Weighted Boolean Formula Games*

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(August 17, 2007)

Abstract

We introduce a new class of *succinct games,* called *weighted boolean formula games*. Here, each player has a set of boolean formulas he wants to get satisfied. The boolean formulas of all players involve a ground set of boolean *variables*, and every player controls some of these variables. The *payoff* of a player is the *weighted* sum of the values of his boolean formulas. For these games, we consider *pure Nash equilibria* [42] and their well-studied refinement of *payoff-dominant equilibria* [30], where every player is no worse-off than in any other pure Nash equilibrium. We study both structural and complexity properties for both *decision* and *search* problems with respect to the two concepts:

- We consider a subclass of weighted boolean formula games, called *mutual weighted boolean formula games,* which make a natural *mutuality* assumption on the payoffs of distinct players. We present a very simple *exact potential* for mutual weighted boolean formula games. We also prove that each *weighted, linear-affine (network) congestion game with player-specific constants* is *polynomial, sound Nash-Harsanyi-Selten homomorphic* to a mutual weighted boolean formula game. In a general way, we prove that each *weighted, linear-affine (network) congestion game with player-specific coefficients and constants* is polynomial, sound Nash-Harasanyi-Selten homomorphic to a weighted boolean formula game. These homomorphisms indicate some of the richness of the new class.
- We present a comprehensive collection of high intractability results. These results show that the computational complexity of decision (and search) problems for both payoff-dominant and pure Nash equilibria in weighted boolean formula games depends in a crucial way on five parameters: *(i)* the number of players; *(ii)* the number of variables per player; *(iii)* the number of boolean formulas per player; *(iv)* the *weights* in the payoff functions (whether *identical* or *non-identical*), and *(v)* the syntax of the boolean formulas. (For example, we prove that deciding the existence of a payoff-dominant equilibrium is Θ_3^P-complete even if weights are identical and there are only *four* players.) Our completeness results show that decision (and search) problems for payoff-dominant equilibria are considerably harder than for pure Nash equilibria (unless the *polynomial hierarchy* collapses).

Due to the space constraints, some technical proofs are shifted to the Appendix.

*This work has been partially supported by the IST Program of the European Union under contract numbers IST-2004-001907 (DELIS) and 15964 (AEOLUS).

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1 Introduction

Motivation and Framework. Deciding the existence of and finding *Nash equilibria* [42] for a *strategic game* are among the most important problems studied in *Algorithmic Game Theory* today – see, for instance, [2, 12, 14, 16, 17, 18, 19, 20, 22, 23, 24, 27, 28, 48]. When the players' *strategy spaces* and *payoffs* are presented explicitly, there is a straightforward polynomial time algorithm to decide the existence of and compute a pure Nash equilibrium. More interesting are the cases where the strategy spaces and the payoffs are presented in a *succinct* way. Interesting examples of *succinct games* include *(unweighted) congestion games* [47], where the payoffs are represented by *payoff functions,* and their even more succinct subclass of *network congestion games* where, in addition, strategy spaces are described succinctly by a graph. The complexity of Nash equilibria for succinct games has been studied very intensively in the last few years – see, e.g., [2, 17, 19, 20, 22, 24, 26, 34, 37, 38, 48].

We introduce *weighted boolean formula games,* abbreviated as WBFG, as an adequate and very general form of succinct games (Definition 2.3). The idea is that each player controls a set of *boolean variables*; different players control disjoint sets of variables. A *strategy* of a player is a truth assignment to his boolean variables. Each player targets a set of constraints expressed by *boolean formulas,* which he wants to get satisfied; naturally, his formulas depend also on variables of other players.* For each formula, there is an (integer) *weight,* which expresses the relative priority of the constraint (for the player). The *payoff* for a player is the *weighted* sum of his satisfied constraints. In an *unweighted boolean formula game,* abbreviated as UWBFG, all weights are 1.

Even though the definition of a WBFG is very simple, any problem in which variables and weighted constraints are distributed among autonomous agents can be formalized as a weighted boolean formula game. Consider, for example, an economic setting in which a *principal* motivates a team of *strategic agents* (or *schedulers*), each coming with a set of (heterogeneous) *tasks*; Each agent seeks to honor a set of *contracts* signed with the principal; however, each contract may involve tasks owned by mulptiple players. For each contract, the agent is incentivized via a *payment* conditioned on the scheduling of all tasks involved in the contract; naturally, each agent tries to maximize his total payment. This setting generalizes the popular model of *(weighted) congestion games* [40, 47], since it allows payments (or *costs*) to depend on combinations of scheduled tasks in an arbitrary way. It is straightforward to formalize this practical example using players, boolean variables, constraints and weights for schedulers, tasks, contracts and payments, respectively.

We shall especially consider a subclass of WBFG, called *mutual weighted boolean formula games* and abbreviated as MWBFG; these add a natural *mutuality* assumption on the constraints targeted by different players. In more detail, it is assumed that whenever some function of a player involves a boolean variable of a second player, then the same function is a constraint for the second player with the *same* weight as well. Mutuality is motivated by real *multi-agent systems,* where a constraint involving several *attributes* typically concerns (in a uniform way) all agents featuring them.

In a (pure) Nash equilibrium [42], no player can increase his payoff by changing the values of his variables. In a *payoff-dominant equilibrium* [30], every player is *no-worse-off* than in any other Nash equilibrium; so, this is stable outcome that *payoff-dominates* all other stable outcomes. Payoff-dominance is a well-know refinement of Nash equilibrium that has been studied extensively in Game Theory. Games admitting payoff-dominance have been intuitively called *games of common interests* (cf. [4]); Colman and Bacharach [15, Section 1] mention the abstract classes of *unanimity games* [33] and *matching games* [5] as the simplest exemplars of them. We shall study the structure and complexity of both payoff-dominant and pure Nash equilibria in WBFGs.

State-of-the-Art. In the *Artificial Intelligence* literature, (game-like) situations involving agents controlling the values of certain subsets of boolean variables are not uncommon (cf. [11]). Yakoo *et al.* introduced *Distributed Constraint Satisfaction Problems* (DCSP) [55], where agents can trade boolean variables in an auction environment in order to satisfy a certain set of propositional formulas which are distributed among the agents; agents are *not* strategic, and there are no weights attached to the formulas. Instead, DCSP seek to develop *asynchronous* and *concurrent, distributed algorithms* converging to a *consistent* combination of agent actions. Studied in the literature have been three formalisms of succinct games similar to (but different than) WBFGs:

- (Boolean) Circuit games: Those were introduced by Schoenebeck and Vadhan [48]. In a *circuit game,* players still control disjoint sets of *variables,* but each player's payoff is given by a **single** *boolean circuit* and there are **no** weights. Note that a WBFG can be encoded as a circuit game since our utility functions (given by some boolean formulas and weights) can be evaluated by a *single* boolean circuit. Hence, WBFG make a very **restricted** subclass of circuit games. *Boolean circuit games* are the special case of circuit games where each player controls a **single** boolean variable.

  Recall that the best-known upper bound for the *formula size* $\mathsf{L}(f)$ of a Boolean function f in terms of its *(boolean) circuit size* $\mathsf{C}(f)$ is $\mathsf{L}(f) = \mathcal{O}(2^{\mathsf{C}(f)})$ [35, 45]. (In fact, the straightforward *depth-preserving*

*A boolean formula is the special case of a *(boolean) circuit* where every *boolean gate* has *fan-out* one; so, a boolean formula is a circuit whose underlying graph is a tree.

conversion of a boolean circuit into an equivalent formula may potentially blow up the size exponentially since pieces of the circuit must be repeated.)† So, there is no known polynomial time transformation of a circuit game into a boolean formula game where each player has a **single** (equivalent) formula. It is nevertheless possible to transform a boolean circuit into a polynomial size set of boolean *clauses*; this requires introducing of new (polynomially many) boolean variables which express the correctness of the computations by the individual gates. Hence, there is a polynomial time transformation of a circuit game into a boolean formula game where each player has a polynomial number of clauses. Nevertheless, we aim at WBFGs where the number of boolean formulas (in particular, clauses) per player is a (small) constant.

- Turing machine games: Álvarez *et al.* [2] study three different levels (*forms*) of succinct representations of strategic games. In the *implicit form,* payoff functions are represented by a deterministic Turing Machine (DTM) computing the payoffs, and strategies are described succinctly. In the *general form,* payoff functions are represented by a DTM and strategy spaces are listed explicitly. For each form, there are two cases: in the *non-uniform* case, the payoff functions are represented by a tuple $\langle M, 1^t \rangle$, where M is a DTM and t is a natural number bounding its computation time; in the *uniform* case, the payoff functions are represented by a (polynomial time) DTM M. Álvarez *et al.* [2] present completeness results on the decision problem for pure Nash equilibria. Their proofs are based on a simple construction of a *gadget game* [2, Section 2]. It is straightforward to see that the payoff functions of the gadget game may be expressed as an instance of a WBFG with $r = 5$. Recall the folklore facts that Turing machine computations with t steps can be encoded as a boolean circuit of size $\mathcal{O}(t^2)$, and that boolean circuits can be evaluated by Turing machines in polynomial time (cf. [6]). Hence, Turing machine games in implicit form and circuit games are *equivalent.* It follows from our previous discussion on the relation of WBFG to circuit games that there holds an identical relation of WBFG to Turing machine games in implicit form.

  In the *explicit form,* payoffs are explicitly listed. The decision problem for pure Nash equilibria in Turing machine games in explicit form is $\mathcal{P}$-complete [2, Theorem 3]. However, it is *not* possible to obtain from a succinctly represented WBFG such an explicit form in polynomial time. To emphasize, we are interested in WBFG that represent *exponentially large* strategy spaces with *succinct* size (i.e., the size of formulas).

- Boolean games: Those were introduced by Harrenstein *et al.* [29] in the context of a logical consequence relation defined in terms of Nash equilibrium; they were further extended and studied in [8, 9, 10, 21] from a computational point of view. The formulation of boolean games by Bonzon *et al.* in [10] is very similar to WBFG: each player still wishes to satisfy a particular set of boolean formulas, but the preferences of each player over his formulas were not defined by means of additional weights attached to the formulas (but on the basis of *logical strength*).

The problem of singling out some "best" Nash equilibrium is probably as old as the concept of Nash equilibrium itself [42]. The corresponding stream of game-theoretic research is called *equilibrium selection* (cf. [30]). To address this problem, some *refinement* of the concept is employed, which provides some criterion to discard implausible and to select plausible when a game exhibits multiple Nash equilibria. Besides payoff-dominance, there are several, well-studied refinements of pure Nash equilibria, such as *dominating equilibrium, Pareto-optimality, risk-dominance* [30], *social-welfare maximization* and *strong equilibrium* [3]. Due to their strength, such refinements are usually unlikely to exist. For example, to the best of our knowledge, the *only* pragmatic class of strategic games for which the well-studied Pareto-optimal equilibria are *always* guaranteed to exist are certain, very special *two-person exchange models* [13], which model an exchange economy with just two traders!

Concurrent Work. Independently and concurrently to our work, Biló [7] introduced and studied *satisfiability games*; these are *almost identical* to our MWBFG, except that associated with each player i is some integer $l_i \geq 1$ such that his strategy set is contained in $\{0,1\}^{l_i}$, while it is equal to $\{0,1\}^{l_i}$ in WBFG;‡ however, this difference is *not* essential in general. Their restricted subclass of *unconstrained satisfiability problems* [7] coincides with the class of MWBFG. Studied by Biló [7] are also the so called *satisfiability games with player-specific payoffs*; these correspond to WBFG with the additional *semi-mutuality* assumption: whenever some function of a player involves a variable of another player, then the same function is a constraint for the different player as well, but with a possibly different weight.

Summary of Results and Significance. We present two types of results. First, we identify structural properties for both WBFG and its rich subclass of MWBFG. Second, we present a comprehensive collection of complexity results about payoff-dominant and pure Nash equilibria. More specifically, we investigate how the complexity of their decision and search problems depends on five natural parameters: *(i)* the number of players

†Nevertheless, the largest *shown* difference between formula size and boolean circuit size is only $\mathsf{L}(f) = \Omega(n^2 \lg^{-1} n)$ and $\mathsf{C}(f) = 2n + o(n)$, where f is the *storage access function* for indirect addressing [46].

‡For both, it can be assumed, without loss of generality, that all numbers l_i, $i \in [n]$, are equal, since one can always "pad" with extra variables that do not enter the formulas.

m; *(ii)* the (maximum) number of variables per player k; *(iii)* the (maximum) number of boolean formulas r weight-summed into each payoff function; *(iv)* the *weights* for the payoff functions (that is, whether *weighted* or *unweighted*), and *(v)* the syntax of the boolean formulas. Each of the parameters m, k, and r can be chosen to be fixed as a specific natural number or can be chosen to be not fixed. We discover that the choice of these parameters may have a crucial impact on complexity. In all cases, corresponding results for the search problem follow from those for the decision problem. (We have not considered search problems for *mixed* Nash equilibria.)

Structural results: MWBFG is an *exact potential* game [41] (Theorem 3.1); so, the decision problem about pure Nash equilibria for these games is trivial and the search problem is in $\mathcal{PLS}$ [32]. We next consider the relation between (mutual) WBFGs and another class of succinct games, namely *weighted, linear-affine congestion games with player-specific (coefficients and) constants* [38]. We prove that every weighted, linear-affine congestion game with player-specific coefficients and constants is *polynomial, sound monomorphic* to a WBFG (Theorem 3.3). In particular, this implies that every weighted, linear-affine congestion game with player-specific constants is polynomial, sound monomorphic to a MWBFG (Corollary 3.5). We also prove that the same hold for weighted, linear-affine *network* congestion games with player-specific constants (Theorem 3.6 and 3.7). We warn the reader that although a network congestion game is a congestion game, these results do *not* follow from Theorem 3.3. This is so since a *network* of size n can succinctly describe a strategy space of size 2^n; hence, transforming a network congestion game into a congestion game may incur an *exponential* blow-up of the instance, and the obvious reduction does not work. Since the search problem for pure Nash equilibria is $\mathcal{PLS}$-complete for weighted, asymmetric network congestion games (with player-specific constants) [38, Theorem 5], Corollary 3.7 implies that the search problem for pure Nash equilibria in MWBFG is also $\mathcal{PLS}$-complete (Corollary 3.8).

