Voronoi Games on Cycle Graphs*

Marios Mavronicolas[1], Burkhard Monien[2], Vicky G. Papadopoulou[1], and Florian Schoppmann[2,3]

[1] Department of Computer Science, University of Cyprus, Nicosia CY-1678, Cyprus
{mavronic,viki}@ucy.ac.cy
[2] Faculty of Computer Science, Electrical Engineering and Mathematics,
University of Paderborn, Fürstenallee 11, 33102 Paderborn, Germany.
{bm,fschopp}@uni-paderborn.de
[3] International Graduate School of Dynamic Intelligent Systems

Abstract. In a Voronoi game, there is a finite number of players who each chooses a point in some metric space. A player's utility is the total measure of all points that are closer to him than to any other player, where points equidistant to several players are split up evenly among the closest players. In a recent paper, Dürr and Thang (2007) considered discrete Voronoi games on graphs, with a particular focus on pure Nash equilibria. They also looked at *Voronoi games on cycle graphs* with n nodes and k players. In this paper, we prove a new characterization of all Nash equilibria for these games. We then use this result to establish that Nash equilibria exist if and only if $k \leq \frac{2n}{3}$ or $k \geq n$. Finally, we give exact bounds of $\frac{9}{4}$ and 1 for the prices of anarchy and stability, respectively. Essentially, this makes Voronoi games on cycle graphs – to the best of our knowledge – the first class of Voronoi games completely understood.

1 Introduction

1.1 Motivation and Framework

In a *Voronoi game*, there is a finite number of players and an associated metric measurable space. Each player has to choose a point in the space, and all choices are made simultaneously. The utility of a player is the measure of all points that are closer to him than to any other player, plus an equal share of the points that are equidistant (and closest) to him and others. Voronoi games are related to (but different than) the extensively studied *facility location problem*, where the goal is to minimize some combination of serving and facility opening costs (cf. [10]). In particular, one can regard them as a model of competitive sellers seeking to maximize their market share by strategic positioning in the market.

Voronoi games on continuous spaces (typically, a 2-dimensional rectangle) have been studied widely. Most papers have considered the existence and computation of a *winning strategy* (or a *best strategy*) for a player, under the assumption that the players alternate in choosing (multiple) points in the space (see, e.g., [1, 3, 7]). Work on Hotelling's model [8] has also considered a generalization of the Voronoi games studied here, in that each player chooses both a point in a continuous space and a price (typically, a line or a line segment); see, e.g., [4, 5]. That work focused on *price equilibria* for a chosen set of points.

In a recent work, Dürr and Thang [6] considered a discrete version of the Voronoi games, played on (undirected) graphs. They focused on the associated Nash equilibria, i.e., the stable states of the game in which no player can improve his utility by unilaterally switching to a different strategy. They also considered *Voronoi games on cycle graphs* and gave a characterization of all Nash equilibria. However, it turns out that their characterization is not correct and requires some

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non-trivial modifications. In this work, we completely settle all relevant questions regarding Nash equilibria of Voronoi games on cycle graphs; we give an exact characterization, a necessary and sufficient existence criterion, and exact prices of anarchy and stability.

1.2 Related Work

The paper most closely related to our work is by Dürr and Thang [6] who established several essential results for Voronoi games on graphs. They gave a relatively simple graph which does not allow for a Nash equilibrium even if there are only two players. In fact, they also showed that deciding the existence of a Nash equilibrium for general graphs and arbitrary many players is NP-hard. Dürr and Thang [6] defined *social cost* of a profile as the sum of distances, over all nodes, to the nearest player. With this definition, they studied what they termed the *social cost discrepancy*, i.e., the maximum ratio between the social costs of any two Nash equilibria. For connected graphs, they showed an upper bound on the social cost discrepancy of $O(\sqrt{kn})$, and gave a construction scheme for graphs with social cost discrepancy of at least $\Omega(\sqrt{n/k})$.

1.3 Contribution and Significance

The contribution of this paper and its structure are as follows:

– In Section 2, we prove that a strategy profile is a Nash equilibrium if and only if no more than two players have the same strategy, the distance between two strategies is at most twice the minimum utility of any player, and three other technical conditions hold.
  We remark here that an algebraic characterization of Nash equilibria on cycle graphs was already given in [6, Lemma 2]. Yet, it turns out that their result contains mistakes (see Appendix A.2). Fixing these mistakes is non-trivial and leads to a different set of conditions.
– In Section 3, we show that a Voronoi game on cycle graph with n nodes and $k \leq n$ players has a Nash equilibrium if and only if $k \leq \frac{2n}{3}$ or $k = n$. If that condition is fulfilled, then the strategy profile that locates all players equidistantly on the cycle (up to rounding) is a Nash equilibrium.
– In Section 4, we prove that profiles with (almost) equidistantly located players have optimal social cost. Furthermore, no Nash equilibrium has social cost greater than $\frac{9}{4}$ times the optimal cost. If $\frac{1}{2} \cdot \lfloor \frac{2n}{k} \rfloor$ is not an odd integer, then the upper bound improves to 2. To obtain these results, we devise and employ carefully constructed optimization problems so that best and worst Nash equilibria coincide with global minima or maxima, respectively. We give families of Voronoi games on cycle graphs where the aforementioned ratios are attained exactly. Hence, these factors are also exact bounds on the price of anarchy. Clearly, the price of stability is 1.

We believe that our combinatorial constructions and proof techniques will spawn further interest; we also hope that they will be applicable to other instances of Voronoi games on metric spaces.

1.4 The Model

Notation. For $n \in \mathbb{N}_0$, let $[n] := \{1, \ldots, n\}$ and $[n]_0 := [n] \cup \{0\}$. Given a vector $\boldsymbol{v}$, we denote its components by $\boldsymbol{v} = (v_1, v_2, \ldots)$. As customary in the game theoretic literature, we make use of the notation $(\boldsymbol{v}_{-i}, v'_i)$ to denote the vector where the i-th component of $\boldsymbol{v}$ is replaced by v'_i.

We begin with a general definition of Voronoi games on connected undirected graphs without edge weights.

