof of Case 2(a) is not useful here. As explained in Algorithm 4, the failure detector returns a set with the identities (pid) of all the processors it regards as active, as well as the identity of the local coordinator of each of these processors. As per the algorithm’s notation, the coordinator of processor $p_k$ is given by crd($k$). Since $p_i$’s failure detector regards $p_k$ as active, then crd($k$) is indeed updated (remember that $p_i$ receives the token with $p_k$’s crd($k$) infinitely often from $p_k$), otherwise $p_k$ is removed from FD and is not a valid coordinator for $p_i$. But $p_k$ does not consider itself as the coordinator (by the assumption of Case 2(b)), and thus it holds that crd($k$) $\neq k$. Therefore, in the first iteration after $p_i$ receives crd($k$) $\neq k$, one of the last two conditions of line 6 fails (depending on what is the view status that $p_i$ has in rep[$i$]) so $p_k$ is not considered as the coordinator and by the assumption of definitive suspicions we reach to the result. It is also important to note that $p_k$ never again satisfies all the conditions of line 9 to create a new view.

We now define the notion of “propose” more rigorously to be used in the sequel.

**Definition 4.2** Processor $p_\ell \in P$ with status Propose, is said to propose a view $\text{prop}V_\ell$, if in a complete iteration of Algorithm 4, $p_\ell$ either satisfies $\text{valCrd}_\ell = \{p_\ell\}$ or satisfies all the conditions of line 9 to create $\text{prop}V_\ell$. A proposal is completed when $\text{prop}V_\ell$ is propagated through lines 24–27 to all the members of $FD_\ell$.

The above definition does not imply that $p_\ell$ will continue proposing the view $\text{prop}V$, since the replicas received from other processors may force $p_\ell$ to either exclude itself from $\text{valCrd}_\ell$ or create a new view (see Lemma 4.3). If the view is installed, then the proposal procedure will stop, although $\text{prop}V_\ell$ will still be sent as part of the replica propagation at the end of each iteration. Also note that the origins of such a proposed view are not defined. Indeed it is possible for a view that was not created by $p_\ell$ but bears $p_\ell$’s creator identity to come from an arbitrary state and be proposed, as long as all the conditions of lines 6 and 7 are met.

**Lemma 4.2** If the conditions of Definition 4.1 hold throughout an execution $R$ of Algorithm 4, then starting from an arbitrary configuration in which there is no global coordinator, the system reaches a configuration in which at least one processor with a supporting majority will propose a view (with “propose” defined as in Definition 4.2).

**Proof.** By Definition 4.1, at least one processor with supporting majority exists. Denote one such processor as $p_\ell$. Assume for contradiction that throughout $R$, no processor $p_\ell$ with supporting majority proposes a view. $p_\ell$ either has a local coordinator (that is not global) or does not have a coordinator.

**Case 1: $p_\ell$ does not have a coordinator (noCrd$\ell$ = True).** If $p_\ell$ does not propose a view (as per the “propose” Definition 4.2), this is because it does not hold a proposal that is suitable and it does not satisfy some condition of line 9 which would allow it to create a new view. The first condition of line 9, $(|FD| > |n/2|)$ is always satisfied by our assumption that $p_\ell$ is not suspected by a majority throughout $R$. In the second condition, both (i) $|(\text{valCrd}_\ell | \neq 1) \land (|\{p_i \in FD_\ell : p_\ell \in \text{rep}[i].FD_\ell \land \text{rep}[i].\text{noCrd}\} > |n/2|)$ and (ii) $(\text{valCrd}_\ell = \{p_\ell\}) \land (FD_\ell \neq \text{prop}V_\ell).\text{set}) \land (|\{p_i \in FD : \text{rep}[i].\text{prop}V = \text{prop}V\} > |n/2|)$ must fail due to our assumption that $p_\ell$ never proposes. Indeed (ii) fails since noCrd$\ell$ = True $\Rightarrow$ valCrd$\ell$ $\neq \{p_\ell\}$. If the first expression also fails, this implies that throughout $R$, $p_\ell$ does not know of a majority of processors with noCrd$\ell$ = True and so it cannot propose a new view.