Complexity results for payoff-dominant equilibria (Theorem 4.2): We present the **first** complexity results about payoff-dominant equilibria. We first consider the case where m is not fixed and $k \geq 1$ is fixed. For unweighted formulas with $r \geq 1$ fixed or not fixed, the problem is Θ_2^P-complete (Case (2)); for weighted formulas with r not fixed, the problem is Δ_2^P-complete (Case (1)). We next consider the case where k is not fixed and $m \geq 4$ is fixed or not fixed. For unweighted formulas with r not fixed, the problem is Θ_3^P-complete (Case (4)); for weighted formulas with r not fixed, the problem is Δ_3^P-complete (Case (3)). These complexity results about payoff-dominant equilibria in WBFG indicate that allowing an arbitrary number of variables per player has a stronger impact on their complexity than allowing an arbitrary number of players. The decision problem for payoff-dominant equilibria with a *non-fixed* number of players, established complete for Θ_2^P in Theorem 4.2 (Case (2)), is one of the very rare, truly natural complete problems for Θ_2^P; in fact, we feel that it seconds the problem of *Dodgson Election for Lewis Carroll's 1876 Voting System,* established Θ_2^P-complete in [31]. Even more, we feel that the decision problem for payoff-dominant equilibria with a *non-fixed* number of variables of player, established Θ_3^P-complete in Theorem 4.2 (Case (4)), is the *first* truly natural complete problem for Θ_3^P.

Complexity results for pure Nash equilibria (Theorem 5.2): We first consider the case where m is not fixed and $k \geq 1$ is fixed. For (weighted) formulas with $r \geq 1$ fixed or not fixed, the problem is $\mathcal{NP}$-complete (Case (1)); for (weighted) clauses with $r \geq 2$ fixed or not fixed, the problem is $\mathcal{NP}$-complete (Case (2)); for (weighted) clauses with $r = 1$, it is in $\mathcal{P}$ (Case (3)). We next consider the case that k is not fixed and $m \geq 2$ is fixed or not fixed. For (weighted) formulas with $r \geq 1$ fixed or not fixed, the problem is Σ_2^P-complete (Case (4)).

Related Work and Comparison. Since WBFG have a restricted structure, our completeness proofs (for decision problems about pure Nash equilibria) have required more detailed arguments than the ones in [2, 48].

- (Boolean) Circuit games: Recall that boolean formula games form a restricted subclass of boolean circuit games. Observe also that all upper bounds established in this paper for boolean formula games are obviously also valid for circuit games. It is shown [48, Theorem 6.1] that the decision problem in *two-player* circuit games is Σ_2^P-complete; this follows trivially from Theorem 5.2 (Case (4)). Furthermore, it is shown [48, Theorem 6.2] that the decision problem in boolean circuit games is $\mathcal{NP}$-complete; this follows trivially from Theorem 5.2 (Case (1)).
- Turing machine games: Álvarez *et al.* [2] prove that the problem is $\mathcal{NP}$-complete for strategic games in general form for both the non-uniform [2, Theorem 2] and the uniform [2, Theorem 5] cases. Specifically, it follows from either [2, Theorem 2] or [2, Theorem 5] that if m is not fixed, then the problem for (weighted) boolean formulas is $\mathcal{NP}$-complete when $k \geq 1$ is fixed and $r \geq 5$ is fixed or not fixed. This implied result is *weaker* than Theorem 5.2. (Case (1)). Furthermore, Álvarez *et al.* [2] prove that the problem is Σ_2^P-complete for strategic games in implicit form for both the non-uniform [2, Theorem 1 and Corollary 1] and the uniform [2, Theorem 4] cases. Specifically, it follows from [2, Corollary 1] that if k is not fixed, then the problem for (weighted) boolean formulas is Σ_2^P-complete when $m \geq 3$ is fixed or not fixed and $r \geq 5$ is fixed or not fixed. This implied result is *incomparable* to Theorem 5.2 (Case (4)).
- Boolean games: Bonzon *et al.* [10, Proposition 5] had *independently* proved a stronger version of Case (4) in Theorem 5.2 (which holds for $m \geq 3$ fixed or not fixed) with $m \geq 2$ (fixed or not fixed); furthermore,

their result applies to *zero-sum* (two-player) games. Bonzon *et al.* [10, Proposition 6] prove that in the case where k is not fixed, the decision problem for boolean formula games with $m \geq 2$ fixed or not fixed is $\mathcal{NP}$-complete when the formula of each player is in DNF (that is, it is a disjuction of conjunctions of literals). This provides an interesting complement to Theorem 5.2 (Case (2)), which establishes a $\mathcal{NP}$-completeness for formulas with a different restriction.

- Satisfiability games: In his concurrent work, Biló [7] considers *restricted satisfiability games,* where the strategy set of each player is the set of strategies in which the player is allowed to set to 1 one and only one of his variables. It is proved [7, Theorem 3] that the class of restricted satisfiability games with player specific payoffs where all functions are conjunctive encompasses *all* strategic games. Furthermore, Biló [7, Theorem 1] proves that every satisfiability game is an unweighted congestion game. Since every unweighted congestion game is (isomorphic to) an exact potential game and vice versa [41, 47], this result is equivalent to Theorem 3.1. However, Theorem 3.1 provides an exact potential for a MWBFG, which is very simple and intuitive, and it may have further applications. Indeed, this exact potential provided in Theorem 3.1 is explicitly represented with size polynomial in the size of the game. In contrast, the isomorphic potential game constructed from an unweighted congestion game in [41, 51] has exponential size since its resource set is the strategy space of the unweighted congestion game; hence, its exact potential needs also exponential size for its explicit representation.

Some completeness results for pure Nash equilibria in other classes of succinct games have been shown for *graphical games* in [19, 23, 34], and for *weighted (network) congestion games* [40] and *local-effect games* [37] in [20]. The impact of the precise form of bounded rationality (e.g., number of available strategies, size of *influence neighborhood,* symmetries, etc.) on the complexity of pure Nash equilibria for *general* games has been investigated in [12, 23, 28].

2 Background and Framework

Notation and Preliminaries. For a set S, denote as $\mathcal{P}(S)$ the *power set* of S; denote as $|S|$ the cardinality of S. For an integer $n \geq 1$, denote as $[n] = \{1, \ldots, n\}$. Denote as $\geq_{cw}$ the *component-wise ordering* relation on vectors. Denote as $\geq_{le}$ the *lexicographic ordering* relation on boolean vectors. We shall sometimes abbreviate *lexicographically maximum* as *lmax*. For a boolean vector $\mathbf{x}$, $|\mathbf{x}|$ denotes the natural number with binary representation $\mathbf{x}$.

A ***strategic game*** (or ***game*** for short) is a triple $\Gamma = \langle m, (S_i)_{i\in[m]}, (u_i)_{i\in[m]}\rangle$, where m is the number of players, S_i is the ***strategy space*** of player $i \in [m]$, and $u_i : S_1 \times \ldots \times S_m \to \mathbb{R}$ is the ***payoff function*** of player $i \in [m]$. The game Γ is ***finite*** if all strategy spaces are finite; all games considered in this paper will be assumed to be finite. For the game Γ, denote $S = S_1 \times \ldots \times S_m$. To demonstrate reference to Γ, we shall sometimes write $S(\Gamma)$ for Γ. A ***profile*** is a tuple of strategies $\mathbf{s} = \langle s_1, \ldots, s_m\rangle$, one for each player; denote as $\mathbf{s}_{-i}$ the partial profile resulting from eliminating the strategy of player i from $\mathbf{s}$. Given a profile $\mathbf{s}$, a player $i \in [m]$ and a strategy $t \in S_i$, denote as $(\mathbf{s}_{-i}, t) = \langle s_1, \ldots, s_{i-1}, t, s_{i+1}, \ldots, s_m\rangle$; so, $(\mathbf{s}_{-i}, t)$ results by substituting in the profile $\mathbf{s}$ the strategy s_i of player i with t. Associated in the natural way with a profile $\mathbf{s}$ is the ***payoff vector***.

A profile $\mathbf{s} \in S$ is a ***(pure) Nash equilibrium*** [42] for the game Γ if for each player $i \in [m]$, for each strategy $t \in S_i$, $u_i(\mathbf{s}) \geq u_i(\mathbf{s}_{-i}, t)$. Denote as $\mathsf{NE}(\Gamma)$ the set of Nash equilibria of Γ. A (pure) Nash equilibrium $\mathbf{s}$ is called a ***payoff-dominant equilibrium*** for Γ if for each (pure) Nash equilibrium $\mathbf{s}'$, for each player $i \in [m]$, $u_i(\mathbf{s}) \geq u_i(\mathbf{s}')$. Denote as $\mathsf{PD}(\Gamma)$ the set of payoff-dominant equilibria for Γ. To compare, a (pure) Nash equilibrium $\mathbf{s}$ for Γ is ***Pareto-optimal*** for Γ if for each (pure) Nash equilibrium $\mathbf{s}'$, there is a player $i \in [m]$ such that $u_i(\mathbf{s}) \geq u_i(\mathbf{s}')$. Clearly, a payoff-dominant equilibrium is Pareto-optimal (but not vice versa).

Maps. Consider two strategic games $\Gamma = \langle m, (S_i)_{i\in[m]}, (u_i)_{i\in[m]}\rangle$ and $\Gamma' = \langle m, (S'_i)_{i\in[m]}, (u'_i)_{i\in[m]}\rangle$ with the same number of players. A *player map* (or *player bijection*) $\pi : [m] \to [m]$ identifies player $i \in [m]$ for Γ with player $\pi(i) \in [m]$ for Γ'. An *action map* is an m-tuple of *action bijections* $\boldsymbol{\phi} = (\phi_i)_{i\in[m]}$ such that each ϕ_i is a bijection $\phi_i : S_i \to S'_{\pi(i)}$; so, the bijection ϕ_i identifies action $s_i \in S_i$ with action $\phi_i(s_i) \in S'_{\pi(i)}$. A ***bijection pair*** from Γ to Γ' is a pair $\langle \pi, \boldsymbol{\phi}\rangle$ of a player map and an action map. The map $\langle \pi, \boldsymbol{\phi}\rangle$ maps profiles from S to profiles in S' in the natural way; that is, for a profile $\mathbf{s} \in S$, $\langle \pi, \boldsymbol{\phi}\rangle(\mathbf{s}) = \mathbf{s}'$ where for each $i \in [m]$, $s'_{\pi(i)} = \phi_i(s_i)$. A ***Harsanyi-Selten isomorphism*** [30] (from Γ to Γ') is a map $\langle \pi, \boldsymbol{\phi}\rangle$ such that for each player $i \in [m]$, there are constants $\gamma_i > 0$ and δ_i such that for each profile $\mathbf{s} \in S$, $u_{\pi(i)}(\langle \pi, \boldsymbol{\phi}\rangle(\mathbf{s})) = \gamma_i\, u_i(\mathbf{s}) + \delta_i$; then, say that Γ is ***Harsanyi-Selten isomorphic*** to Γ'. Defined earlier by Nash [42], a ***strong isomorphism*** is the special case of a Harsanyi-Selten isomorphism where for each player $i \in [m]$, $\delta_i = 0$. We shall use a relaxation of the Harsanyi-Selten isomorphism which we call a ***Harsanyi-Selten monomorphism***; there, the action map is relaxed to be an m-tuple of *action injections* (which need not be surjective), and the bijection pair $\langle \pi, \boldsymbol{\phi}\rangle$ becomes a ***monomorphism***.

For our purposes, we shall consider an extension of the Harsanyi-Selten monomorphism (from games) to classes of games, which takes computation into account. Consider two classes of strategic games $\mathcal{C}$ and $\mathcal{C}'$. Say that the class $\mathcal{C}$ is ***Harsanyi-Selten monomorphic*** to the class $\mathcal{C}'$ if every game $\Gamma \in \mathcal{C}$ is Harsanyi-Selten monomorphic to some game $\Gamma' \in \mathcal{C}'$, which can be computed from Γ via a map $\lambda : \mathcal{C} \rightarrow \mathcal{C}'$. (For each particular game $\Gamma \in \mathcal{C}$, λ and $\langle \pi, \boldsymbol{\phi} \rangle$ induce together a corresponding map, denoted as $\lambda \circ \langle \pi, \boldsymbol{\phi} \rangle$ by abuse of notation, which maps each profile $\mathbf{s} \in S(\Gamma)$ to the profile $\langle \pi, \boldsymbol{\phi} \rangle(\mathbf{s}) \in S(\lambda(\Gamma))$; denote as $\lambda \circ \langle \pi, \boldsymbol{\phi} \rangle(S(\Gamma))$ the resulting set of images of profiles in $S(\Gamma)$.) Say that the class $\mathcal{C}$ is ***polynomial Harsanyi-Selten monomorphic*** to the class $\mathcal{C}'$ if *(i)* $\mathcal{C}$ is Harsanyi-Selten monomorphic to $\mathcal{C}'$, *(ii)* the map $\lambda : \mathcal{C} \rightarrow \mathcal{C}$ is polynomial time, and *(iii)* for each pair of a game $\Gamma \in \mathcal{C}$ and its image $\lambda(\Gamma) \in \mathcal{C}'$, the map $\langle \pi, \boldsymbol{\phi} \rangle$ can be computed in polynomial time. Clearly, a Harsanyi-Selten isomorphism from Γ to Γ' induces a bijection from $\mathsf{NE}(\Gamma)$ to $\mathsf{NE}(\Gamma')$; a Harsanyi-Selten monomorphism from Γ to Γ' induces an injection from $\mathsf{NE}(\Gamma)$ to $\mathsf{NE}(\Gamma')$. We now define:

Definition 2.1 (Polynomial Sound Monomorphism) *A* ***polynomial sound monomorphism*** *from $\mathcal{C}$ to $\mathcal{C}'$ is a triple $\langle \lambda, \langle \pi, \boldsymbol{\phi} \rangle, \psi \rangle$ where:*

(1) *The class $\mathcal{C}$ is polynomial Harsanyi-Selten monomorphic to the class $\mathcal{C}'$ via the map $\lambda : \mathcal{C} \rightarrow \mathcal{C}'$ and the (Harsanyi-Selten) monomorphism $\langle \pi, \boldsymbol{\phi} \rangle$.*

(2) *For each game $\Gamma \in \mathcal{C}$, ψ_Γ is a function $\psi_\Gamma : \mathsf{NE}(\lambda((\Gamma))) \rightarrow \mathsf{NE}(\Gamma)$; that is, ψ_Γ maps a Nash equilibrium for the game $\lambda(\Gamma) \in \mathcal{C}'$ to a Nash equilibrium for Γ. Then, $\psi := \bigcup_{\Gamma \in \mathcal{C}} \psi_\Gamma$, Furthermore, ψ is a polynomial time map.*

(3) (Soundness Condition) *For each game $\Gamma \in \mathcal{C}$, $\mathsf{NE}(\lambda(\Gamma)) \subseteq \lambda \circ \langle \pi, \boldsymbol{\phi} \rangle(S(\Gamma))$; that is, a Nash equilibrium for the image game $\lambda(\Gamma)$ is necessarily the image (under $\lambda \circ \langle \pi, \boldsymbol{\phi} \rangle$) of some profile of Γ.*

Note that Condition (3) requires that that for any game $\Gamma \in \mathcal{C}$, the Harsanyi-Selten monomorphism $\langle \pi, \boldsymbol{\phi} \rangle$ from Γ to Γ' (from Condition (1)) induces indeed a bijection from $\mathsf{NE}(\Gamma)$ to $\mathsf{NE}(\Gamma')$. Definition 2.1 extends a recent definition of ***Nash homomorphism*** due to Abbott *et al.* [1] (cf. [27]):

Definition 2.2 *A* ***Nash homomorphism*** [1] *from $\mathcal{C}$ to $\mathcal{C}'$ is a pair $\langle \lambda, \psi \rangle$ where:*

$(1)'$ *The map $\lambda : \mathcal{C} \rightarrow \mathcal{C}'$ is a homomorphism.*

$(2)'$ *For each game $\Gamma \in \mathcal{C}$, ψ_Γ is a function $\psi_\Gamma : \mathsf{NE}(\lambda(\Gamma)) \rightarrow \mathsf{NE}(\Gamma)$; that is, ψ_Γ maps a Nash equilibrium for the game $\lambda(\Gamma) \in \mathcal{C}'$ to a Nash equilibrium for Γ. Then, $\psi := \bigcup_{\Gamma \in \mathcal{C}} \psi_\Gamma$. Furthermore, ψ is a polynomial time map.*

The extension (Definition 2.1) imposes *(i)* the requirement that the class $\mathcal{C}$ be polynomial Harsanyi-Selten monomorphic to the class $\mathcal{C}'$ – note that (1)' only requires that $\mathcal{C}$ be homomorphic to $\mathcal{C}'$, and *(ii)* Condition (3).