Definition 1. *A* Voronoi game on a connected graph *is specified by a graph $G = (V, E)$ and the number of players $k \in \mathbb{N}$. The strategic game is then completed as follows:*

- *The* strategy set *of each player is V; so, the set of strategy profiles is $\mathscr{S} := V^k$.*
- *The* utility function $u_i : \mathscr{S} \to \mathbb{R}$ *of a player $i \in [k]$ is defined as follows: Let the* distance $\operatorname{dist} : V \times V \to \mathbb{N}_0$ *be defined such that* $\operatorname{dist}(v, w)$ *is the length of a shortest path connecting v, w in G. Moreover, for any node $v \in V$, the function $F_v : \mathscr{S} \to 2^{[k]}$, $F_v(\boldsymbol{s}) := \arg\min_{i \in [k]} \operatorname{dist}(v, s_i)$, maps a strategy profile to the set of players closest to v. Then,*

$$u_i(\boldsymbol{s}) := \sum_{v \in V : i \in F_v(\boldsymbol{s})} \frac{1}{|F_v(\boldsymbol{s})|} .$$

In order to evaluate the quality of a strategy profile $\boldsymbol{s} \in \mathscr{S}$, *social cost* is defined as

$$\operatorname{SC}(\boldsymbol{s}) := \sum_{v \in V} \min_{i \in [k]} \operatorname{dist}(v, s_i) .$$

Note that this is the same definition as used by Dürr and Thang [6]. The *optimum social cost* (or just the optimum) associated to a game is $\operatorname{OPT} := \inf_{\boldsymbol{s} \in \mathscr{S}} \operatorname{SC}(\boldsymbol{s})$.

We are interested in profiles called *Nash equilibria*, where no player has an incentive to unilaterally deviate. That is, $\boldsymbol{s} \in \mathscr{S}$ is a Nash equilibrium if and only if for all $i \in [k]$ it holds that $u_i(\boldsymbol{s}_{-i}, s_i) \le u_i(\boldsymbol{s})$. If such a profile exists in a game, to what degree can social cost deteriorate due to player's selfish behavior? Several metrics have been proposed to capture this question: The *price of anarchy* [9] is the worst-case ratio between a Nash equilibrium and the optimum, i.e., $\operatorname{PoA} = \sup_{\boldsymbol{s} \text{ is NE}} \frac{\operatorname{SC}(\boldsymbol{s})}{\operatorname{OPT}}$. The *price of stability* [2] is the best-case ratio between a Nash equilibrium and the optimum, i.e., $\operatorname{PoS} = \inf_{\boldsymbol{s} \text{ is NE}} \frac{\operatorname{SC}(\boldsymbol{s})}{\operatorname{OPT}}$. Finally, the *social cost discrepancy* [6] measures the maximum ratio between worst and best Nash equilibria, i.e., $\operatorname{SCD} = \sup_{\boldsymbol{s}, \boldsymbol{s}' \text{ are NE}} \frac{\operatorname{SC}(\boldsymbol{s})}{\operatorname{SC}(\boldsymbol{s}')}$. For these ratios, $\frac{0}{0}$ is defined as 1 and, for any $x > 0$, $\frac{x}{0}$ is defined as ∞.

In this paper, we consider *Voronoi games on cycle graphs*. A cycle graph is a graph $G = (V, E)$ where $V = \mathbb{Z}_n$ is the set of congruence classes modulo n, for some $n \in \mathbb{N}$, and $E := \{(x, x+1) : x \in \mathbb{Z}_n\}$. Clearly, a Voronoi game on a cycle graph is thus fully specified by the number of nodes n and the number of players k. As an abbreviation we use $\mathcal{C}(n, k)$. We will assume $k \le n$ throughout the rest of this paper as otherwise the games have a trivial structure. (In particular, whenever all nodes are used and the difference in the number of players on any two nodes is at most 1, this profile is a Nash equilibrium with zero social cost.)

We use a representation of strategy profiles that is convenient in the context of cycle graphs and which was also used in [6]. Define the *support* of a strategy profile $\boldsymbol{s} \in \mathscr{S}$ as the set of all chosen strategies, i.e., $\operatorname{supp} : \mathscr{S} \to 2^V$, $\operatorname{supp}(\boldsymbol{s}) := \{s_1, \dots, s_k\}$. Now fix a profile $\boldsymbol{s}$. Then, define $\ell := |\operatorname{supp}(\boldsymbol{s})|$ and $\theta_0 < \dots < \theta_{\ell-1}$ such that $\{\theta_i\}_{i \in \mathbb{Z}_\ell} = \operatorname{supp}(\boldsymbol{s})$. Note here that the choice of $V = \mathbb{Z}_n$ gives a fixed ordering of the nodes. Now, for $i \in \mathbb{Z}_\ell$:

- Let $d_i := (n + \theta_{i+1} - \theta_i) \bmod n$; so, d_i is the distance from θ_i to θ_{i+1}.
- Denote by c_i the number of players with strategy θ_i. Clearly, up to rotation and renumbering of the players, $\boldsymbol{s}$ is uniquely determined by ℓ, $\boldsymbol{d} = (d_i)_{i \in \mathbb{Z}_\ell}$, and $\boldsymbol{c} = (c_i)_{i \in \mathbb{Z}_\ell}$.
- Denote by v_i the utility of each player with strategy θ_i.
- Following [6], we define for all $i \in \mathbb{Z}_\ell$ the unique numbers $a_i \in \mathbb{N}$, $b_i \in \{0, 1\}$ by $d_i - 1 = 2 \cdot a_i + b_i$.

With these definition, the utility of a player with strategy θ_i is obviously

$$v_i = \frac{b_{i-1}}{c_{i-1} + c_i} + \frac{a_{i-1} + 1 + a_i}{c_i} + \frac{b_i}{c_i + c_{i+1}} .$$

Throughout, we use the set of congruence classes modulo ℓ for indexing; i.e., $c_i = c_{i+\ell}$ and $d_i = d_{i+\ell}$ for all $i \in \mathbb{Z}$. Note that for better readability, we do not reflect the dependency between $\boldsymbol{s}$ and $\ell, \boldsymbol{d}, \boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b}$ in our notation. This should be always clear from the context.

2 Characterization of Nash Equilibria

In this section, we prove an exact characterization of all Nash equilibria for the Voronoi games on a cycle with $n \in \mathbb{N}$ nodes and $k \in [n]$ players.