Let’s assume that only one processor $p_j \in P_{maj}(\ell) \subseteq FD_\ell$ is required to switch from noCrd$\ell$ = False to True in order for $p_\ell$ to gain a majority of processors without a coordinator. But if noCrd$\ell$ = False then $p_j$ must already have a coordinator, say $p_k$. We have the following two subcases:

**Case 1(a):** $p_k$ does not have a supporting majority. Lemma 4.1 guarantees that $p_k$ stops considering $p_k$ as its local coordinator. Thus $p_j$ eventually goes to noCrd$\ell$ = True and by the propagation of its replica, $p_\ell$ receives the required majority to go into proposing a view. But this contradicts our initial assumption, so we are lead to the following case.

**Case 1(b):** $p_k$ has a supporting majority and a view proposal $\text{prop}V_\ell$ from the initial arbitrary configuration but is not the global coordinator. But this implies that the Lemma trivially holds, and so the
following case must be true.

Case 2: **pℓ** has a coordinator, say **pk**. The two subcases of whether **pk** has a supporting majority or not, are identical to the two subcases 1(a) and 1(b) concerning *pk* that we studied above. Thus, it must be that either **pℓ** will eventually propose a label, or that **pk** has a proposed view, thus contradicting our assumption and so the lemma follows. ■

Lemma 4.2 establishes that at least one processor with supporting majority will propose a view in the absence of a valid coordinator. We now move to prove that such a processor will only propose one view, unless it experiences changes in its *FD* that render the view proposal’s membership obsolete. The lemma also proves that any two processors with supporting majority will not create views in order to compete for the coordinatorship.

**Lemma 4.3 (Closure and Convergence)** If the conditions of Definition 4.1 hold throughout an execution R of Algorithm 4, then starting from an arbitrary configuration, the system reaches a configuration in which any processor **pℓ** with a supporting majority proposes a view *propVℓ*, and cannot create a new proposed view in R unless *FDℓ ≠ propVℓ.set* and a majority of processors has adopted *propVℓ*. As a consequence, the system reaches a configuration in which one processor with supporting majority is the global coordinator until the end of the execution.

**Proof.** We distinguish the following cases:

**Case 1:** Only one processor with supporting majority exists. Assume there is only a single processor **pℓ** that has a supporting majority throughout R. According to Lemma 4.2, **pℓ** must eventually propose a view *propVℓ*, based on the current *FDℓ* reading (line 5) which becomes the *propVℓ.set*. By Lemma 4.1, any other processor without a supporting majority will eventually stop being the local coordinator of any **pj** ∈ *propVℓ.set* and since such processors do not have a supporting majority, the first condition of line 9 will prevent them from proposing.

Processor **pℓ** continuously proposes *propVℓ* until all processors in *propVℓ.set* have sent a replica showing that they have adopted *propVℓ* as their *propV*. Every processor that is alive throughout R and in *FDℓ* should receive this replica through the self-stabilizing reliable communication. The only condition that may prevent **pj** to adopt *propVℓ* is if for some **pr** ∈ *repj[*ℓ*]*.propVℓ.set* it holds that **pℓ** ∉ *repj[*ℓ*].FD* (line 6). Plainly put, **pj** believes that **pℓ** suspects **pr**.

- **Case 1(a):** If **pj**’s information is correct about **pr**, then **pr** ∉ *Pmaj(*ℓ*). Thus at some point **pℓ** will suspect **pr** and exclude **pr** from *FDℓ*.

- **Case 1(b):** If **pj**’s information is false –remnant of some arbitrary state– then **pℓ** ∈ *FDr* and since **pr**, by the last condition of line 26, sends *repj[*ℓ*]* infinitely often to **pj**, then *repj[*ℓ*]* will be corrected and **pj** will accept *propVℓ*.

Since **pℓ** has a majority *Pmaj(*ℓ*) ⊆ *propVℓ.set*, then at least a majority of processors have received *propVℓ* and eventually accept it. If some processor **pj** ∈ *propVℓ* does not adopt **pℓ**’s proposal in R, it is eventually removed from *FDℓ* and thus does not belong to the supporting majority of **pℓ** (as detailed in Case 1(a) above). By the above we note that **pℓ** is able to get at least the supporting majority *Pmaj(*ℓ*)* to accept its view if not all of the members in *propVℓ.set*. In the last case it can proceed to the installation of the view. If there is any change in the failure detector of **pℓ** before it installs a view, **pℓ** can satisfy the second case of line 9, to create a new updated view. Note that in the mean time no processor other than **pℓ** can satisfy the conditions of that line, and thus it is the only processor that can propose and become the coordinator.

Thus **pℓ** eventually becomes the coordinator if it is the single majority-supported processor.