Potential Games and Congestion Games. Fix a positive vector $\mathbf{b} = \langle b_1, \ldots, b_n \rangle$. Then, a **b*-potential*** for the game Γ is a function $\Phi : S \rightarrow \mathbb{R}$ such that for each profile $\mathbf{s} \in S$, for each player $i \in [m]$ and strategy $s_i' \in S_i$, $u_i(\mathbf{s}_{-i}, s_i') - u_i(\mathbf{s}) = b_i(\Phi(\mathbf{s}_{-i}, s_i') - \Phi(\mathbf{s}))$. A ***vector potential game*** is a game that admits a **w**-potential for some (non-negative) vector **w**. A finite vector potential game has a pure Nash equilibrium [41], which is a local maximizer of the vector potential. An ***exact potential game*** (or ***potential game*** for short) is a **b**-potential game for some constant vector **b**; such a **b**-potential is called an ***exact potential*** (or ***potential*** for short).. Note that if a game Γ is Harsanyi-Selten monomorphic to a (vector) potential game Γ', then Γ is a vector potential game; hence, to prove that a game is vector potential, it suffices to provide a Harsanyi-Selten monomorphism (from it) to a (vector) potential game.

A ***weighted, linear-affine congestion game with player-specific constants*** [38] is a game $\Gamma = \langle m, (S_i)_{i \in [m]}, (u_i)_{i \in [m]} \rangle$ such that: (1) There is an integer $k \geq 2$ such that for each player $i \in [m]$, $S_i \subseteq \mathcal{P}(\{1, 2, \ldots, k\})$. (Equivalently, $S_i \subseteq \{0, 1\}^k$.) (2) There exist families of integers $(\beta_e)_{e \in [k]}$ with $\beta_e \geq 0$ (the *coefficients*), $(\gamma_{ie})_{i \in [m], e \in [k]}$ with $\gamma_{ie} \geq 0$ (the *constants*), and $(w_i)_{i \in [m]}$ with $w_i \geq 1$ (the *weights*) such that for each profile $\mathbf{s} = \langle s_1, \ldots, s_m \rangle$, for each player $i \in [m]$, $u_i(\mathbf{s}) = -\sum_{e \in s_i} \left(\beta_e \cdot \sum_{j \in [m] | e \in s_j} w_j + \gamma_{ie} \right)$. Denote as $\mathsf{WLACGwPSC}$ the class of weighted, linear-affine congestion games with player-specific constants. Clearly, $\mathsf{WLACGwPSC}$ contains the class of *weighted, linear-affine congestion games* [25] where the constants $(\gamma_e)_{e \in [k]}$ are no more player-specific; it is also contained in the class $\mathsf{WLACGwPSC}^2$ of *weighted, linear-affine congestion games with player-specific coefficients and constants* [26], which, in turn, is contained in the general class of *weighted congestion games with player-specific payoff functions* [40]. It is known that $\mathsf{WLACGwPSC}$ admit a vector potential and a pure Nash equilibrium [38, Theorem 6 and Corollary 7]; in contrast, $\mathsf{WLACGwPSC}^2$ do *not* necessarily admit a pure Nash equilibrium [26, Theorem 2].

Weighted Boolean Formula Games. We are now ready for our main definition.

Definition 2.3 (Weighted Boolean Formula Game) *Fix a triple of integers $m \geq 2$, $k \geq 1$ and $r \geq 1$. A game $\Gamma = \langle m, (S_i)_{i\in[m]}, (u_i)_{i\in[m]}\rangle$, is called a **weighted** (m,k,r)**-boolean formula game** (or **weighted boolean formula game** for short) if* (1) *for each player $i \in [m]$, $S_i = \{0,1\}^k$ and* (2) *there is a set $F_i = \{(f,\alpha) \mid f$ is a (km)-ary boolean formula and $\alpha \in \mathbb{N}\}$ with $|F_i| \leq r$ such that for each profile $\langle s_1,\dots,s_m\rangle \in S$, $u_i(s_1,\dots,s_m) = \sum_{(f,\alpha)\in F_i} \alpha \cdot f(s_1,\dots,s_m)$.*

We also write $\Gamma = \langle m,k,r,(F_i)_{i\in[m]}\rangle$. Denote $F = \bigcup_{i\in[m]} F_i$. We use WBFG as an abbreviation for a weighted boolean formula game. An (m,k,r)***-boolean formula game*** is the special case of a weighted (m,k,r)-boolean formula game $\Gamma = \langle m,k,r,(F_i)_{i\in[m]}\rangle$ such that for each pair $(f,\alpha) \in F$, $\alpha = 1$. A ***(weighted)*** (m,k,r)***-boolean clause game*** is the special case of a (weighted) (m,k,r)-boolean formula game $\Gamma = \langle m,k,r,(F_i)_{i\in[m]}\rangle$ such that for each pair $(f,\alpha) \in F$, f is a *clause*. We now formulate a restricted class of WBFGs.

Definition 2.4 *A weighted boolean formula game $\Gamma = \langle m,k,r,(F_i)_{i\in[m]}\rangle$, is called **mutual** if the following holds: For each pair $(f,\alpha) \in F$, if f depends on a variable of player $i \in [m]$, then $(f,\alpha) \in F_i$.*

So, in a mutual weighted boolean formula game, for each pair $(f,\alpha) \in F_i$, if (f,α) depends on a variable x_l with $l \neq m$, then $(f,\alpha) \in F_l$ as well. A mutual weighted boolean formula game will be denoted as MWBFG.

Decision and Search Problems. Let $m \in \{2,3,\dots\}$, $k \in \{1,2,\dots\}$ and $r \in \{1,2,\dots\}$. We formulate and study the following decision problems regarding payoff-dominant and pure Nash equilibria:

PROBLEM:	PROBLEM:	GIVEN:
$\mathsf{WBF\text{-}PD}_d(m,k,r)$	$\mathsf{WBF\text{-}NASH}_d(m,k,r)$	A weighted (m,k,r)-boolean formula game Γ.
$\mathsf{BF\text{-}PD}_d(m,k,r)$	$\mathsf{BF\text{-}NASH}_d(m,k,r)$	An (m,k,r)-boolean formula game Γ.
$\mathsf{WBC\text{-}PD}_d(m,k,r)$	$\mathsf{WBC\text{-}NASH}_d(m,k,r)$	A weighted (m,k,r)-boolean clause game Γ.
$\mathsf{BC\text{-}PD}_d(m,k,r)$	$\mathsf{BC\text{-}NASH}_d(m,k,r)$	An (m,k,r)-boolean clause game Γ.
QUESTION: Is $\mathsf{PD}(\Gamma) \neq \emptyset$?	QUESTION: Is $\mathsf{NASH}(\Gamma) \neq \emptyset$?	

We shall consider corresponding search problems $\mathsf{WBF\text{-}PD\text{-}NASH}_s(m,k,r)$, $\mathsf{BF\text{-}PD\text{-}NASH}_s(m,k,r)$, $\mathsf{WBC\text{-}PD\text{-}NASH}_s(m,k,r)$, $\mathsf{BC\text{-}PD\text{-}NASH}_s(m,k,r)$ and $\mathsf{WBF\text{-}NASH}_s(m,k,r)$, $\mathsf{BF\text{-}NASH}_s(m,k,r)$, $\mathsf{WBC\text{-}NASH}_s(m,k,r)$, $\mathsf{BC\text{-}NASH}_s(m,k,r)$. We shall often consider the case where some of the parameters m, k, and r are not restricted to a fixed value. In this case, such a parameter gets the value $*$. For example, for $k \in \{1,2,\dots\}$ and $r \in \{1,2,\dots\}$, we define $\mathsf{BF\text{-}NASH}_d(*,k,r) = \bigcup_{m\geq 2} \mathsf{BF\text{-}NASH}_d(m,k,r)$. Denote as $\mathsf{MWBF\text{-}NASH}_s(*,*,*)$ the search problem for pure Nash equilibria in MWBFG when none of the parameters m, k and r are fixed.

Complexity Theory. We assume some basic familiarity of the reader with the central complexity classes as articulated, for example, in the textbook of Papadimitriou [43]. Specifically, we shall treat $\mathcal{P}$, *polynomial local search* $\mathcal{PLS}$ [32], $\mathcal{NP}$, the *polynomial hierarchy* $\mathcal{PH}$ [39, 49] and $\mathcal{PSPACE}$. In particular, we will encounter $\Delta_2^P = \mathcal{P}^{\mathcal{NP}}$, $\Sigma_2^P = \mathcal{NP}^{\mathcal{NP}}$ and $\Delta_3^P = \mathcal{P}^{\mathcal{NP}^{\mathcal{NP}}}$; the *bounded query classes* $\Theta_2^P = \mathcal{P}^{\mathcal{NP}[\log n]}$ [44] (the class of all languages that can be decided via parallel access to $\mathcal{NP}$) and $\Theta_3^P = \mathcal{P}^{\Sigma_2^P[\log n]} = \mathcal{P}^{\mathcal{NP}^{\mathcal{NP}}[\log n]}$ [53] (the class of all languages that can be decided via parallel access to Σ_2^P) at the (initial) intermediate levels of $\mathcal{PH}$; the *function classes* $\mathrm{F}\mathcal{P}$, $\mathrm{F}\mathcal{NP}$ and $\mathrm{F}\Sigma_2^P$, of all function problems associated with languages in $\mathcal{P}$, $\mathcal{NP}$ and Σ_2^P, respectively. (Clearly, $\mathcal{NP} \subseteq \Theta_2^P \subseteq \Delta_2^P \subseteq \Sigma_2^P \subseteq \Theta_3^P \subseteq \Delta_3^P$.)

We recall some prominent decision problems (in the form of their underlying languages), which we shall use in our later reductions: In what follows, H is a propositional formula; each of $\mathbf{x}$ and $\mathbf{y}$ is a vector of n boolean variables; C is a boolean clause.

$$
\begin{aligned}
\mathsf{SAT} &= \{H(\mathbf{x}) \mid \exists \mathbf{a} \in \{0,1\}^n (H(\mathbf{a}) = 1)\} \\
\mathsf{CNFSAT} &= \{H(\mathbf{x}) \mid H(\mathbf{x}) \text{ has conjunctive normal form and } \exists \mathbf{a} \in \{0,1\}^n (H(\mathbf{a}) = 1)\} \\
\Sigma_2\text{-}\mathsf{QBF} &= \{H(\mathbf{x},\mathbf{y}) \mid \exists \mathbf{a} \in \{0,1\}^n \forall \mathbf{b} \in \{0,1\}^n (H(\mathbf{a},\mathbf{b}) = 1)\} \\
\Delta_2\text{-}\mathsf{QBF} &= \{H(\mathbf{x}) \mid \text{ the lexmax } \mathbf{a} \text{ with } (H(\mathbf{a}) = 1) \text{ has } a_n = 1\} \\
\Theta_2\text{-}\mathsf{QBF} &= \{\langle H(\mathbf{x}), 1^m\rangle \mid \text{ the lmax } \mathbf{a} \text{ with } (H(\mathbf{a}) = 1) \;\&\; |\mathbf{a}| \leq m \text{ has } a_n = 1\} \\
\Delta_3\text{-}\mathsf{QBF} &= \{H(\mathbf{x},\mathbf{y}) \mid \text{ the lmax } \mathbf{a} \text{ with } \forall \mathbf{b} \in \{0,1\}^k \; (H(\mathbf{a},\mathbf{b}) = 1) \text{ has } a_n = 1\} \\
\Theta_3\text{-}\mathsf{QBF} &= \{\langle H(\mathbf{x}), 1^m\rangle \mid \text{ the lmax } \mathbf{a} \text{ with } \forall \mathbf{b} \in \{0,1\}^k \; (H(\mathbf{a},\mathbf{b}) = 1) \text{ and } |\mathbf{a}| \leq m \text{ has } a_n = 1\}
\end{aligned}
$$

SAT and CNFSAT are the prototypical $\mathcal{NP}$-complete problems. Σ_2-QBF was shown Σ_2^P-complete by Stockmeyer [49] and Wrathall [54]. Δ_2-QBF was shown Δ_2^P-complete by Krentel [36] and Wagner [52]. Θ_2-QBF was shown Θ_2^P-complete by Wagner [52], and similarly, Θ_3-QBF is Θ_3^P-complete. It is an easy consequence of results by Vollmer and Wagner [50] that Δ_3-QBF is Δ_3^P-complete. Consider a restricted subclass Σ_2-RQBF of Σ_2-QBF to the domain $\mathsf{R} = \{H(\mathbf{x}, \mathbf{y}) \mid \forall \mathbf{a} \in \{0,1\}^n \exists \mathbf{b} \in \{0,1\}^n (H(\mathbf{a}, \mathbf{b}) = 1)\}$ thus,

$$\Sigma_2\text{-}\mathsf{RQBF} \quad = \quad \{H(\mathbf{x}, \mathbf{y}) \in \mathsf{R} \mid \exists \mathbf{a} \in \{0,1\}^n \forall \mathbf{b} \in \{0,1\}^n (H(\mathbf{a}, \mathbf{b}) = 1)\}\,;$$

So, Σ_2-RQBF is the restriction of Σ_2-QBF to all formulas $H(\mathbf{x}, \mathbf{y})$ such that $\forall \mathbf{a} \in \{0,1\}^n \exists \mathbf{b} \in \{0,1\}^n (H(\mathbf{a}, \mathbf{b}) = 1)$. It is straightforward to prove that Σ_2-RQBF is Σ_2^P-complete. (See Appendix.) Similarly, we consider restricted subclasses

$$\begin{array}{rcl} \Delta_3\text{-}\mathsf{RQBF} & = & \{H(\mathbf{x}, \mathbf{y}) \in \mathsf{R} \mid \text{ the lmax } \mathbf{a} \text{ with } \forall \mathbf{b} \in \{0,1\}^k \ (H(\mathbf{a}, \mathbf{b}) = 1) \text{ has } a_n = 1\} \\ \Theta_3\text{-}\mathsf{RQBF} & = & \{\langle H(\mathbf{x}), 1^m \rangle \in \mathsf{R} \times \{1^m\} \mid \text{ the lmax } \mathbf{a} \text{ with } \forall \mathbf{b} \in \{0,1\}^k \ (H(\mathbf{a}, \mathbf{b}) = 1) \ \& \ |\mathbf{a}| \leq m \text{ has } a_n = 1\} \end{array}$$

of Δ_3-QBF and Θ_3-QBF, respectively. It is again straightforward to prove that Δ_3-RQBF is Δ_3^P-complete and Θ_3-RQBF is Θ_3^P-complete.