Theorem 1 (Strong characterization). *Consider $\mathcal{C}(n,k)$ where $n \in \mathbb{N}$, $k \in [n]$. A strategy profile $\boldsymbol{s} \in \mathscr{S}$ with minimum utility $\gamma := \min_{i \in \mathbb{Z}_\ell}\{v_i\}$ is a Nash equilibrium if and only if the following holds for all $i \in \mathbb{Z}_\ell$:*

S1. $c_i \leq 2$
S2. $d_i \leq 2\gamma$
S3. $c_i \neq c_{i+1} \Longrightarrow \lfloor 2\gamma \rfloor$ *odd*
S4. $c_i = 1,\ d_{i-1} = d_i = 2\gamma \Longrightarrow 2\gamma$ *odd*
S5. $c_i = c_{i+1} = 1,\ d_{i-1} + d_i = d_{i+1} = 2\gamma \Longrightarrow 2\gamma$ *odd*
$c_i = c_{i-1} = 1,\ d_{i-1} = d_i + d_{i+1} = 2\gamma \Longrightarrow 2\gamma$ *odd*

For the proof, we need the following lemma:

Lemma 1. *If property (S2) of Theorem 1 is fulfilled then $\forall i \in \mathbb{Z}_\ell, c_i = 2 : d_{i-1} = d_i = \lfloor 2\gamma \rfloor$. If additionally (S1) and (S3) are fulfilled then also $\forall i \in \mathbb{Z}_\ell, c_i = 2 : v_i = \gamma$ and $2\gamma = \lfloor 2\gamma \rfloor \in \mathbb{N}$.*

Proof (of Lemma 1). First consider the case that there is an $i \in \mathbb{Z}_\ell$ with $c_i = 2$. We show that $d_{i-1} = d_i = \lfloor 2\gamma \rfloor$. By way of contradiction, assume the converse. Due to property (S2), this means that, w.l.o.g., $d_i = 2a_i + b_i + 1 \leq \lfloor 2\gamma \rfloor - 1$ and $d_{i-1} \leq \lfloor 2\gamma \rfloor$. Then,

$$\frac{a_i}{2} \leq \frac{\lfloor 2\gamma \rfloor}{4} - \frac{b_i}{4} - \frac{1}{2} \quad \text{and} \quad \frac{a_{i-1}}{2} \leq \frac{\lfloor 2\gamma \rfloor}{4} - \frac{b_{i-1}}{4} - \frac{1}{4}\,.$$

This contradicts the definition of γ because

$$v_i \leq \frac{b_{i-1}}{3} + \frac{a_{i-1} + a_i + 1}{2} + \frac{b_i}{3} \leq \frac{\lfloor 2\gamma \rfloor}{2} + \frac{b_{i-1}}{12} + \frac{b_i}{12} - \frac{1}{4} \leq \gamma - \frac{1}{12}\,.$$

If $\lfloor 2\gamma \rfloor$ is even then it follows, by a simple induction using property (S3), that n is a multiple of $\lfloor 2\gamma \rfloor$ and so $\forall i \in \mathbb{Z}_\ell : v_i = \frac{\lfloor 2\gamma \rfloor}{2} = \gamma$. Otherwise, if $\lfloor 2\gamma \rfloor$ is odd, then $v_i = \frac{\lfloor 2\gamma \rfloor}{2} \leq \gamma$, so $\frac{\lfloor 2\gamma \rfloor}{2} = \gamma$. Finally, if $\forall i \in \mathbb{Z}_\ell : c_i = 1$, then clearly $2\gamma \in \mathbb{N}$. Consequently, properties (S1)–(S3) imply in all cases that $2\gamma = \lfloor 2\gamma \rfloor$. □

Proof (of Theorem 1). We start with a weak characterization that essentially states the definition of a Nash equilibrium in the context of Voronoi games on cycle graphs. Note that, in order to deal with parity issues, we find it convenient to mix in Boolean arithmetic and identify $1 \equiv$ *true* and $0 \equiv$ *false*. For instance, if $b, b' \in \{0,1\}$, then $b \leftrightarrow b' = 1$ if $b = b'$, and 0 otherwise. Similarly, $b \vee b = 1$ if $b = 1$ or $b' = 1$, and 0 otherwise.

Claim 1 (Weak characterization). The strategy profile $\boldsymbol{s}$ is a Nash equilibrium if and only if the following holds:

W1. No player being on a node alone can improve by moving to a neighboring node not in the support (and thereby swapping parity of the distances to neighboring strategies), i.e.,

$$\begin{aligned} \forall i \in \mathbb{Z}_\ell, c_i = 1 : (&b_{i-1} = b_i = 1 \quad \Longrightarrow c_{i-1} = c_{i+1} = 1) \wedge \\ (&b_{i-1} = 1, b_i = 0 \Longrightarrow c_{i-1} \leq c_{i+1}) \wedge \\ (&b_{i-1} = 0, b_i = 1 \Longrightarrow c_{i-1} \geq c_{i+1})\,. \end{aligned}$$

W2. No player can improve by moving to a node that is not in the support (for the cases not covered by (W1)), i.e.,

$$\forall i, j \in \mathbb{Z}_\ell : v_i \geq a_j + \frac{\neg b_j}{\min\{c_{j-1}, c_j\} + 1} + b_j \,.$$

W3. No player can improve by copying an arbitrary non-neighboring strategy, i.e.,

$$\forall i, j \in \mathbb{Z}_\ell, j \notin \{i-1, i+1\} : v_i \geq \frac{b_{j-1}}{c_{j-1} + c_j + 1} + \frac{a_{j-1} + 1 + a_j}{c_j + 1} + \frac{b_j}{c_j + c_{j+1} + 1} \,.$$

W4. No player sharing a node can improve by copying a neighboring strategy, i.e.,

$$\forall i \in \mathbb{Z}_\ell, c_i \geq 2 : v_i \geq \frac{b_i}{c_i + c_{i+1}} + \frac{a_i + 1 + a_{i+1}}{c_{i+1} + 1} + \frac{b_{i+1}}{c_{i+1} + c_{i+2} + 1} \,,$$

with a corresponding inequality for moving to θ_{i-1} instead of θ_{i+1}.