**Case 2:** More than one processor with supporting majority. Consider two processors **pℓ**, **pℓ′** that have a supporting majority such that each creates a view (line 9). By the correctness of our counter algorithm, *inc()* returns two distinct and ordered counters to use as view identifiers. Without loss of generality, we assume that *propVℓ* proposed by **pℓ** has the greatest identifier of all the counters created by calls to *inc()*.

We identify the following four subcases:

- **Case 2(a):** **pℓ** ∈ *FDℓ* ∧ **pℓ′** ∈ *FDℓ*. In this case **pℓ′** will propose its view *propVℓ′*, and wait for all **pi** ∈ *propVℓ.set* to adopt it (line 10). Whenever **pℓ** receives *propVℓ′*, it will store it but will not adopt it,
Proof. Consider a finite prefix \( R' \) of \( R \). Assume that in this prefix Lemma 4.3 holds, i.e., we reach a configuration in which a processor \( p_t \) has a supporting majority and is the global coordinator with view \( v \). We define a multicast round to be a sequence of ordered events: (i) \( fetch() \) input and propagate to coordinator, (ii) coordinator propagates the collected messages of this round, (iii) messages are delivered and (iv) all view members \( apply() \) side effects. The VS property is preserved between two consecutive rounds \( r, r' \) that may belong to different views \( v, v' \) (with possibly identical coordinators \( p_t, p_t' \)) respectively, if and only if \( \forall p_i \in v.set \cap v'.set \) it holds that every \( rep[i].input \) at round \( r \) is in \( rep[i'] . msg[i] \) of round \( r' \). Our proof is progressive: Claim 4.5 proves that VS is preserved between any two consecutive
multicast rounds. Claim 4.6 that VS is preserved in two consecutive views with the same coordinator and Claim 4.7 preservation in two consecutive view installations where the coordinator changes.

Claim 4.5 VS is preserved between \(r \) and \(r' \) where \(v = v' \).

**Proof.** Suppose that there exists an input and a related message \(m \) in round \(r \) that is not delivered within \(r \). We follow the multicast round \(r \). First observe the following.

**Remark:** Within any multicast round, the coordinator executes lines 12 to 14 only once and a follower executes lines 18 to 21 only once, because the conditions are only satisfied the first time that the coordinator’s local copy of the replica changes the round number.

By our Remark we notice that \(fetch() \) is called only once per round to collect input from the environment. This cannot be changed/overwritten since followers can never access \(rep[i] \leftarrow rep[ℓ] \) of line 20 that is the only line modifying the \(input \) field, unless they receive a new round number greater than the one they currently hold. We notice that the followers have produced side effects for the previous round (using \(apply() \)) based on the messages and state of the previous round. Similarly, the coordinator executes \(fetch() \) exactly once and only before it populates the \(msg \) array and after it has produced the side effects for the environment that were based on the previous messages (line 12). Line 13 populates the \(msg \) array with messages and including \(m \). The coordinator \(p_ℓ \) then continuously propagates its current replica but cannot change it by the Remark and until condition \((∀ j ∈ v.set : rep[i].(view, status, rnd) = (view_i, status_i, rnd_i)) \) (line 10) holds again. This ensures that the coordinator will change its \(msg \) array only when every follower has executed line 20 which allows the aforementioned condition to hold.

Any follower that keeps a previous round number does not allow the coordinator to move to the next round. If the coordinator moves to a new round, it is implied that \(rep[i] \leftarrow rep[ℓ] \) and thus message \(m \) was received by any follower \(p_i \), by our assumptions that the replica is propagated infinitely often and the data links are stabilizing. Thus, by the assumptions, any message \(m \) is certainly delivered within the view and round it was sent in, and thus the virtual synchrony property is preserved, whilst at the same time common state replication is achieved.

Claim 4.6 VS is preserved between \(r \) and \(r' \) where \(v \neq v' \) and \(p_ℓ = p_μ \).

**Proof.** We now turn to the case where from one configuration \(c_{safe} \) we move to a new \(c'_{safe} \) that has a different view \(v' \) but has the same coordinator \(p_ℓ \). Once \(p_ℓ \) is in an iteration where the condition \(FD \neq propV.set \) of line 9 holds, a view change is required. Since \(p_ℓ \) is the global coordinator holds, no other processor can satisfy the condition \((|\{p_k ∈ FD_ℓ : prop[k].propV = propV_ℓ\}| > \lfloor n/2 \rfloor) \) of line 9, and so only \(p_ℓ \). For more on why this holds one can refer to Lemma 4.2. Processor \(p_ℓ \) creates a new \(propV \) with a new view ID taken from the increment counter algorithm, which is greater than the previous established view ID in \(v.ID \). The last condition of line 10 guarantees that \(p_ℓ \) will not execute lines 12 to 16 and thus will not change its \(rep.(state, input, msg) \) fields, until all the expected followers of the proposed view have sent their replicas. Followers that receive the proposal will accept it, since none of the conditions that existed change and so the new view proposal enforces that \(valCrd = \{p_ℓ\} \).