3 Structure

We prove:

Theorem 3.1 (**MWBFG is Exact Potential**) *Consider the* MWBFG $\Gamma = \langle m, k, r, (F_i)_{i \in [m]} \rangle$. *Then, the function* $\Phi : (\{0,1\}^k)^m \rightarrow \mathbb{R}$ *with* $\Phi(\mathbf{s}) = \sum_{\langle f, \alpha \rangle \in F} \alpha \cdot f(\mathbf{s})$ *is an exact potential for* Γ.

Proof: Consider an arbitrary profile $\mathbf{s} \in S$ and a strategy $t_i \in \{0,1\}^k$ of player $i \in [m]$. Then,

$$\begin{aligned} & \Phi(\mathbf{s}_{-i}, t_i) - \Phi(\mathbf{s}) \\ = \ & \sum_{\langle f, \alpha \rangle \in F} \alpha \cdot f(\mathbf{s}_{-i}, t_i) - \sum_{\langle f, \alpha \rangle \in F} \alpha \cdot f(\mathbf{s}) \\ = \ & \sum_{\langle f, \alpha \rangle \in F_i} \alpha \cdot f(\mathbf{s}_{-i}, t_i) + \sum_{\langle f, \alpha \rangle \in F \setminus F_i} \alpha \cdot f(\mathbf{s}_{-i}, t_i) - \sum_{\langle f, \alpha \rangle \in F_i} \alpha \cdot f(\mathbf{s}) - \sum_{\langle f, \alpha \rangle \in F \setminus F_i} \alpha \cdot f(\mathbf{s}) \\ = \ & \sum_{\langle f, \alpha \rangle \in F_i} \alpha \cdot f(\mathbf{s}_{-i}, t_i) - \sum_{\langle f, \alpha \rangle \in F_i} \alpha \cdot f(\mathbf{s}) + \sum_{\langle f, \alpha \rangle \in F \setminus F_i} \alpha \cdot (f(\mathbf{s}_{-i}, t_i) - f(\mathbf{s}))\,. \end{aligned}$$

Since Γ is a mutual weighted boolean formula game, it follows that for each pair $\langle f, \alpha \rangle \in F \setminus F_i$, $f(s_1, \ldots, s_m)$ does *not* depend on s_i; hence, for each pair $\langle f, \alpha \rangle \in F \setminus F_i$, $f(\mathbf{s}_{-i}, t_i) = f(\mathbf{s})$. Hence,

$$\Phi(\mathbf{s}_{-i}, t_i) - \Phi(\mathbf{s}) \quad = \quad \sum_{\langle f, \alpha \rangle \in F_i} \alpha \cdot f(\mathbf{s}_{-i}, t_i) - \sum_{\langle f, \alpha \rangle \in F_i} \alpha \cdot f(\mathbf{s}) \quad = \quad u_i(\mathbf{s}_{-i}, t_i) - u_i(\mathbf{s})\,;$$

hence, Φ is an exact potential for Γ, as needed. ■

An inspection to the proof reveals that the assumption that player variables and formulas are boolean is *not* essential: mutuality alone suffices for the existence of an exact potential. Theorem 3.1 implies:

Corollary 3.2 (1) *Every* MWBFG *has a pure Nash equilibrium;* (2) MWBF-NASH$_d(*, *, *) \in \mathcal{PLS}$.

We now prove:

Theorem 3.3 *The class of weighted, linear-affine congestion games with player-specific coefficients and constants is polynomial, sound monomorphic to the class of weighted boolean formula games.*

In the proof, we will identify a set $t \subseteq \{1, \ldots, k\}$ with the *characteristic vector* $\langle \chi_t(1), \ldots, \chi_t(k) \rangle$, where χ_t is the *characteristic function* for t: for each $e \in [k]$, $\chi_t(e) = 1$ if $e \in t$ and 0 otherwise. For a boolean variable x, set $x^{\chi_t(e)} = x$ if $\chi_t(e) = 1$ and $\overline{x}$ otherwise.

Proof: Here is a polynomial, sound monomorphism $\langle \lambda, \langle \pi, \phi \rangle, \psi \rangle$. from WLACGwPSC^2 to WBFG. We first define the action of λ on any WLACGwPSC^2 $\Gamma = \langle m, (S_i)_{i \in [m]}, (u_i)_{i \in [m]} \rangle$, with $S_i \subseteq \{0,1\}^k$ for each $i \in [m]$.

Each player $i \in [n]$ has variables $x_i = \langle x_{i1}, \ldots, x_{ik} \rangle$; so, $S'_i = \{0,1\}^k$. The set F_i consists of:

	Boolean formula	Weight
For each resource $e \in [k]$:	$f_{ie}(x_1, \ldots, x_m) = \overline{x_{ie}}$	$\alpha_{ie} = \delta_{ie} \cdot w_i$
For each player $j \in [m]$ and resource $e \in [k]$:	$f_{ije}(x_1, \ldots, x_m) = \overline{x_{ie}} \bigvee \overline{x_{je}}$	$\alpha_{ije} = \beta_{ie} \cdot w_i \cdot w_j$
For each strategy $t \in S_i$ (so, $t \subseteq \{1, \ldots, k\}$):	$f_{it}(x_1, \ldots, x_m) = \bigwedge_{e \in [k]} x_{ie}^{\chi_t(e)}$	$\alpha_{it} = w_i \cdot \sum_{e \in [k]} \left(\beta_{ie} \cdot \sum_{j \in [m]} w_j + \delta_{ie} \right) + 1$

Set π and $\boldsymbol{\phi}$ to be the identity maps, respectively. Furthermore, set ψ_Γ to be the identity map; so, ψ_Γ maps a Nash equilibrium $\mathbf{s}'$ (for Γ') to itself. We now show Conditions (1), (2) and (3) in Definition 2.1. Clearly, both maps λ and $\langle \pi, \boldsymbol{\phi} \rangle$ are polynomial. Furthermore, since for each player $i \in [m]$, $S_i \subseteq S'_i$, the map $\langle \pi, \boldsymbol{\phi} \rangle$ is a homomorphism. For Condition (1), we proceed to show that $\langle \pi, \boldsymbol{\phi} \rangle$ is Harsanyi-Selten. Fix any profile $\mathbf{s}' = \langle s'_1, \ldots, s'_m \rangle$ (for Γ'), where for each player $i \in [m]$, $s_i = \langle s'_{i1}, \ldots, s'_{ik} \rangle$. (Note that it need not be the case that $\mathbf{s}'$ is a profile for Γ.) Note that for each player $i \in [m]$,

$$\begin{aligned} u_{\pi(i)}(\mathbf{s}') &= u'_i(\mathbf{s}') \\ &= \sum_{e \in [k]} \alpha_{ie} \cdot f_{ie}(\mathbf{s}') + \sum_{j \in [m]} \sum_{e \in [k]} \alpha_{ije} \cdot f_{ije}(\mathbf{s}') + \sum_{t \in S_i} \alpha_{it} \cdot f_{it}(\mathbf{s}') \\ &= \underbrace{\sum_{e \in [k]} \alpha_{ie} \cdot (1 - \overline{f_{ie}(\mathbf{s}')})}_{\Sigma_1(\mathbf{s}')} + \underbrace{\sum_{j \in [m]} \sum_{e \in [k]} \alpha_{ije} \cdot (1 - \overline{f_{ije}(\mathbf{s}')})}_{\Sigma_2(\mathbf{s}')} + \underbrace{\sum_{t \in S_i} \alpha_{it} \cdot f_{it}(\mathbf{s}')}_{\Sigma_3(\mathbf{s}')} . \end{aligned}$$

- For $\Sigma_1(\mathbf{s}')$, note that

$$\Sigma_1(\mathbf{s}') = \sum_{e \in [k]} \delta_{ie} \cdot w_i \cdot (1 - s'_{ie}) = \sum_{e \in [k]} \delta_{ie} \cdot w_i - \sum_{e \in [k]} \delta_{ie} \cdot w_i \cdot s'_{ie} = w_i \sum_{e \in [k]} \delta_{ie} - w_i \sum_{e \in s'_i} \delta_{ie} .$$

- For $\Sigma_2(\mathbf{s}')$, note that

$$\Sigma_2(\mathbf{s}') = \sum_{j \in [m]} \sum_{e \in [k]} \beta_{ie} \cdot w_i \cdot w_j - \sum_{j \in [m]} \sum_{e \in [k]} \beta_{ie} \cdot w_i \cdot w_j \left((s_{ie} \bigwedge s_{je} \right) .$$

  Note that $s'_{ie} \bigwedge s'_{je} = 1$ if and only if $e \in s'_i$ and $e \in s'_j$. Hence,

$$\Sigma_2(\mathbf{s}') = w_i \sum_{j \in [m]} \sum_{e \in [k]} \beta_{ie} \cdot w_j - w_i \cdot \sum_{e \in s'_i} \sum_{j | e \in s'_j} \beta_{ie} \cdot w_j .$$

Hence,

$$\begin{aligned} \Sigma_1(\mathbf{s}') + \Sigma_2(\mathbf{s}') &= w_i \sum_{e \in [k]} \delta_{ie} - w_i \sum_{e \in s'_i} \delta_{ie} + w_i \sum_{j \in [m]} \sum_{e \in [k]} \beta_{ie} \cdot w_j - w_i \cdot \sum_{e \in s'_i} \sum_{j | e \in s'_j} \beta_{ie} \cdot w_j \\ &= w_i \sum_{e \in [k]} \left(\beta_{ie} \cdot \sum_{j \in [m]} w_j + \delta_{ie} \right) - w_i \sum_{e \in s'_i} \left(\beta_{ie} \cdot \sum_{j | e \in s'_j} w_j + \delta_{ie} \right) . \end{aligned}$$

- For $\Sigma_3(\mathbf{s}')$, note that

$$\begin{aligned} \Sigma_3(\mathbf{s}') &= \sum_{t \in S_i} \left(w_i \cdot \sum_{e \in [k]} \left(\beta_{ie} \cdot \sum_{j \in [m]} w_j + \delta_{ie} \right) + 1 \right) \cdot \bigwedge_{e \in [k]} s_{ie}^{\chi_t(e)} \\ &= \left(w_i \cdot \sum_{e \in [k]} \left(\beta_{ie} \cdot \sum_{j \in [m]} w_j + \delta_{ie} \right) + 1 \right) \cdot \sum_{t \in S_i} \bigwedge_{e \in [k]} (s'_{ie})^{\chi_t(e)} . \end{aligned}$$

  We observe that for each $t \in S_i$, $\bigwedge_{e \in [k]} (s'_{ie})^{\chi_t(e)} = 1$ if and only if ($e \in s_i$ if and only if $e \in t$) if and only if $s'_i = t$. Thus, $\sum_{t \in S_i} \bigwedge_{e \in [k]} (s'_{ie})^{\chi_t(e)} = \sum_{t \in S_i} (s'_i = t)$, where $(s'_i = t) = 1$ if and only if $s'_i = t$. Since $\sum_{t \in S_i} (s'_i = t) = \chi_{S_i}(s'_i)$, it follows that

$$\Sigma_3(\mathbf{s}') = \left(w_i \cdot \sum_{e \in [k]} \left(\beta_{ie} \cdot \sum_{j \in [m]} w_j + \delta_{ie} \right) + 1 \right) \cdot \chi_{S_i}(s'_i) .$$

We continue to prove:

Lemma 3.4 *The map $\langle \pi, \boldsymbol{\phi} \rangle$ is Harsanyi-Selten. and sound.*

Lemma 3.4 completes the proof of Condition (1) and establishes Condition (3). To prove Condition (2), note that Lemma 3.4 implies that the map $\langle \pi, \boldsymbol{\phi} \rangle$ induces a Harsanyi-Selten bijection from $\mathsf{NE}(\Gamma)$ to $\mathsf{NE}(\Gamma')$. Since both π and $\boldsymbol{\phi}$ are identity, this bijection is also identity as well as its inverse. So, the identity map is a bijection from $\mathsf{NE}(\Gamma')$ to $\mathsf{NE}(\Gamma)$. Since Ψ_Γ is this identity map, Condition (2) follows. The proof that $\langle \lambda, \langle \pi, \boldsymbol{\phi} \rangle, \psi \rangle$ is a polynomial, sound Nash-Harsanyi-Selten homomorphism is now complete. ■

An inspection to the proof of Theorem 3.3 reveals that if we had *player-independent* coefficients $(\beta_e)_{e \in [k]}$ (as opposed to player-specific coefficients $(\beta_{ie})_{i \in [m], e \in [k]}$) in the original game Γ, the resulting WBFG $\lambda(\Gamma)$ would be mutual. Hence, Theorem 3.3 immediately implies:

Corollary 3.5 *The class of weighted, linear-affine congestion games with player-specific constants is polynomial, sound monomorphic to the class of mutual weighted boolean formula games.*

We continue to prove:

Theorem 3.6 *The class of weighted, linear-affine network congestion games with player-specific coefficients and constants is polynomial, sound monomorphic to the class of weighted boolean formula games.*

The proof of Theorem 3.6 follows the same structure as the proof of Theorem 3.3. The only difference in the construction is that for each player $i \in [m]$, for each strategy $t \in S_i$, we replace the formula $f_{it}(x_1, \ldots, x_m) = \bigwedge_{e \in [k]} x_{ie}^{\chi_t(e)}$ (which describes the strategy t of player i) by another (polynomial size) formula that describes the admissible paths for player i in the network (from his *source* to his *destination*). Besides a (single) *simple* path from source to destination, this formula may yield a collection of *cycles* that are disjoint from the single path and mutually disjoint as well. However, these cases cannot correspond to Nash equilibria since the payoff of player i can be increased by eliminating these disjoint cycles. We omit further proof details. Similar to Corollary 3.5, we obtain:

Corollary 3.7 *The class of weighted, linear-affine network congestion games with player-specific constants is polynomial, sound monomorphic to the class of mutual weighted boolean formula games.*

Since the search problem for pure Nash equilibria in weighted, asymmetric network congestion games with player-specific constants is $\mathcal{PLS}$-complete [38, Theorem 5], Corollary 3.1 and Corollary 3.7 immediately imply:

Corollary 3.8 $\mathsf{MWBF\text{-}NASH}_d(*, *, *)$ *is $\mathcal{PLS}$-complete.*

By Theorem 3.1, Corollary 3.7 immediately implies a known result from Mavronicolas *et al.* [38]: Every weighted, linear-affine congestion game with player-specific constants has a vector potential.