W5. No player being on a node alone can improve by copying a neighboring strategy, i.e.,

$$\forall i \in \mathbb{Z}_\ell, c_i = 1 : v_i \geq \frac{b_{i-1} \leftrightarrow b_i}{c_{i-1} + 1 + c_{i+1}} + \frac{a_{i-1} + a_i + a_{i+1} + b_{i-1} \vee b_i + 1}{1 + c_{i+1}} + \frac{b_{i+1}}{1 + c_{i+1} + c_{i+2}} \,,$$

with a corresponding inequality for moving to θ_{i-1} instead of θ_{i+1}.

Proof (of claim). Conditions (W1)–(W5) are exhaustive. ∎

We now continue by proving necessity ("$\Longrightarrow$"). Note that (S1) and (S2) have also been stated in [6, Lemma 2 (i), (ii)]. For completeness and since their proof contained mistakes (cf. Appendix A.2), we reestablish the claims here.

(S1) Assume by way of contradiction that there is some $i \in \mathbb{Z}_\ell$ with $c_i \geq 3$. W.l.o.g., assume $d_i \geq d_{i-1}$, i.e., also $a_i \geq a_{i-1}$ and $(b_{i-1} > b_i \Longrightarrow a_{i-1} < a_i)$. Since $v_i \geq 1$, it must hold that $a_i \geq 1$. Consider now the move by some player with strategy θ_i to node $\theta_i + 1$.
Since $\frac{b_{i-1}}{c_{i-1}+c_i} + \frac{a_{i-1}}{c_i} \leq \frac{b_i}{c_{i-1}+c_i} + \frac{a_i}{c_i}$ and $2a_i + 1 \leq c_i a_i$, his old utility v_i is at most

$$v_i = \frac{b_{i-1}}{c_{i-1} + c_i} + \frac{a_{i-1} + 1 + a_i}{c_i} + \frac{b_i}{c_i + c_{i+1}} \leq a_i + \frac{b_i}{2} \,,$$

whereas his new utility is

$$v' = a_i + b_i + \frac{\neg b_i}{1 + c_{i+1}} > v_i \,.$$

This is a contradiction to the profile being a Nash equilibrium.

(S2) We first show that $d_i \leq \lfloor 2\gamma \rfloor + 1$: Otherwise, there is some $d_i \geq \lfloor 2\gamma \rfloor + 2$ and a player with utility γ could move to node $\theta_i + 1$ and thus improve his utility to at least

$$\left\lfloor \frac{d_i}{2} \right\rfloor \geq \left\lfloor \frac{\lfloor 2\gamma \rfloor}{2} \right\rfloor + 1 = \lfloor \gamma \rfloor + 1 > \gamma \,.$$

Now assume $d_i = \lfloor 2\gamma \rfloor + 1$. Then, $c_i = 2$ because otherwise, if $c_i = 1$, a player with utility γ could change his strategy to $\theta_{i+1} - 1$ and thus achieve a new utility of

$$\frac{d_i}{2} = \frac{\lfloor 2\gamma \rfloor}{2} + \frac{1}{2} > \gamma \,.$$

The argument can be repeated correspondingly to obtain $c_{i+1} = 2$. Now note that $v_{i+1} \geq \frac{d_i}{2}$ because this is what a player with strategy θ_{i+1} could otherwise improve to, when moving to $\theta_i + 1$. It follows that $d_{i+1} = \lfloor 2\gamma \rfloor + 1 = d_i$. Inductively, we get for all $j \in \mathbb{Z}_\ell$ that $d_j = d_i$ and $c_j = c_i$. Then, n has to be a multiple of d_i, and for all $j \in \mathbb{Z}_\ell$ it holds that $v_j = \frac{d_j}{2} > \gamma$. Clearly, a contradiction.

(S3) W.l.o.g., assume that $c_i = 2$, $\lfloor 2\gamma \rfloor$ even, and $c_{i+1} = 1$. By Lemma 1 we have that $d_i = \lfloor 2\gamma \rfloor$, so $b_i = 1$. We get a contradiction to condition (W1) of Claim 1 both when $c_{i+2} = 1$ and $c_{i+2} = 2$ (in which case also $b_{i+1} = 1$).

(S4) Assume, by way of contradiction, that $c_i = 1$ and $d_{i-1} = d_i = 2\gamma$ even. Due to Lemma 1, it then follows that $c_{i-1} = c_{i+1} = 1$. Moreover, $a_{i-1} = a_i = \gamma - 1$. Hence, a player with utility γ could move to node θ_i and so improve his utility to (at least)

$$\frac{1}{3} + \frac{a_{i-1} + a_i + 1}{2} + \frac{1}{3} = \gamma + \frac{1}{6}\,.$$

(S5) We only show the first implication as the second one is symmetric. Assume, by way of contradiction, that $c_i = c_{i+1} = 1$ and $d_{i-1} + d_i = d_{i+1} = 2\gamma$ even. Due to Lemma 1, it then follows that $c_{i-1} = c_{i+2} = 1$ and $v_i = \frac{d_{i-1}+d_i}{2} = \gamma$. Moreover, $a_{i-1} + a_i = \gamma - (b_{i-1} \vee b_i) - 1$ and $a_{i+1} = \gamma - 1$. Hence, the player with strategy θ_i could move to θ_{i+1} and so improve his utility to

$$\frac{1}{3} + \frac{a_{i-1} + a_i + a_{i+1} + b_{i-1} \vee b_i + 1}{2} + \frac{1}{3} = \gamma + \frac{1}{6}\,.$$

In the remainder of the proof, we establish that the conditions are indeed sufficient ("$\Longleftarrow$"): Clearly, we have to verify all conditions of Claim 1.

(W1) Assume $c_i = 1$. Then, if d_{i-1} and d_i are even, it holds by condition (S3) that $c_{i-1} = c_{i+1} = 1$. Similarly, if d_{i-1} is even and d_i odd, then (S3) implies $c_{i-1} = 1 \leq c_{i+1}$. Correspondingly, d_{i-1} odd and d_i even implies $c_{i-1} \geq c_{i+1}$.