Moreover, the proposal satisfies the condition of line 17 and the followers of the view enter status \(Propose \) leading to the installation of the view. What is important is that virtual synchrony is preserved since no follower is changing \(rep.(state, input, msg) \) during this procedure, and moreover each sends its replica to \(p_ℓ \) by line 26. Once the replicas of all the followers have been collected, the coordinator creates a consolidated \(state \) and \(msg \) array of all messages that were either delivered or pending, \(p_ℓ \)’s new replica is communicated to the followers who adopt this state as their own (line 22). Thus virtual synchrony is preserved and once all the processors have replicated the state of the coordinator, a new series of multicast rounds can begin by producing the side effects required by the input collected before the view change.

Claim 4.7 VS is preserved between \(r \) and \(r' \) where \(v \neq v' \) and \(p_ℓ \neq p_μ \).
Proof. We assume that \( p_\ell \) had a supporting majority throughout \( R' \). We define a matching suffix \( R'' \) to prefix \( R' \), such that \( R'' \) results from the loss of supporting majority by \( p_\ell \). Notice that since Definition 4.1 is required to hold, then some other processor with supporting majority \( p_{\ell'} \), will by Lemma 4.2 propose the view \( v' \) with the highest view ID. We note that by the intersection property and the fact that a view set can only be formed by a majority set, \( \exists p_i \in v \cap v' \). Thus, the “knowledge” of the system, \((state_{\ell'}, input_{\ell'}, msg)\) is retained within the majority.

As detailed in step 2, if a processor \( p_i \) had \( noCrd = True \) for some time or was in status Propose it did not incur any changes to its replica. If it entered the Install phase, then this implies that the proposing processor has created a consolidated state that \( p_i \) has replicated. What is noteworthy is that whether in status Propose or Install, if the proposer collapses (becomes inactive or suspected), the virtual synchrony property is preserved. It follows that, once status Multicast is reached by all followers, the system can start a practically infinite number of multicast rounds.

Thus, by the self-stabilization property of all the components of the system (counter increment algorithm, the data links, the failure detector and multicast) a legal execution is reached in which the virtual synchrony property is guaranteed and common state replication is preserved...

4.4 Algorithm Complexity

The local memory for this algorithm consists of \( n \) copies of two labels, of the encapsulated state (say of size \( |S| \) bits) and of other lesser size variables. These give a space complexity of order \( O(n \beta \log \beta + n|S|) \); recall that \( \beta = n^3 \text{cap} + 2n^2 - 2n \). Stabilization time can be provided by a bound on view creations. It is, therefore, implicit that stabilization is dependent upon the stabilization of the counter algorithm, i.e., \( O(n \cdot \beta \cdot t) \), before processors can issue views with identifiers that can be totally ordered. When this is satisfied, then Lemma 4.3 suggests that \( O(n) \) view creations are required to acquire a coordinator, namely, in the worse case where every processor is a proposer. Once a coordinator is established then Theorem 4.4 guarantees that there can be practically infinite multicast rounds (0 to \( 2^\tau \)).

5 Conclusion

State-machine replication (SMR) is a service that simulates finite automata by letting the participating processors to periodically exchange messages about their current state as well as the last input that has led to this shared state. Thus, the processors can verify that they are in sync with each other. A well-known way to emulate SMRs is to use reliable multicast algorithms that guarantee virtual synchrony [4, 19]. To this respect, we have presented the first practically-self-stabilizing algorithm that guarantees virtual synchrony, and used it to obtain a practically-self-stabilizing SMR emulation; within this emulation, the system progresses in more extreme asynchronous executions in contrast to consensus-based SMRs, like the one in [7]. One of the key components of the virtual synchrony algorithm is a novel practically-self-stabilizing counter algorithm, that establishes an efficient practically unbounded counter, which in turn can be directly used to implement a practically-self-stabilizing MWMR register emulation; this extends the work in [1] that implements SWMR registers and can also be considered simpler and more communication efficient than the MWMR register implementation presented in [7].

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References