4 Payoff-Dominant Equilibria

We show:

Theorem 4.1 (Upper Complexity Bounds for Payoff-Dominant Equilibria). *Let $m \in \{2, 3, \ldots\}$, $k \in \{1, 2, \ldots\}$ and $r \in \{1, 2, \ldots, *\}$. Then:* (1) $\mathsf{WBF\text{-}PD}_d(m, k, r) \in \mathcal{P}$. (2) $\mathsf{BF\text{-}PD}_d(*, k, r) \in \Theta_2^P$. (3) $\mathsf{WBF\text{-}PD}_d(*, k, r) \in \Delta_2^P$. (4) $\mathsf{BF\text{-}PD}_d(*, *, r) \in \Theta_3^P$. (5) $\mathsf{WBF\text{-}PD}_d(*, *, r) \in \Delta_3^P$.

We now show:

Theorem 4.2 (Completeness Results for Payoff-Dominant Equilibria) *We have:*

(1) $\mathsf{WBF\text{-}PD}_d(*, k, *)$ *is* Δ_2^P*-complete for* $k \in \{1, 2, \ldots\}$.
(2) $\mathsf{BF\text{-}PD}_d(*, k, r)$ *is* Θ_2^P*-complete for* $k \in \{1, 2, \ldots\}$ *and* $r \in \{1, 2, \ldots, *\}$.
(3) $\mathsf{WBF\text{-}PD}_d(m, *, *)$ *is* Δ_3^P*-complete for* $m \in \{4, 5, \ldots, *\}$.
(4) $\mathsf{BF\text{-}PD}_d(m, *, *)$ *is* Θ_3^P*-complete for* $m \in \{4, 5, \ldots, *\}$.

Proof: We consider each case separately:

(1) It suffices to prove that $\mathsf{WBF\text{-}PD}_d(*, 1, *)$ is Δ_2^P-complete. Membership in Δ_2^P follows from Theorem 4.1 (case (3)). To show Δ_2^P-hardness, we establish that $\Delta_2\text{-}\mathsf{QBF} \leq_m^P \mathsf{WBF\text{-}PD}_d(*, 1, *)$.

Consider a propositional formula $H(\mathbf{x})$ with $\mathbf{x} = \langle x_1, \ldots, x_n \rangle$. Assume, without loss of generality, that $H(0^{n-1}1, \mathbf{b}) = 1$ for all vectors $\mathbf{b} \in \{0,1\}^n$. Construct a weighted $(n+4, 1, n)$- boolean formula game Γ_H as follows, where $\mathbf{z} = \langle z_1, z_2, z_3, z_4 \rangle$:

Player	Variable	Boolean formulas	Weights
$i \in [n]$	x_i	$f_i(\mathbf{x}, \mathbf{z}) = 0$	$\alpha_i = 1$
$n+1$	z_1	$f_{n+1}(\mathbf{x}, \mathbf{z}) = H(\mathbf{x}) \bigvee (z_1 \oplus z_2)$	$\alpha_{n+1} = 1$
$n+2$	z_2	$f_{n+1}(\mathbf{x}, \mathbf{z}) = H(\mathbf{x}) \bigvee (z_1 \oplus z_2 \oplus 1)$	$\alpha_{n+2} = 1$
$n+3$	z_3	$f_{n+3,i}(\mathbf{x}, \mathbf{z}) = x_i$, $i \in [n]$	$\alpha_{n+3,i} = 2^{n-i}$, $i \in [n]$
$n+4$	z_4	$f_{n+4}(\mathbf{x}, \mathbf{z}) = x_n$	$\alpha_{n+3} = 1$

We use $\mathbf{a}$ and $\mathbf{b}$ to denote vectors of values for the boolean variables in $\mathbf{x}$ and $\mathbf{z}$, respectively. We remark that the payoffs of all players $i \in [n]$ are *constant*; the payoffs of players $n+3$ and $n+4$ are *independent* of their strategies. Hence, neither players $i \in [n]$ nor players $n+3$ and $n+4$ matter for a pure Nash equilibrium. Now the proof is completed by showing:

Lemma 4.3 *1. Assume that* $H(\mathbf{a}) = 1$. *Then, for all vectors* $\mathbf{b} \in \{0,1\}^4$, $(\mathbf{a}, \mathbf{b})$ *is a Nash equilibrium for* Γ_H *with* $u(\mathbf{a}, \mathbf{b}) = \langle \underbrace{0, \ldots, 0}_{n}, 1, 1, \sum_{i \in [n]} a_i \cdot 2^{n-i}, a_n \rangle$.

2. Assume that $H(\mathbf{a}) = 0$. *Then, for all vectors* $\mathbf{b} \in \{0,1\}^4$, $(\mathbf{a}, \mathbf{b})$ *is not a Nash equilibrium for* Γ_H.

3. $H(\mathbf{x}) \in \Delta_2\text{-}\mathsf{QBF}$ *if and only if* Γ_H *has a best pure Nash equilibrium.*

(2) Membership in Θ_2^P follows from Theorem 4.1 (case (2)). For the hardness, the proof follows the same structure as for case (1). The only difference in the reduction is that player $n+3$ is replaced by players $(n+3, l)$ for $l \in [n]$ with formula $(|\mathbf{x}| \geq l)$. Proof details are omitted.

(3) It suffices to prove that $\mathsf{WBF\text{-}PD}_d(4, *, *)$ is Δ_2^P-complete. For the sake of clarity, we first prove the weaker claim that $\mathsf{WBF\text{-}PD}_d(5, *, *)$ is Δ_3^P-complete. At the end, we will extend the claim to $\mathsf{WBF\text{-}PD}_d(4, *, *)$. Membership in Δ_3^P follows from Theorem 4.1 (case (5)). To prove that $\mathsf{WBF\text{-}PD}_d(5, *, *)$ is Δ_3^P-hard, we establish that $\Delta_3\text{-}\mathsf{QBF} \leq_m^P \mathsf{WBF\text{-}PD}_d(5, *, *)$.

Consider a propositional formula $H(\mathbf{x}, \mathbf{y})$ with $\mathbf{x} = \langle x_1, \ldots, x_k \rangle$ and $\mathbf{y} = \langle y_1, \ldots, y_k \rangle$. Assume, without loss of generality, that *(i)* for every vector $\mathbf{b} \in \{0,1\}^n$, $H(0^{n-1}1, \mathbf{b}) = 1$. Recall that we can assume, without loss of generality, that *(ii)* for every vector $\mathbf{a} \in \{0,1\}^n$, there is some vector $\mathbf{b}_{\mathbf{a}}^1$ such that $H(\mathbf{a}, \mathbf{b}_{\mathbf{a}}^1) = 1$. Construct a weighted $(5, n, n)$-boolean formula game Γ_H as follows:

Player	Variables	Boolean formulas	Weights
1	$\mathbf{x}$	$f_1(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{v}) = 0$	$\alpha_1 = 1$
2	$\mathbf{y}$	$f_2(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{v}) = H(\mathbf{x}, \mathbf{y}) \oplus H(\mathbf{x}, \mathbf{z})$	$\alpha_2 = 1$
3	$\mathbf{z}$	$f_3(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{v}) = H(\mathbf{x}, \mathbf{y}) \oplus H(\mathbf{x}, \mathbf{z}) \oplus 1$	$\alpha_3 = 1$
4	v_1	$f_{4i}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{v}) = x_i$ for $i \in [n]$	$\alpha_{4i} = 2^{n-i}$ for $i \in [n]$
5	v_2	$f_5(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{v}) = x_k$	$\alpha_5 = 1$

Now the proof is completed by showing:

Lemma 4.4 *1. Consider a vector* $\mathbf{a} \in \{0,1\}^n$ *such that for all vectors* $\mathbf{b} \in \{0,1\}^n$, $H(\mathbf{a}, \mathbf{b}) = 1$. *Then, for every triple* $(\mathbf{b}, \mathbf{c}, \mathbf{d})$, $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ *is a pure Nash equilibrium for* Γ_H *with payoff vector* $u(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = \langle 0, 0, 1, \sum_{j \in [n]} a_j \cdot 2^{n-j}, a_n \rangle$.

2. Consider a vector $\mathbf{a} \in \{0,1\}^n$ *such that there is some vector* $\mathbf{b}_{\mathbf{a}}^0 \in \{0,1\}^n$ *such that* $H(\mathbf{a}, \mathbf{b}_{\mathbf{a}}^0) = 0$. *Then for every triple* $(\mathbf{b}, \mathbf{c}, \mathbf{d})$ *with* $\mathbf{d} = \langle d_1, d_2 \rangle$, $(\mathbf{a}, \mathbf{d}, \mathbf{c}, \mathbf{d})$ *is not a pure Nash equilibrium for* Γ_H.

3. $H(\mathbf{x}, \mathbf{y}) \in \Delta_3\text{-}\mathsf{QBF}$ *if and only if* Γ_H *has a payoff-dominant Nash equilibrium.*

To extend the claim to $\mathsf{WBF\text{-}PD}_d(4,*,*)$, we eliminate player 5 and transfer the boolean formula f_5 to player 2. Details of the proof that this modification works are omitted. Hence, $\mathsf{WBF\text{-}PD}_d(4,*,*)$ is Δ_3^P-complete.

(4) Membership in Θ_3^P follows from Theorem 4.1 (case (4)). The hardness proof follows the same structure as for case (3). The only difference in the reduction is that player 4 has the formulas $(|\mathbf{x}| \geq 1), (|\mathbf{x}| \geq 2), \ldots, (|\mathbf{x}| \geq n)$. Details of the proof that this modification works are omitted.

The proof is now complete. ■

Open Problem 4.1 *Find the complexity of* $\mathsf{WBF\text{-}PD}_d(*,k,r)$ *for* $k \in \{1,2,\ldots,*\}$ *and* $r \in \{1,2,\ldots,*\}$. *Find the complexities of* $\mathsf{BF\text{-}PD}_d(m,*,r)$ *and* $\mathsf{BF\text{-}PD}_d(m,*,r)$ *for* $m \in \{2,3,\ldots,*\}$ *and* $r \in \{1,2,\ldots,*\}$. *Find the complexity of* $\mathsf{BF\text{-}PD}_d(m,*,*)$ *for* $m \in \{2,3\}$.

5 Pure Nash Equilibria

We observe:

Proposition 5.1 (Upper Bounds for Pure Nash Equilibria) *Let* $m \in \{2,3,\ldots\}$, $k \in \{1,2,\ldots\}$ *and* $r \in \{1,2,\ldots,*\}$. *Then:*
(1) $\mathsf{WBF\text{-}NASH}_d(m,k,r) \in \mathcal{P}$ *(and* $\mathsf{WBF\text{-}NASH}_s(m,k,r) \in \mathrm{F}\mathcal{P}$*).*
(2) $\mathsf{WBF\text{-}NASH}_d(*,k,r) \in \mathcal{NP}$ *(and* $\mathsf{WBF\text{-}NASH}_s(*,k,r) \in \mathrm{F}\mathcal{NP}$*).*
(3) $\mathsf{WBF\text{-}NASH}_d(m,*,r) \in \Sigma_2^P$ *(and* $\mathsf{WBF\text{-}NASH}_s(m,*,r) \in \mathrm{F}\Sigma_2^P$*).*
(4) $\mathsf{WBF\text{-}NASH}_d(*,*,r) \in \Sigma_2^P$ *(and* $\mathsf{WBF\text{-}NASH}_s(*,*,r) \in \mathrm{F}\Sigma_2^P$*).*

We show:

Theorem 5.2 (Completeness Results for Pure Nash Equilibria) *We have:*
(1) *For* $k \in \{1,2,\ldots\}$ *and* $r \in \{1,2,\ldots,*\}$, $\mathsf{BF\text{-}NASH}_d(*,k,r)$ *is* $\mathcal{NP}$*-complete.*
(2) *For* $k \in \{1,2,\ldots\}$ *and* $r \in \{2,3,\ldots,*\}$,
$\mathsf{BC\text{-}NASH}_d(*,k,r)$ *is* $\mathcal{NP}$*-complete.*
(3) *For* $k \in \{1,2,\ldots,*\}$, $\mathsf{WBC\text{-}NASH}_d(*,k,1) \in \mathcal{P}$. *In fact, every weighted* $(*,k,1)$*-boolean clause game has a pure Nash equilibrium.*
(4) *For* $m \in \{3,4,\ldots,*\}$ *and* $r \in \{1,2,\ldots,*\}$,
$\mathsf{BF\text{-}NASH}_d(m,*,r)$ *is* Σ_2^P*-complete*

Proof: We consider each case separately:

(1) It suffices to prove that $\mathsf{BF\text{-}NASH}_d(*,1,1)$ is $\mathcal{NP}$-complete (and $\mathsf{BF\text{-}NASH}_s(*,1,1)$ is $\mathrm{F}\mathcal{NP}$-complete). Membership follows from Proposition 5.1 (case (2)). To prove that $\mathsf{BF\text{-}NASH}_d(*,1,1)$ is $\mathcal{NP}$-hard, we establish a reduction from SAT to $\mathsf{BF\text{-}NASH}_d(*,1,1)$. Consider a propositional formula $H(x_1,\ldots,x_n)$. Construct a BFG $\Gamma_H = \langle n+2, (\{0,1\})_{i\in[n+2]}, (F_i)_{i\in[n+2]}\rangle$ as follows:

Player	Variable	Boolean function
$i \in [n]$	x_i	$f_i(\mathbf{x}) = 0$
$n+1$	x_{n+1}	$f_{n+1}(\mathbf{x}) = H(x_1,\ldots,x_n) \bigvee (x_{n+1} \oplus x_{n+2})$
$n+2$	x_{n+2}	$f_{n+2}(\mathbf{x}) = H(x_1,\ldots,x_n) \bigvee (x_{n+1} \oplus x_{n+2} \oplus 1)$

We need to show that $H(x_1,\ldots,x_n)$ is satisfiable if and only if Γ_H has a pure Nash equilibrium. To complete the proof, we show:

Lemma 5.3 *For all* a_{n+1}, a_{n+2}, $H(a_1,\ldots a_n) = 1$ *if and only if* $\langle a_1,\ldots a_n, a_{n+1}, a_{n+2}\rangle$ *is a pure Nash equilibrium of* Γ_H.