(W2) Condition (S2) implies that if a player moves to a node that is not in the support, then his new utility is at most $\frac{2\gamma}{2} = \gamma$.

(W3) Due to Lemma 1, a player could only improve by copying a non-neighboring strategy θ_j if $c_j = 1$ and $d_{j-1} = d_j = 2\gamma$. Then 2γ is odd due to condition (S4), hence $v' = \gamma$.

(W4) The same argument as for (W3) applies.

(W5) Let $i \in \mathbb{Z}_\ell$ and consider the unique player $p \in [n]$ with strategy $s_p = \theta_i$. Let v' be his new utility if he moved to θ_{i+1}. Assume for the moment that $c_{i+1} = 1$. Then,

$$v_i = a_{i-1} + a_i + \frac{b_{i-1}}{c_{i-1}+1} + \frac{b_i}{2} + 1$$

and

$$v' = \frac{a_{i-1} + a_i}{2} + \frac{a_{i+1}}{2} + \frac{b_{i-1} \leftrightarrow b_i}{c_{i-1}+2} + \frac{b_{i-1} \vee b_i}{2} + \frac{b_{i+1}}{2 + c_{i+2}} + \frac{1}{2}\,.$$

We now argue that it is sufficient to show the claim for $c_{i+1} = 1$. Otherwise, if $c_{i+1} = 2$, the old utility of player p would be $v_i - \frac{b_i}{2} + \frac{b_i}{3}$ and his new utility (after moving to θ_{i+1}) would be at most $v' - \frac{1}{2} + \frac{1}{3}$; hence, the gain in utility cannot be larger than in the case $c_{i+1} = 1$.

Since $c_i = 1$, we have $c_{i-1} = 1$ or $b_{i-1} = 0$ due to condition (S3) and Lemma 1. Hence, it is sufficient to consider the case $b_{i-1} = 0$. Otherwise, if $b_{i-1} = 1$, then $c_{i-1} = 1$ and the utility of player p would remain v_i when moving him to $\theta_i - 1$ (to change parity).

We have now

$$v_i = a_{i-1} + a_i + \frac{b_i}{2} + 1 \tag{1}$$

and

$$v' = \frac{a_{i-1} + a_i}{2} + \frac{a_{i+1}}{2} + \frac{\neg b_i}{c_{i-1}+2} + \frac{b_i}{2} + \frac{b_{i+1}}{2 + c_{i+2}} + \frac{1}{2}\,. \tag{2}$$

Since $b_{i-1} = 0$ and $c_i = c_{i+1} = 1$, it holds that $d_{i-1} + d_i = 2v_i \geq 2\gamma$. Consequently, there are two cases:

– $d_{i-1} + d_i = 2\gamma$
  Due to Lemma 1, the move could only improve p's utility if $d_{i-1} + d_i = d_{i+1} = 2\gamma$. Then 2γ is odd due to condition (S5), so $v' = \gamma = v_i$.
– $d_{i-1} + d_i > 2\gamma$
  Since $b_{i-1} = 0$ and $c_{i+1} = 1$, we have

$$v_i = \frac{d_{i-1} + d_i}{2} \geq \frac{2\gamma + 1}{2}, \quad \text{i.e.,} \quad 2\gamma \leq 2v_i - 1\,.$$

  Now condition (S2) implies $d_{i+1} = 2a_{i+1} + b_{i+1} + 1 \leq 2\gamma \leq 2v_i - 1 = 2(a_{i-1} + a_i + \frac{b_i}{2} + \frac{1}{2})$, so

$$\frac{a_{i+1}}{2} \leq \frac{a_{i-1} + a_i}{2} + \frac{b_i}{4} - \frac{b_{i+1}}{4}\,.$$

  Inserting into (2) yields

$$\begin{aligned} v' &\leq a_{i-1} + a_i + \frac{b_i}{4} - \frac{b_{i+1}}{4} + \frac{\neg b_i}{c_{i-1} + 2} + \frac{b_i}{2} + \frac{b_{i+1}}{2 + c_{i+2}} + \frac{1}{2} \\ &\leq a_{i-1} + a_i + \frac{3b_i}{4} + \frac{\neg b_i}{3} + \frac{7}{12}\,. \end{aligned}$$

  Hence $v' \leq a_{i-1} + a_i + \frac{11}{12} < v_i$ if $b_i = 0$ and $v' \leq a_{i-1} + a_i + \frac{4}{3} < v_i$ if $b_i = 1$.

Due to symmetry, we have hence shown that no player using a node alone may improve by moving to a neighboring strategy. □

3 Existence of Nash Equilibria

In this section, we give a condition for the existence of Nash equilibria in cycle graphs that is both necessary and sufficient. This condition only depends on the ratio between the number of players and the number of nodes in the cycle graph.

Theorem 2. *The Voronoi game $\mathcal{C}(n, k)$ does not have a Nash equilibrium if $\frac{2n}{3} < k < n$.*

Proof. By way of contradiction, let $\frac{2n}{3} < k < n$ and assume there is a Nash equilibrium. Note that $n \geq 4$ and $k \geq 3$. Clearly, $1 \leq \gamma \leq \frac{n}{k} < \frac{3}{2}$, so Theorem 1 together with Lemma 1 implies $\gamma = 1$. Hence, no two players may have the same strategy as otherwise (by the same Lemma) it holds for all $i \in \mathbb{Z}_\ell$ that $c_i = 2$ and $d_i = 2$. This implies $k = n$ (and n even). A contradiction.

Consequently, we have that $\ell = k$ and for all $i \in \mathbb{Z}_\ell$ that $c_i = 1$. Since $k > \frac{2n}{3}$, there has to be some $i \in \mathbb{Z}_\ell$ with $d_{i-1} = d_i = 1$ and $d_{i+1} = 2$. This is a contradiction to condition (S5) of Theorem 1, as 2γ is even. Specifically, it would hold that $v_i = 1$ but when switching to strategy θ_{i+1}, the player with strategy θ_i would improve to at least $\frac{1}{3} + \frac{1}{2} + \frac{1}{3} = \frac{7}{6} > 1$. (See Figure 1 in Appendix A.2 for an illustration.) □

Definition 2. *A strategy profile with distances $(d_i)_{i \in \mathbb{Z}_\ell}$ is called* standard *if $\forall i \in \mathbb{Z}_\ell : d_i \in \{\lfloor \frac{n}{k} \rfloor, \lceil \frac{n}{k} \rceil\}$.*

Theorem 3. *If $k \leq \frac{2n}{3}$ or $k = n$, then the Voronoi game $\mathcal{C}(n, k)$ has a standard strategy profile which is a Nash equilibrium.*

Proof. If $k = n$, then $\boldsymbol{s} = (0, 1, \ldots, n-1)$, i.e., $\ell = n$, $(c_i)_{i \in [n]} = (d_i)_{i \in [n]} = (1, \ldots, 1)$ is trivially a standard Nash equilibrium.