(2) It suffices to prove that $\mathsf{BC\text{-}NASH}_d(*,1,2)$ is $\mathcal{NP}$-complete (and $\mathsf{BC\text{-}NASH}_s(*,1,2)$ is $\mathrm{F}\mathcal{NP}$-complete). Membership follows from Proposition 5.1 (case (2)). To prove that $\mathsf{BC\text{-}NASH}_d(*,1,2)$ is $\mathcal{NP}$-hard, we establish a reduction from $\mathsf{CNF\text{-}SAT}$ to $\mathsf{BC\text{-}NASH}_d(*,1,2)$. Consider a propositional formula $H(x_1,\ldots,x_n) = \bigwedge_{j\in[m]} C_j(x_1,\ldots,x_n)$ in CNF. Construct a BCG $\Gamma_H = \langle n+2m, 1, 2, \{C_{jk}\}_{j\in[n+2m],k\in[2]}\rangle$ as follows:

Player	Variable	Clause $C_{j1}(\mathbf{x},\mathbf{y},\mathbf{z})$	Clause $C_{j2}(\mathbf{x},\mathbf{y},\mathbf{z})$
$j \in [n]$	x_j	0	0
$n+j$ with $j \in [m]$	y_j	$C_j(\mathbf{x}) \bigvee y_j \bigvee z_j$	$C_j(\mathbf{x}) \bigvee \overline{y_j} \bigvee \overline{z_j}$
$n+m+j$ with $j \in [m]$	z_j	$C_j(\mathbf{x}) \bigvee y_j \bigvee \overline{z_j}$	$C_j(\mathbf{x}) \bigvee \overline{y_j} \bigvee z_j$

In the sequel, we will use vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ to denote boolean values for the boolean variables in the vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, respectively. We remark that all players $j \in [n]$ do not matter for a pure Nash equilibrium since their payoffs are *constant*. To complete the proof, we show:

Lemma 5.4 *For all* $\mathbf{b}, \mathbf{c}$, $H(\mathbf{a}) = 1$ *if and only if* $\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle$ *is a pure Nash equilibrium for* Γ_H.

(3) Consider any game Γ as input to WBC-NASH$_d(*, k, 1)$. Then, Γ has a Nash equilibrium where each player $i \in [m]$ chooses his variables $\mathbf{x}^i = \langle x_{i1}, \ldots, x_{ik} \rangle \in \{0,1\}^k$ as follows: if x_{ij} appears unnegated in his clause function f_i, then $x_{ij} := 1$, else $x_{ij} := 0$. So, WBC-NASH$_d(*, k, 1) \in \mathcal{P}$.

(4) It suffices to prove that BF-NASH$_d(3, *, 1)$ is Σ_2^P-complete (and BF-NASH$_s(3, *, 1)$ is $\mathrm{F}\Sigma_2^P$-complete). Membership follows from Proposition 5.1 (case (3)). To prove that BF-NASH$_d(3, *, 1)$ is Σ_2^P-hard, we establish a reduction from Σ_2-RQBF to BF-NASH$_d(3, *, 1)$. Consider a propositional formula $H(\mathbf{x}, \mathbf{y})$ with $\mathbf{x} = \langle x_1, \ldots, x_n \rangle$ and $\mathbf{y} = \langle y_1, \ldots, y_n \rangle$. Construct a BFG $\Gamma_H = \langle 3, n, 1, \{f_i\}_{i \in [3]} \rangle$ as follows:

Player	Variables	Boolean formula
1	$\mathbf{x} \in \{0,1\}^n$	$f_1(\mathbf{x},\mathbf{y},\mathbf{z}) = 0$
2	$\mathbf{y} \in \{0,1\}^n$	$f_2(\mathbf{x},\mathbf{y},\mathbf{z}) = H(\mathbf{x},\mathbf{y}) \oplus H(\mathbf{x},\mathbf{z})$
3	$\mathbf{z} \in \{0,1\}^n$	$f_3(\mathbf{x},\mathbf{y},\mathbf{z}) = H(\mathbf{x},\mathbf{y}) \oplus H(\mathbf{x},\mathbf{z}) \oplus 1$

We need to show that $H(\mathbf{x}, \mathbf{y}) \in \Sigma_2$-RQBF if and only if Γ_H has a pure Nash equilibrium.[§] To complete the proof, we show:

Lemma 5.5 *For all* $\mathbf{c}$, $\forall \mathbf{b}(H(\mathbf{a}, \mathbf{b}) = 1)$ *if and only if* $\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle$ *is a pure Nash equilibrium of* Γ_H.

The proof is now complete. ■

Open Problem 5.1 *Find the complexities of* BF-NASH$_d(2, *, r)$*,* BF-NASH$_d(2, *, r)$*,* WBF-NASH$_d(2, *, r)$ *and* WBF-NASH$_d(2, *, r)$ *for* $r \in \{1, 2, \ldots, *\}$.

Open Problem 5.2 *Find the complexities of* BC-NASH$_d(m, *, r)$*,* BC-NASH$_d(m, *, r)$*,* WBC-NASH$_d(m, *, r)$ *and* WBC-NASH$_d(m, *, r)$ *for* $m \in \{2, 3, \ldots, *\}$ *and* $r \in \{1, 2, \ldots, *\}$.

6 Open Problems

Our work raises far more interesting open problems than it answers; we mention a few of them here. On the most concrete level, what is the complexity of other refinements of pure Nash equilibria (e.g., Pareto-optimal equilibria, dominating equilibria, etc.) in WBFGs? We feel that our work opens up *Pandora's box* with regard to investigating the *selection problem* for (pure) Nash equilibria [30] from the point of view of computational complexity. Our results identified natural refinements of pure Nash equilibrium (including itself) that are complete for some of the lowest levels of $\mathcal{PH}$ (not exceeding the third level). Is there for *any* level k of $\mathcal{PH}$, some (natural) refinement of pure Nash equilibrium that is complete there (for some of Σ_k^P, Δ_k^P or Π_k^P)? More ambitiously, is there some (natural) refinement of pure Nash equilibrium that is complete for $\mathcal{PSPACE}$?

Theorem 3.1 determines a sufficient condition (namely, *mutuality*) for a WBFG to admit an exact potential; a corresponding necessary condition is missing. Finally, Corollaries 3.5 and 3.7 identify two classes of succinct games that are polynomially embeddable in the class of MWBFGs. Which other classes of succinct games are so embeddable?

Acknowledgements. We would like to thank Paul Spirakis and Karsten Tiemann for many helpful discussions and comments on earlier versions of our work.

[§] We **warn** the reader against the formula $G(\mathbf{x}, \mathbf{y}) \equiv 0$ for all $\mathbf{x}$ and $\mathbf{y}$. Notice that in the constructed game Γ_G, $f_1 \equiv 0$, $f_2 \equiv 0$ and $f_3 \equiv 1$; so, every profile is a pure Nash equilibrium for Γ_G. However, this is *not* a contradiction, since $G \notin \mathsf{R}$, which implies that G is *not* a valid input for Σ_2-RQBF (even though $G \notin \Sigma_2$-RQBF). In fact, we used reduction from Σ_2-RQBF (as opposed to Σ_2-QBF) in order to eliminate such degenerate formulas from consideration.

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A Proof that Σ_2-RQBF is Σ_2^P-complete

By reduction from Σ_2-QBF. Given a propositional formula $H(\mathbf{x}, \mathbf{y})$, construct the propositional formula $H'(\mathbf{x}; u, \mathbf{y}; v) = (H(\mathbf{x}, \mathbf{y}) \bigwedge v) \bigvee \overline{v}$. We prove that $H(\mathbf{x}, \mathbf{y}) \in \Sigma_2$-QBF if and only if $H'(\mathbf{x}; u, \mathbf{y}; v) \in \Sigma_2$-RQBF.

- Assume first that $H(\mathbf{x}, \mathbf{y}) \in \Sigma_2$-QBF. So there is $\mathbf{a}_0$ such that for all $\mathbf{b}$, $H(\mathbf{a}_0, \mathbf{b}) = 1$
  - Recall $\mathbf{a}_0$ and c_0. Fix arbitrary $\mathbf{b}$ and d. Then, $H(\mathbf{a}_0; c_0, \mathbf{b}; d) = (H(\mathbf{a}_0, \mathbf{b}) \bigwedge d) \bigvee \overline{d} = (1 \bigwedge d) \bigvee \overline{d} = 1$.
  - Fix arbitrary $\mathbf{a}$ and c. Set $\mathbf{b} := \mathbf{b}_0$ (where $\mathbf{b}_0$ is arbitrary) and $d := 1$. Then, $H(\mathbf{a}; c, \mathbf{b}_0; 1) = 1$.

  So, $H'(\mathbf{x}; u, \mathbf{y}; v) \in \Sigma_2$-RQBF.
- Assume now that $H'(\mathbf{x}; u, \mathbf{y}; v) \in \Sigma_2$-RQBF. Then, there is some $\mathbf{a}_0$ and c_0 such that for all $\mathbf{b}$ and d, $H'(\mathbf{a}_0; c_0, \mathbf{b}; d) = 1$. Set $d := 1$. It follows that for all $\mathbf{b}$, $H'(\mathbf{a}_0; c_0, \mathbf{b}; d) = H(\mathbf{a}_0, \mathbf{b}) = 1$. This implies that $H(\mathbf{x}, \mathbf{y}) \in \Sigma_2$-QBF.

B Proof of Lemma 3.4

Consider any profile $\mathbf{s}' \in S'(\Gamma')$ such that $\mathbf{s}' = \langle \pi, \boldsymbol{\phi} \rangle(\mathbf{s})$ for some profile $\mathbf{s} \in S(\Gamma)$. Hence, for each player $i \in [m]$, $\chi_{S_i}(s_i) = 1$. Since $\langle \pi, \boldsymbol{\phi} \rangle$ is the identity map, $\mathbf{s}' = \mathbf{s}$. It follows that for each player $i \in [m]$, $\chi_{S_i}(s'_i) = 1$ as well. Hence,

$$\begin{aligned}
u'_i(\langle \pi, \boldsymbol{\phi} \rangle \mathbf{s}) &= w_i \sum_{e \in [k]} \left(\beta_{ie} \sum_{j \in [m]} w_j + \delta_{ie} \right) - w_i \sum_{e \in s'_i} \left(\beta_{ie} \sum_{j | e \in s'_j} w_j + \delta_{ie} \right) + w_i \sum_{e \in [k]} \left(\beta_{ie} \sum_{j \in [m]} w_j + \delta_{ie} \right) + 1 \\
&= -w_i \sum_{e \in s'_i} \left(\beta_{ie} \sum_{j | e \in s'_j} w_j + \delta_{ie} \right) . + 2\, w_i \sum_{e \in [k]} \left(\beta_{ie} \sum_{j \in [m]} w_j + \delta_{ie} \right) + 1 \\
&= -w_i \sum_{e \in s'_i} \left(\beta_{ie} \sum_{j | e \in s_j} w_j + \delta_{ie} \right) + 2\, w_i \sum_{e \in [k]} \left(\beta_{ie} \sum_{j \in [m]} w_j + \delta_{ie} \right) + 1 \\
&= w_i \cdot u_i(\mathbf{s}) + 2\, w_i \sum_{e \in [k]} \left(\beta_{ie} \sum_{j \in [m]} w_j + \delta_{ie} \right) + 1 \,.
\end{aligned}$$

Since $w_i > 0$, it follows that $\langle \pi, \boldsymbol{\phi} \rangle$ is Harsanyi-Selten.

We continue to prove that $\langle \pi, \boldsymbol{\phi} \rangle$ is sound. Consider any profile $\mathbf{s}' \in S'(\Gamma')$ such that there is no profile $\mathbf{s} \in S(\Gamma)$ such that $\mathbf{s}' \in \langle \pi, \boldsymbol{\phi} \rangle(\mathbf{s})$. Since π and $\boldsymbol{\phi}$ are the identity maps, it follows that $\mathbf{s}' \notin S(\Gamma)$. Hence, there is a player $i \in [m]$ such that $s'_i \notin S_i$; so, $\chi_{S_i}(s'_i) = 0$. It follows that

$$u'_i(\mathbf{s}') \quad \leq \quad w_i \cdot \sum_{e \in [k]} \left(\beta_{ie} \cdot \sum_{j \in [m]} w_j + \delta_{ie} \right) .$$

Assume now that player i switches from strategy $s'_i \notin S_i$ to strategy $s''_i \in S_i$; thus, $\chi_{S_i}(s''_i) = 1$. Hence,

$$\begin{aligned}
u'_i(\mathbf{s}'_{-i}, s''_i) &= 2w_i \cdot \sum_{e \in [k]} \left(\beta_{ie} \cdot \sum_{j \in [m]} w_j + \delta_{ie} \right) + 1 - w_i \cdot \sum_{e \in s''_i} \left(\beta_{ie} \cdot \sum_{j \in [m] | e \in s'_j} w_j + \delta_{ie} \right) \\
&\geq 2w_i \cdot \sum_{e \in [k]} \left(\beta_{ie} \cdot \sum_{j \in [m]} w_j + \delta_{ie} \right) + 1 - w_i \cdot \sum_{e \in [k]} \left(\beta_{ie} \cdot \sum_{j \in [m]} w_j + \delta_{ie} \right) \\
&= w_i \cdot \sum_{e \in [k]} \left(\beta_{ie} \cdot \sum_{j \in [m]} w_j + \delta_{ie} \right) + 1 \,.
\end{aligned}$$

Hence, $u'_i(\mathbf{s}'_{-i}, s''_i) > u'_i(\mathbf{s}')$ and $\mathbf{s}'$ is not a Nash equilibrium for Γ'. The Soundness Condition follows.