Consider now the case $k < \frac{2n}{3}$. Define $p \in \mathbb{N}_0$, $q \in [k-1]_0$ by $n = p \cdot k + q$. Denote $r := \min\{q, k-q\}$. Define a profile by $\ell = k$, and for all $i \in \mathbb{Z}_\ell$,

$$d_i := \begin{cases} p & \text{if } i \in \{1,3,\ldots,2r-1\} \cup \{2q+1, 2q+2, \ldots, k\} \\ p+1 & \text{if } i \in \{2,4,\ldots,2r\} \cup \{2(k-q)+1, 2(k-q)+2, \ldots, k\} \end{cases},$$
$$c_i := 1\,.$$

Note here that either $2q+1 > k$ or $2(k-q)+1 > k$. Hence, if $q \le \frac{k}{2}$, then

$$(d_i)_{i\in\mathbb{Z}_\ell} = (\underbrace{p, p+1, p, p+1, \ldots, p, p+1}_{2q \text{ elements}}, p, p, \ldots, p)$$

and otherwise

$$(d_i)_{i\in\mathbb{Z}_\ell} = (\underbrace{p, p+1, p, p+1, \ldots, p, p+1}_{2(k-q) \text{ elements}}, p+1, p+1, \ldots, p+1)\,.$$

Clearly, both are valid profiles because $\sum_{i\in\mathbb{Z}_\ell} d_i = p \cdot k + q = n$. Moreover, let again $\gamma := \min_{i\in\mathbb{Z}_\ell}\{v_i\}$ be the minimum utility of any player. Then $1 \le p \le \gamma < p+1 \le 2p \le 2\gamma$, so conditions (S1)–(S3) of Theorem 1 are fulfilled. In order to verify also conditions (S4) and (S5), we show that $p+1 < 2\gamma$: If $\frac{n}{2} < k \le \frac{2n}{3}$ then $p = 1$ and $q \ge \frac{k}{2}$; so $\gamma = \frac{3}{2}$. Hence, $p+1 = 2 < 3 = 2\gamma$. Otherwise, if $k \le \frac{n}{2}$, then $p \ge 2$ and so $p+1 < 2p \le 2\gamma$. □

4 Social Cost and the Prices of Anarchy and Stability

In this section, we first show that standard profiles are optimal; hence, if $k \le \frac{2n}{3}$ or $k = n$, then the price of stability is 1. We then continue by proving that the price of anarchy is at most $\frac{9}{4}$. Furthermore, we give families of Voronoi games on cycle graphs where these ratios are attained exactly.

Consider the following optimization problem on a vector $\boldsymbol{\lambda} \in \mathbb{N}^n$, where $n \in \mathbb{N}$:

$$\begin{aligned} \text{Minimize } & \textstyle\sum_{i=1}^n i \cdot x_i \\ \text{subject to } & \textstyle\sum_{i=1}^n x_i = n \\ & 0 \le x_i \le \lambda_i \quad \forall i \in [n] \\ \text{where } & x_i \in \mathbb{N}_0 \qquad \forall i \in [n] \end{aligned} \tag{3}$$

Lemma 2. *Let $\boldsymbol{\lambda} \in \mathbb{N}^n$ and define $r := \min\{i \in [n] : \sum_{j=1}^i \lambda_j \ge n\}$. Then, the unique optimal solution of* (3) *is $\boldsymbol{x}^* := (\lambda_1, \ldots, \lambda_{r-1}, n - \sum_{i=0}^{r-1} \lambda_i, 0, \ldots, 0) \in \mathbb{N}_0^n$.*

Proof. Let $\boldsymbol{y} \in \mathbb{N}^n$ be another optimal solution. It is sufficient to show that for all $i \in [r]$ it holds that $x_i^* = y_i$. By way of contradiction, assume the converse; i.e., there is some $i \in [r]$ with $y_i < x_i^* \le \lambda_i$. Since $\boldsymbol{x}^*, \boldsymbol{y}$ are feasible solutions, this means that there is some $j \in \{r, r+1, \ldots, n\}$ with $y_i > x_i^* \ge 0$. Hence, $\boldsymbol{y}$ cannot be optimal as $(\boldsymbol{y}_{-i,j}, y_i + 1, y_j - 1)$ would be a feasible better solution. □

Theorem 4. *A standard strategy profile has optimal social cost.*

Proof. Consider the Voronoi game $\mathcal{C}(n,k)$. We first observe the following relationship between the optimization problem (3) on $\boldsymbol{\lambda} := (k, 2k, 2k, \ldots, 2k) \in \mathbb{N}^n$ and profiles with optimal social cost. For any strategy profile $\boldsymbol{s} \in \mathscr{S}$ define $\boldsymbol{x}(\boldsymbol{s}) \in \mathbb{N}_0^n$ by $x_i(\boldsymbol{s}) := |\{u \in \mathbb{Z}_n : \min_{j\in[k]} \operatorname{dist}(s_j, u) = i-1\}|$. It is easy to see that, for all $\boldsymbol{s} \in \mathscr{S}$, $\boldsymbol{x}(\boldsymbol{s})$ is a feasible solution to optimization problem

(3) (on vector $\boldsymbol{\lambda}$) and $\mathrm{SC}(\boldsymbol{s}) = \sum_{i=1}^{n} i \cdot x_i(\boldsymbol{s})$. Hence, if $\boldsymbol{x}(\boldsymbol{s})$ is an optimal solution to (3) then $\boldsymbol{s}$ is a profile with optimal social cost.

Now let $\boldsymbol{s} \in \mathscr{S}$ be a standard profile. By definition, $\ell = k$, and for all $i \in [k]$ it holds that $c_i = 1$ and $d_i \in \{\lfloor \frac{n}{k} \rfloor, \lceil \frac{n}{k} \rceil\}$. Hence, since $\frac{1}{2} \cdot \lceil \frac{n}{k} \rceil \leq \lceil \frac{n}{2k} \rceil$, we have for all $u \in \mathbb{Z}_n$ that $\min_{j \in [k]} \mathrm{dist}(s_j, u) \leq \lfloor \frac{1}{2} \cdot (\lceil \frac{n}{k} \rceil + 1) \rfloor \leq \lfloor \lceil \frac{n}{2k} \rceil + \frac{1}{2} \rfloor \leq \lfloor \frac{n}{2k} \rfloor + 1$. Moreover, $x_1(\boldsymbol{s}) = k$, and for all $i \in \{2, \ldots, \lfloor \frac{n}{2k} \rfloor\}$ we have $x_i(\boldsymbol{s}) = 2k$. Hence, according to Lemma 2, $\boldsymbol{x}(\boldsymbol{s})$ is the optimal solution to (3). By the above observation, it then follows that $\boldsymbol{s}$ has optimal social cost. □

We will now determine tight upper bounds for the social cost of worst Nash equilibria. Therefore, consider the following optimization problem on a tuple (n, μ, f) where $n \in \mathbb{N}$, $\mu \in \mathbb{N}$, and $f : \mathbb{R} \to \mathbb{R}$ is a function.

$$\begin{aligned} \text{Maximize } & \textstyle\sum_{i=1}^{\ell} f(d_i) \\ \text{subject to } & \textstyle\sum_{i=1}^{\ell} d_i = n \\ & 1 \leq d_i \leq \mu \quad \forall i \in [\ell] \\ \text{where } & \ell, d_i \in \mathbb{N} \qquad \forall i \in [\ell] \end{aligned} \tag{4}$$

Recall that a function f is *superadditive* if it satisfies $f(x+y) \geq f(x) + f(y)$ for all of its domain. We prove:

Lemma 3. *Let $n \in \mathbb{N}$, $\mu \in [n] \setminus \{1\}$, and f be a superadditive function. Then, $(\ell^*, \boldsymbol{d}^*)$ with $\ell^* = \lceil \frac{n}{\mu} \rceil \in \mathbb{N}$ and $\boldsymbol{d}^* = (\mu, \ldots, \mu, n - (\ell^* - 1) \cdot \mu) \in \mathbb{N}^{\ell^*}$ is an optimal solution of (4).*

Proof. By way of contradiction, assume an optimal solution is $(\ell', \boldsymbol{d}')$ with $\sum_{i=1}^{\ell'} f(d'_i) > \sum_{i=1}^{\ell^*} f(d^*_i)$. One of the following holds:

- There are $i, j \in [\ell']$ with $1 < d'_i \leq d'_j < \mu$. Then, since f is superadditive, replacing d'_i, d'_j by $d'_i - 1, d'_j + 1$ gives a better solution.
- There are $i, j \in [\ell']$ with $d'_i + d'_j \leq \mu$. Then, since f is superadditive, decreasing ℓ' by 1 and replacing d'_i, d'_j by $d'_i + d'_j$ gives again a better solution.

(If none of the two conditions held, then all except one d'_i, where $i \in [\ell']$, would be equal to μ.) Hence, we have shown that $(\ell', \boldsymbol{d}')$ cannot be optimal. □

In the following, let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be an auxiliary function by the social cost corresponding (only) to the distance between two strategies; define f by

$$f(x) := \begin{cases} \frac{x^2}{4} & \text{if } x \in \mathbb{N}_0 \text{ and } x \text{ is even} \\ \frac{x^2-1}{4} & \text{if } x \in \mathbb{N} \text{ and } x \text{ is odd,} \end{cases}$$

and by linear interpolation for all other points. That is, if $x \in \mathbb{R}_{>0} \setminus \mathbb{N}$, then $f(x) := (\lceil x \rceil - x) \cdot f(\lfloor x \rfloor) + (x - \lfloor x \rfloor) \cdot f(\lceil x \rceil)$. By definition, the social cost of a strategy profile is $\sum_{i=1}^{\ell} f(d_i)$. It is straightforward to verify that for all $x \in \mathbb{R}_{\geq 0}$, $f(x) \geq \frac{x^2-1}{4}$ (see appendix). Hence, for all $x, y \geq 1$, we have $f(x) + f(y) \leq \frac{x^2+y^2}{4} \leq \frac{x^2+2xy+y^2-1}{4} = \frac{(x+y)^2-1}{4} \leq f(x+y)$. If $x < 1$ or $y < 1$, then $f(x) + f(y) \leq f(x+y)$ holds anyway because $f(0) = f(1) = 0$. It follows that f is superadditive. Note also that

$$f(2x) = \begin{cases} x^2 = 4f(x) & \text{if } x \in \mathbb{N} \text{ is even} \\ x^2 = 4f(x) + 1 & \text{if } x \in \mathbb{N} \text{ is odd} \\ x^2 - \frac{1}{4} = 2f(x - \frac{1}{2}) + 2f(x + \frac{1}{2}) = 4f(x) & \text{if } 2x \in \mathbb{N} \text{ is odd}\,. \end{cases}$$

Theorem 5. *Consider an arbitrary Voronoi game $\mathcal{C}(n, k)$ where $k \leq \frac{n}{2}$ and let $\boldsymbol{s} \in \mathscr{S}$ be a Nash equilibrium. Define $\gamma := \frac{1}{2} \cdot \lfloor \frac{2n}{k} \rfloor$. The following holds:*

1. *If* γ *is an odd integer, then* $\mathrm{SC}(\boldsymbol{s}) \leq \frac{9}{4}\,\mathrm{OPT}$.
2. *Otherwise,* $\mathrm{SC}(\boldsymbol{s}) \leq 2\,\mathrm{OPT}$.