C Proof of Lemma 4.1

Fix a weighted boolean formula game $\Gamma = \langle m, k, r, (F_i)_{i\in[m]}\rangle$. Consider the following algorithm A for the decision problem $\mathsf{WBF\text{-}PD}_d(m, k, r)$:

1. For each player $i \in [m]$, compute $\mu_i = \max_{\mathbf{s}\in\mathsf{NE}(\Gamma)} u_i(\mathbf{s})$ by a binary search using queries of the kind:

   (Q_1) "Does Γ have a pure Nash equilibrium $\mathbf{s}$ such that $u_i(\mathbf{s}) \geq \ell$?"

2. Output the answer (YES or NO) returned to the query:

   (Q_2) "Does Γ have a pure Nash equilibrium $\mathbf{s}$ such that for each player $i \in [m]$, $u_i(\mathbf{s}) = \mu_i$?"

Clearly, the algorithm A uses a polynomial number of queries (of the kind (Q_1) or (Q_2)). We proceed to establish upper bounds on the complexity of the algorithm A in each case:

(1) Note that if m and k are fixed, then each kind of query (Q_1) and (Q_2) is a $\mathcal{P}$-query. Hence, $\mathsf{WBF\text{-}PD}_d(m, k, r) \in \mathcal{P}^{\mathcal{P}} = \mathcal{P}$.

(2) Note that if k and r are fixed, then each kind of query (Q_1) and (Q_2) is an $\mathcal{NP}$-query. Since there are no weights, the total number of queries is $O(\lg m)$. Hence, $\mathsf{WBF\text{-}PD}_d(m, k, r) \in \mathcal{P}^{\mathcal{NP}(\lg m)} = \Theta_2^P$.

(3) Note that if k and r are fixed, then each kind of query (Q_1) and (Q_2) is an $\mathcal{NP}$-query. Hence, $\mathsf{WBF\text{-}PD}_d(m, k, r) \in \mathcal{P}^{\mathcal{NP}} = \Delta_2^P$.

(4) Note that if r is fixed, then each kind of query (Q_1) and (Q_2) is a Σ_2^P-query. Since there are no weights, the total number of queries is $O(\lg m)$. Hence, $\mathsf{WBF\text{-}PD}_d(m, k, r) \in \mathcal{P}^{\Sigma_2^P(\lg m)} = \Theta_3^P$.

(5) Note that if r is fixed, then each kind of query (Q_1) and (Q_2) is a Σ_2^P-query. Hence, $\mathsf{WBF\text{-}PD}_d(m, k, r) \in \mathcal{P}^{\Sigma_2^P} = \Delta_3^P$.

The proof is now complete.

D Proof of Lemma 4.3.1

We only need to consider the players $n+1$ and $n+2$. Since $H(\mathbf{a}) = 1$, $u_{n+1}(\mathbf{a}, \mathbf{b}) = u_{n+2}(\mathbf{a}, \mathbf{b}) = 1$, which cannot be further increased. Hence, $\langle \mathbf{a}, \mathbf{b}\rangle$ is a Nash equilibrium for Γ_H with $u(\mathbf{a}, \mathbf{b}) = \langle \underbrace{0, \ldots, 0}_{n}, 1, 1, \sum_{i\in[n]} a_i \cdot 2^{n-i}, a_n\rangle$.

E Proof of Lemma 4.3.2

Since $H(\mathbf{a}, \mathbf{b}) = 0$, $u_1(\mathbf{a}, \mathbf{b}) = b_1 \oplus b_2$ and $u_1(\mathbf{a}, \mathbf{b}) = b_1 \oplus b_2 \oplus 1$. If $b_1 \oplus b_2 = 0$, then player $n+1$ can increase its payoff $u_{n+1}(\mathbf{a}, \mathbf{b}) = 0$ by flipping b_1. If $b_1 \oplus b_2 = 1$, then player $n+2$ can increase its payoff $u_{n+1}(\mathbf{a}, \mathbf{b}) = 0$ by flipping b_2. Hence, $(\mathbf{a}, \mathbf{b})$ is not a Nash equilibrium.

F Proof of Lemma 4.3.3

Assume first that $H(\mathbf{x}) \in \Delta_2$-QBF. Fix the lmax $\mathbf{a} \in \{0, 1\}^n$ such that $H(\mathbf{a}) = 1$. It follows that $a_n = 1$. Lemma 4.3.1 implies that $(\mathbf{a}, \mathbf{b})$ is a Nash equilibrium for Γ_H with $u(\mathbf{a}, \mathbf{b}) = \langle \underbrace{0, \ldots, 0}_{n}, 1, 1, \sum_{i\in[n]} a_i \cdot 2^{n-i}, a_n\rangle$.

We now prove that $(\mathbf{a}, \mathbf{b})$ is a payoff-dominant equilibrium for Γ_H. Fix any Nash equilibrium $(\mathbf{a}', \mathbf{b}')$ for Γ_H. By Lemma 4.3.2, $H(\mathbf{a}') = 1$. Hence, by Lemma 4.3.1, $u(\mathbf{a}, \mathbf{b}) = \langle \underbrace{0, \ldots, 0}_{n}, 1, 1, \sum_{i\in[n]} a'_i \cdot 2^{n-i}, a'_n\rangle$. Since $\mathbf{a}$ is lmax among all vectors $\mathbf{a}'' \in \{0, 1\}^n$ such that $H(\mathbf{a}'') = 1$, $\mathbf{a} \geq_{le} \mathbf{a}'$, Hence, $\sum_{i\in[n]} a_i \cdot 2^{n-i} \geq \sum_{i\in[n]} a'_i \cdot 2^{n-i}$. Since $a_n = 1$, $a_n \geq a'_n$. It follows that $(\mathbf{a}, \mathbf{b})$ is a payoff-dominant equilibrium for Γ_H.

Assume now that $H(\mathbf{x}) \notin \Delta_2$-QBF. Assume, by way of contradiction, that Γ_H has a payoff-dominant equilibrium. Fix the lmax $\mathbf{a} \in \{0,1\}^n$ such that $H(\mathbf{a}) = 1$. By definition of Δ_2-QBF, it follows that $a_k = 0$. Fix now any vector $\mathbf{b}$. Lemma 4.3.1 implies that $(\mathbf{a}, \mathbf{b})$ is a Nash equilibrium for Γ_H with payoff vector $u(\mathbf{a}, \mathbf{b}) = \langle \underbrace{0, \ldots, 0}_{n}, 1, 1, \sum_{i \in [n]} a_i \cdot 2^{n-i}, a_n \rangle$. By Lemma 4.3.2, Lemma 4.3.1 implies that each pure Nash equilibrium has payoff vector $\langle \underbrace{0, \ldots, 0}_{n}, 1, 1, x, y \rangle$, where $x \geq 0$ and $y \in \{0,1\}$. Since $\mathbf{a}$ is lmax, this implies that $(\mathbf{a}, \mathbf{b})$ is a payoff-dominant equilibrium with payoff vector $u(\mathbf{a}, \mathbf{b}) = \langle \underbrace{0, \ldots, 0}_{n}, 1, 1, \sum_{i \in [n]} a_i \cdot 2^{n-i}, 0 \rangle$.

By assumption, it follows that $H(0^{n-1}1, \mathbf{b}) = 1$. Lemma 4.3 implies that $(0^{n-1}1, \mathbf{b})$ is a pure Nash equilibrium for Γ_H with payoff vector $\langle \underbrace{0, \ldots, 0}_{n}, 1, 1, 1, 1 \rangle$. Since $(\mathbf{a}, \mathbf{b})$ is a payoff-dominant equilibrium, it follows that $\langle \underbrace{0, \ldots, 0}_{n}, 1, 1, \sum_{i \in [n]} a_i \cdot 2^{n-i}, 0 \rangle \geq_{cw} \langle \underbrace{0, \ldots, 0}_{n}, 1, 1, 1, 1 \rangle$. A contradiction.

G Proof of Lemma 4.4.1

Fix any arbitrary triple $(\mathbf{b}, \mathbf{c}, \mathbf{d})$. We examine all players:

- For player 2, fix any arbitrary strategy $\mathbf{b}' \in \{0,1\}^n$. The assumption implies that $u_2(\mathbf{a}, \mathbf{b}', \mathbf{c}, \mathbf{d}) = f_2(\mathbf{a}, \mathbf{b}', \mathbf{c}, \mathbf{d}) = H(\mathbf{a}, \mathbf{b}') \oplus H(\mathbf{a}, \mathbf{c}) = 1 \oplus 1 = 0$. So, player 2 cannot unilaterally increase his payoff 0 in $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$.
- For player 3, fix any arbitrary strategy $\mathbf{c}' \in \{0,1\}^n$. The assumption implies that $u_2(\mathbf{a}, \mathbf{b}, \mathbf{c}', \mathbf{d}) = f_2(\mathbf{a}, \mathbf{b}, \mathbf{c}', \mathbf{d}) = H(\mathbf{a}, \mathbf{b}) \oplus H(\mathbf{a}, \mathbf{c}') \oplus 1 = 1 \oplus 1 \oplus 1 = 1$. So, player 2 cannot unilaterally increase his payoff 1 in $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$.
- For player 4, note that each boolean formula f_{4i}, $i \in [n]$, is *independent* of his strategy d_1. So, player 4 cannot unilaterally increase his payoff $\sum_{j \in [n]} a_j \cdot 2^{n-j}$ in $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$.
- For player 5, note that the boolean formula f_5 is *independent* of his strategy d_2. So, player 5 cannot unilaterally increase his payoff a_k in $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$.

It follows that $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ is a pure Nash equilibrium for Γ_H with payoff vector

$$u(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \quad = \quad \langle 0, 0, 1, \sum_{j \in [k]} a_j \cdot 2^{k-j}, a_k \rangle \,,$$

as needed.

H Proof of Lemma 4.4.2

Fix any arbitrary triple $(\mathbf{b}, \mathbf{c}, \mathbf{d})$. We proceed by case analysis on the pair of values $H(\mathbf{a}, \mathbf{b})$ and $H(\mathbf{a}, \mathbf{c})$.

- Assume that $H(\mathbf{a}, \mathbf{b}) = H(\mathbf{a}, \mathbf{c}) = 0$. Then, $u_2(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = f_2(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = H(\mathbf{a}, \mathbf{b}) \oplus H(\mathbf{a}, \mathbf{c}) = 0 \oplus 0 = 0$. However, by assumption *(ii)*, $u_2(\mathbf{a}, \mathbf{b}^1_{\mathbf{a}}, \mathbf{c}, \mathbf{d}) = f_2(\mathbf{a}, \mathbf{b}^1_{\mathbf{a}}, \mathbf{c}, \mathbf{d}) = H(\mathbf{a}, \mathbf{b}^1_{\mathbf{a}}) \oplus H(\mathbf{a}, \mathbf{c}) = 1 \oplus 0 = 1$. So, player 2 can increase his payoff in $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ by changing his strategy $\mathbf{b}$ to $\mathbf{b}^1_{\mathbf{a}}$. Hence, $\langle \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \rangle$ is not a pure Nash equilibrium.
- Assume that $H(\mathbf{a}, \mathbf{b}) = 0$ and $H(\mathbf{a}, \mathbf{c}) = 1$. Then, $u_3(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = f_3(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = H(\mathbf{a}, \mathbf{b}) \oplus H(\mathbf{a}, \mathbf{c}) \oplus 1 = 0 \oplus 1 \oplus 1 = 0$. However, by assumption *(ii)*, $u_3(\mathbf{a}, \mathbf{b}, \mathbf{b}^0_{\mathbf{a}}, \mathbf{d}) = f_3(\mathbf{a}, \mathbf{b}, \mathbf{b}^0_{\mathbf{a}}, \mathbf{d}) = H(\mathbf{a}, \mathbf{b}) \oplus H(\mathbf{a}, \mathbf{b}^0_{\mathbf{a}}) \oplus 1 = 0 \oplus 0 \oplus 1 = 1$. So, player 3 can increase his payoff in $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ by changing his strategy $\mathbf{c}$ to $\mathbf{b}^0_{\mathbf{a}}$. Hence, $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ is not a pure Nash equilibrium.
- Assume that $H(\mathbf{a}, \mathbf{b}) = 1$ and $H(\mathbf{a}, \mathbf{c}) = 0$. Then, $u_3(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = f_3(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = H(\mathbf{a}, \mathbf{b}) \oplus H(\mathbf{a}, \mathbf{c}) \oplus 1 = 1 \oplus 0 \oplus 1 = 0$. However, by assumption *(ii)*, $u_3(\mathbf{a}, \mathbf{b}, \mathbf{b}^1_{\mathbf{a}}, \mathbf{d}) = f_3(\mathbf{a}, \mathbf{b}, \mathbf{b}^1_{\mathbf{a}}, \mathbf{d}) = H(\mathbf{a}, \mathbf{b}) \oplus H(\mathbf{a}, \mathbf{b}^1_{\mathbf{a}}) \oplus 1 = 1 \oplus 1 \oplus 1 = 1$. So, player 3 can increase his payoff in $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ by changing his strategy $\mathbf{c}$ to $\mathbf{b}^1_{\mathbf{a}}$. Hence, $\langle \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \rangle$ is not a pure Nash equilibrium.

- Assume that $H(\mathbf{a},\mathbf{b}) = H(\mathbf{a},\mathbf{c}) = 1$. Then, $u_2(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) = f_2(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) = H(\mathbf{a},\mathbf{b}) \oplus H(\mathbf{a},\mathbf{c}) = 1 \oplus 1 = 0$. However, by assumption *(ii)*, $u_2(\mathbf{a},\mathbf{b}^0_{\mathbf{a}},\mathbf{c},\mathbf{d}) = f_2(\mathbf{a},\mathbf{b}^0_{\mathbf{a}},\mathbf{c},\mathbf{d}) = H(\mathbf{a},\mathbf{b}^0_{\mathbf{a}}) \oplus H(\mathbf{a},\mathbf{c}) = 0 \oplus 1 = 1$. So, player 2 can increase his payoff in $(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$ by changing his strategy $\mathbf{b}$ to $\mathbf{b}^0_{\mathbf{a}}$. Hence, $\langle \mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d} \rangle$ is not a pure Nash equilibrium.

The proof is now complete.

I Proof of Lemma 4.4.3

Assume first that $H(\mathbf{x},\mathbf{y}) \in \Delta_3$-QBF. Choose the lmax $\mathbf{a} \in \{0,1\}^n$ such that for all vectors $\mathbf{b} \in \{0,1\}^n$, $H(\mathbf{a},\mathbf{b}) = 1$ (By assumption *(i)*, such a vector $\mathbf{a}$ exists.) By definition of Δ_3-QBF, it follows that $a_n = 1$. By Lemma 4.4.1, it follows that for every triple $(\mathbf{b},\mathbf{c},\mathbf{d})$, $(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$ is a pure Nash equilibrium for Γ_H with payoff vector $u(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) = \langle 0, 0, 1, \sum_{j\in[n]} a_j \cdot 2^{n-j}, a_n \rangle$.

We now prove that $(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$ is a payoff-dominant Nash equilibrium for Γ_H. Consider any pure Nash equilibrium $(\mathbf{a}',\mathbf{b}',\mathbf{c}',\mathbf{d}')$ for Γ_H. Lemma 4.4.2 implies that for all vectors $\mathbf{b}'' \in \{0,1\}^n$, $H(\mathbf{a},\mathbf{b}'') = 1$. Lemma 4.4.1 implies that the payoff vector for $(\mathbf{a}',\mathbf{b}',\mathbf{c}',\mathbf{d}')$ is $\langle 0, 0, 1, \sum_{j\in[n]} a'_j \cdot 2^{n-j}, a'_n \rangle$. Since $\mathbf{a}$ is the lmax vector such that for all vectors $\mathbf{b} \in \{0,1\}^n$, $H(\mathbf{a},\mathbf{b}) = 1$, it follows that $\sum_{j\in[n]} a_j \cdot 2^{n-j} \geq \sum_{j\in[n]} a'_j \cdot 2^{n-j}$. Since $a_n = 1$, $a_n \geq a'_n$. It follows that $\langle 0, 0, 1, \sum_{j\in[n]} a_j \cdot 2^{n-j}, a_n \rangle \geq_{cw} \langle 0, 0, 1, \sum_{j\in[n]} a'_j \cdot 2^{n-j}, a'_n$. Since $(\mathbf{a}',\mathbf{b}',\mathbf{c}',\mathbf{d}')$ was chosen as an arbitrary pure Nash equilibrium, this implies that $(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$ is payoff-dominant.

Assume now that $H(\mathbf{x},\mathbf{y}) \notin \Delta_3$-QBF. Assume, by way of contradiction, that Γ_H has a payoff-dominant Nash equilibrium. Fix the lmax $\mathbf{a} \in \{0,1\}^n$ such that for all vectors $\mathbf{b} \in \{0,1\}^n$, $H(\mathbf{a},\mathbf{b}) = 1$. By definition of Δ_3-QBF, it follows that $a_n = 0$. Fix any triple $(\mathbf{b},\mathbf{c},\mathbf{d})$. Lemma 4.4.1 implies that $(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$ is a pure Nash equilibrium for Γ_H with payoff vector $\langle 0, 0, 1, \sum_{j\in[n]} a_j \cdot 2^{n-j}, 0 \rangle$. By Lemma 4.4.2, Lemma 4.4.1 implies that that all pure Nash equilibria have payoff vectors $\langle 0, 0, 1, x, y \rangle$ where $x \geq 0$ and $y \in \{0,1\}$. Since $\mathbf{a}$ is lmax, this implies that $(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$ is the best pure Nash equilibrium with payoff vector $\langle 0, 0, 1, \sum_{j\in[n]} a_j \cdot 2^{n-j}, 0 \rangle$.

By assumption *(i)*, it follows that $H(0^{n-1}1,\mathbf{b}) = 1$. Lemma 4.4 implies that $(0^{n-1}1,\mathbf{b},\mathbf{c},\mathbf{d})$ is a pure Nash equilibrium for Γ_H with payoff vector $\langle 0,0,1,1,1 \rangle$. Since $(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$ is the best pure Nash equilibrium, $\langle 0, 0, 1, \sum_{j\in[n]} a_j \cdot 2^{n-j}, 0 \rangle \geq_{cw} \langle 0,0,1,1,1 \rangle$. A contradiction.

J Proof of Lemma 5.3

Assume first that $H(a_1,\ldots,a_n) = 1$. Thus, $f_{n+1}(a_1,\ldots,a_{n+2}) = f_{n+2}(a_1,\ldots,a_{n+2}) = 1$, so that no player can improve its payoff $u_i(a_1,\ldots,a_n,a_{n+1},a_{n+2}) = 1$ for any $a_{n+1}, a_{n+2} \in \{0,1\}$. So, $\langle a_1,\ldots,a_n,a_{n+1},a_{n+2} \rangle$ is a pure Nash equilibrium for Γ_H (for any $a_{n+1}, a_{n+2} \in \{0,1\}$).

Assume now that $\langle a_1,\ldots,a_n,a_{n+1},a_{n+2} \rangle$ is a pure Nash equilibrium for Γ_H. Then:

- Player $n+1$ cannot increase his payoff by switching a_{n+1} to $\overline{a_{n+1}}$. It follows that

$$H(a_1,\ldots,a_n) \bigvee (a_{n+1} \oplus a_{n+2}) \quad \geq \quad H(a_1,\ldots,a_n) \bigvee (\overline{a_{n+1}} \oplus a_{n+2}) \,.$$

  Hence, either $H(a_1,\ldots,a_n) = 1$ or $a_{n+1} \oplus a_{n+2} = 1$.

- Player $n+2$ cannot increase his payoff by switching a_{n+2} to $\overline{a_{n+2}}$. It follows that

$$H(a_1,\ldots,a_n) \bigvee (a_{n+1} \oplus a_{n+2} \oplus 1) \quad \geq \quad H(a_1,\ldots,a_n) \bigvee (a_{n+1} \oplus \overline{a_{n+2}} \oplus 1) \,.$$

  Hence, either $H(a_1,\ldots,a_n) = 1$ or $a_{n+1} \oplus a_{n+2} \oplus 1 = 1$.

It follows that $H(a_1,\ldots,a_n) = 1$, and H is satisfiable.

K Proof of Lemma 5.4

Assume first that $H(\mathbf{a}) = 1$. It follows that for each $j \in [m]$, $C_j(\mathbf{a}) = 1$. Hence, $C_{j1}(\mathbf{a},\mathbf{b},\mathbf{c}) = C_{j2}(\mathbf{a},\mathbf{b},\mathbf{c}) = 1$ for all players $n+j$ with $1 \leq j \leq 2m$. So, the payoff of each player $n+j$, with $1 \leq j \leq 2m$, cannot increase by flipping his variable (y_j or z_j). It follows that $\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle$ is a Nash equilibrium for Γ_H (for any boolean vectors $\mathbf{b}$ and $\mathbf{c}$).

Assume now that $\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle$ is a Nash equilibrium for Γ_H. Fix any index $j \in [m]$. By definition of Nash equilibrium:

- Player $n + j$ cannot increase his payoff by flipping his variable y_j (from b_j to $\overline{b_j}$). Hence,

$$(C_j(\mathbf{a}) \bigvee b_j \bigvee c_j) + (C_j(\mathbf{a}) \bigvee \overline{b_j} \bigvee \overline{c_j}) \quad \geq \quad (C_j(\mathbf{a}) \bigvee \overline{b_j} \bigvee c_j) + (C_j(\mathbf{a}) \bigvee b_j \bigvee \overline{c_j})\,.$$

- Player $n + m + j$ cannot increase his payoff by flipping his variable z_j (from c_j to $\overline{c_j}$). Hence,

$$(C_j(\mathbf{a}) \bigvee b_j \bigvee \overline{c_j}) + (C_j(\mathbf{a}) \bigvee \overline{b_j} \bigvee c_j) \quad \geq \quad (C_j(\mathbf{a}) \bigvee b_j \bigvee c_j) + (C_j(\mathbf{a}) \bigvee \overline{b_j} \bigvee \overline{c_j})\,.$$

The two inequalities imply that

$$(C_j(\mathbf{a}) \bigvee b_j \bigvee c_j) + (C_j(\mathbf{a}) \bigvee \overline{b_j} \bigvee \overline{c_j}) \quad = \quad (C_j(\mathbf{a}) \bigvee \overline{b_j} \bigvee c_j) + (C_j(\mathbf{a}) \bigvee b_j \bigvee \overline{c_j})\,.$$

Denote as $g(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \{0, 1, 2\}$ this common value. We prove:

Lemma K.1 $g(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 2$.

Proof: Assume, by way of contradiction, that $g(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \{0, 1\}$. We proceed by case analysis.

- Assume first that $g(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 0$. It follows that $C_j(\mathbf{a}) \bigvee b_j \bigvee c_J = 0$. Hence, $b_j = 0$. This implies that $C_j(\mathbf{a}) \bigvee \overline{b_j} \bigvee \overline{c_j} = 1$. Hence, $g(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 1$, a contradiction.
- Assume now that $g(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 1$. We proceed by subcase analysis on the value $C_j(\mathbf{a}) \bigvee b_j \bigvee c_j \in \{0, 1\}$.
  - Assume first that $C_j(\mathbf{a}) \bigvee b_j \bigvee c_j = 0$. This implies that $b_j = c_j = 0$. Hence, it follows that $C_j(\mathbf{a}) \bigvee \overline{b_j} \bigvee c_j = C_j(\mathbf{a}) \bigvee b_j \bigvee \overline{c_j} = 1$. Hence, $g(\mathbf{a}, \mathbf{b}, \mathbf{c}) = C_j(\mathbf{a}) \bigvee \overline{b_j} \bigvee c_j + C_j(\mathbf{a}) \bigvee b_j \bigvee \overline{c_j} = 2$. A contradiction.
  - Assume now that $C_j(\mathbf{a}) \bigvee b_j \bigvee c_j = 0$. Since $g(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (C_j(\mathbf{a}) \bigvee b_j \bigvee c_j) + (C_j(\mathbf{a}) \bigvee \overline{b_j} \bigvee \overline{c_j})$ and $g(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 1$ (by assumption), it follows that $(C_j(\mathbf{a}) \bigvee \overline{b_j} \bigvee \overline{c_j}) = 0$. This implies that $b_j = c_j = 1$. Hence, $C_j(\mathbf{a}) \bigvee \overline{b_j} \bigvee c_j = 1$ and $C_j(\mathbf{a}) \bigvee b_j \bigvee \overline{c_j} = 1$ Hence, $g(\mathbf{a}, \mathbf{b}, \mathbf{c}) = C_j(\mathbf{a}) \bigvee \overline{b_j} \bigvee c_j + C_j(\mathbf{a}) \bigvee b_j \bigvee \overline{c_j} = 2$. A contradiction.

the proof is now complete. ■

By definition of g, Lemma K.1 implies that $C_j(\mathbf{a}) \bigvee b_j \bigvee c_j = C_j(\mathbf{a}) \bigvee \overline{b_j} \bigvee \overline{c_j} = C_j(\mathbf{a}) \bigvee \overline{b_j} \bigvee c_j = +C_j(\mathbf{a}) \bigvee b_j \bigvee \overline{c_j} = 1$. It follows that $C_j(\mathbf{a}) = 1$. Since $j \in [m]$ was chosen arbitrarily, this implies that $H(\mathbf{a}) = \bigwedge_{j \in [m]} C_j(\mathbf{a}) = 1$.

L Proof of Lemma 5.5

Assume first that for all vectors $\mathbf{b} \in \{0, 1\}^k$, $H(\mathbf{a}, \mathbf{b}) = 1$. Fix any such vector $\mathbf{a} \in \{0, 1\}^n$.

- Fix now any vector $\mathbf{c} \in \{0, 1\}^n$. Note that the choice of $\mathbf{a}$ implies that for all vectors $\mathbf{b} \in \{0, 1\}^n$, $f_2(\mathbf{a}, \mathbf{b}.\mathbf{c}) = H(\mathbf{a}, \mathbf{b}) \oplus H(\mathbf{a}, \mathbf{c}) = 0$. Hence, player 2 cannot increase his payoff by changing his strategy $\mathbf{b}$.
- Fix now any vector $\mathbf{b} \in \{0, 1\}^n$. Note that the choice of $\mathbf{a}$ implies that for all vectors $\mathbf{c} \in \{0, 1\}^n$, $f_3(\mathbf{a}, \mathbf{b}.\mathbf{c}) = H(\mathbf{a}, \mathbf{b}) \oplus H(\mathbf{a}, \mathbf{c}) \oplus 1 = 1$. Hence, player 3 cannot increase his payoff by changing his strategy $\mathbf{c}$.

Thus, $\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle$ is a pure Nash equilibrium for the strategic game Γ_H (for any arbitrary pair of vectors $\mathbf{b}, \mathbf{c} \in \{0, 1\}^n$).

Assume now that $\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle$ is a pure Nash equilibrium for the game Γ_H. Then:

- Player 2 cannot increase his payoff by switching his strategy $\mathbf{b}$ to any strategy $\mathbf{b}'$. Thus, for all $\mathbf{b}' \in \{0,1\}^n$, $H(\mathbf{a},\mathbf{b}) \oplus H(\mathbf{a},\mathbf{c}) \geq H(\mathbf{a},\mathbf{b}') \oplus H(\mathbf{a},\mathbf{c})$. Two cases are now possible:
  - $H(\mathbf{a},\mathbf{b}) \oplus H(\mathbf{a},\mathbf{c}) = 1$.
  - $H(\mathbf{a},\mathbf{b}) \oplus H(\mathbf{a},\mathbf{c}) = 0$. This implies that for all $\mathbf{b}' \in \{0,1\}^n$, $H(\mathbf{a},\mathbf{b}) = H(\mathbf{a},\mathbf{b}')$.
- Player 3 cannot increase his payoff by switching his strategy $\mathbf{c}$ to any strategy $\mathbf{c}'$. Thus, for all $\mathbf{c}' \in \{0,1\}^n$, $H(\mathbf{a},\mathbf{b}) \oplus H(\mathbf{a},\mathbf{c}) \oplus 1 \geq H(\mathbf{a},\mathbf{b}) \oplus H(\mathbf{a},\mathbf{c}') \oplus 1$. Two cases are now possible:
  - $H(\mathbf{a},\mathbf{b}) \oplus H(\mathbf{a},\mathbf{c}) \oplus = 1$.
  - $H(\mathbf{a},\mathbf{b}) \oplus H(\mathbf{a},\mathbf{c}) \oplus 1 = 0$. This implies that for all $\mathbf{c}' \in \{0,1\}^n$, $H(\mathbf{a},\mathbf{c}) = H(\mathbf{a},\mathbf{c}')$.

Since it is impossible that both $H(\mathbf{a},\mathbf{b}) \oplus H(\mathbf{a},\mathbf{c})$ and $H(\mathbf{a},\mathbf{b}) \oplus H(\mathbf{a},\mathbf{c}) \oplus 1 = 1$, it follows that for all $\mathbf{b}' \in \{0,1\}^n$, $H(\mathbf{a},\mathbf{b}) = H(\mathbf{a},\mathbf{b}')$. Recall that, by assumption, there is a $\mathbf{b}^1_{\mathbf{a}} \in \{0,1\}^n$ such that $H(\mathsf{a},\mathbf{b}^1_{\mathbf{a}}) = 1$. It follows that for all $\mathbf{b}' \in \{0,1\}^n$, $H(\mathbf{a},\mathbf{b}') = 1$.