Proof. Theorem 1 and Lemma 1 imply that, in any Nash equilibrium, the minimum utility of all players can be no more than γ. Hence, the maximum distance between two strategies is 2γ. Now let $\boldsymbol{s}$ be a strategy profile with $\ell = \lceil \frac{n}{2\gamma} \rceil$ and $\boldsymbol{d} = (2\gamma, \ldots, 2\gamma, n - (\ell - 1) \cdot 2\gamma)$. Due to Lemma 3 (with $\mu := 2\gamma$), no Nash equilibrium can have social cost larger than $\mathrm{SC}(\boldsymbol{s})$.

Let $p \in \mathbb{N}_0, q \in [k-1]_0$ be defined by $n = p \cdot k + q$. Similarly, let $t \in \mathbb{N}_0, u \in [2\gamma - 1]_0$ be defined by $n = t \cdot 2\gamma + u$. Clearly,

$$\mathrm{SC}(\boldsymbol{s}) \leq t \cdot f(2\gamma) + f(u)\,.$$

Finally, in order to compare $\mathrm{SC}(\boldsymbol{s})$ with OPT, define $v \in \mathbb{N}_0, w \in [0, 2\gamma)$: If $\gamma \in \mathbb{N}$, then by $q = v \cdot 2\gamma + w$ and otherwise (if $2\gamma \in \mathbb{N}$ is odd) by $(q - \frac{k}{2}) = v \cdot 2\gamma + w$. Note here that $2w \in \mathbb{N}_0$ and (2γ odd $\Longrightarrow q - \frac{k}{2} \geq 0$).

Claim 2.

$$\mathrm{SC}(\boldsymbol{s}) \leq \left(\frac{k}{2} + v\right) \cdot f(2\gamma) + w \cdot \frac{\gamma}{2}$$

Proof (of claim). If k is even, then $n \equiv q \mod 2\gamma$. Hence, it must hold that $t = \frac{k}{2} + v$ and $u = w$. Otherwise, if k is odd, then ($t = \frac{k+1}{2} + v$ and $u = w - \gamma$) or ($t = \frac{k-1}{2} + v$ and $u = w + \gamma$). Now observe that for any $x \in [0, 2\gamma)$, it holds that $f(x) \leq \frac{x^2}{4} < x \cdot \frac{\gamma}{2}$. Thus, $f(w) \leq w \cdot \frac{\gamma}{2}$,

$$\frac{1}{2} \cdot f(2\gamma) + f(w - \gamma) < \frac{\gamma^2}{2} + (w - \gamma) \cdot \frac{\gamma}{2} = w \cdot \frac{\gamma}{2}\,, \quad \text{and}$$
$$-\frac{1}{2} \cdot f(2\gamma) + f(w + \gamma) < -\frac{\gamma^2}{2} + (w + \gamma) \cdot \frac{\gamma}{2} = w \cdot \frac{\gamma}{2}\,.$$

The claim follows. ■

In the following, we now examine the optimal cost. Note first that if $\gamma \in \mathbb{N}$, then

$$\mathrm{OPT} = k \cdot f(\gamma) + q \cdot \left\lfloor \frac{\gamma + 1}{2} \right\rfloor\,.$$

Consider the the following cases:

– γ is even
  Since

$$q \cdot \left\lfloor \frac{\gamma + 1}{2} \right\rfloor = v \cdot \gamma^2 + w \cdot \frac{\gamma}{2} = v \cdot f(2\gamma) + w \cdot \frac{\gamma}{2}\,,$$

  we have

$$\mathrm{OPT} = k \cdot f(\gamma) + v \cdot f(2\gamma) + w \cdot \frac{\gamma}{2} = \left(\frac{k}{4} + v\right) \cdot f(2\gamma) + w \cdot \frac{\gamma}{2} \geq \frac{1}{2} \cdot \mathrm{SC}(\boldsymbol{s})\,.$$

– γ is odd
  Since now

$$q \cdot \left\lfloor \frac{\gamma + 1}{2} \right\rfloor = v \cdot \gamma^2 + w \cdot \frac{\gamma}{2} + \frac{q}{2} = v \cdot f(2\gamma) + w \cdot \frac{\gamma}{2} + \frac{q}{2}\,,$$

we have

$$\begin{aligned}\mathrm{OPT} &= k\cdot f(\gamma)+v\cdot f(2\gamma)+w\cdot\frac{\gamma}{2}+\frac{q}{2}=\left(\frac{k}{4}+v\right)\cdot f(2\gamma)-\frac{k}{4}+w\cdot\frac{\gamma}{2}+\frac{q}{2}\\ &\geq \frac{1}{2}\cdot \mathrm{SC}(\boldsymbol{s})-\frac{k}{4}\,.\end{aligned}$$

Now, a trivial bound is always $\mathrm{OPT}\geq n-k$. Since $k\leq \frac{n}{\gamma}$, as otherwise $\gamma=\frac{1}{2}\cdot\lfloor\frac{2n}{k}\rfloor>\frac{n}{k}$, this implies $\mathrm{OPT}\geq(\gamma-1)\cdot\frac{n}{\gamma}\geq(\gamma-1)\cdot k$. Finally, due to $k\leq\frac{n}{2}$ and since γ is odd, we have $\gamma\geq 3$; so $\mathrm{OPT}\geq 2k$ and

$$\mathrm{SC}(\boldsymbol{s})\leq\frac{9}{4}\,\mathrm{OPT}\,.$$

– $2\gamma\in\mathbb{N}$ is odd
Then $p=\gamma-\frac{1}{2}$ and $q\geq\frac{k}{2}$. Note that $n=\gamma\cdot k+(q-\frac{k}{2})$. If $p=\lfloor\gamma\rfloor$ is even, then

$$2\cdot(f(p+1)-f(\gamma))=\frac{(p+1)^2-1-p^2}{4}=\frac{p}{2}=\frac{\gamma}{2}-\frac{1}{4}\,.$$

Thus,

$$\mathrm{OPT}=k\cdot f(\gamma)+\left(q-\frac{k}{2}\right)\cdot\left(\frac{\gamma}{2}-\frac{1}{4}\right)=\left(\frac{k}{4}+v\right)\cdot f(2\gamma)+w\cdot\frac{\gamma}{2}-\frac{q-\frac{k}{2}}{4}\,.$$

Moreover,

$$\begin{aligned}\mathrm{SC}(\boldsymbol{s}) &\leq\left(\frac{k}{2}+v\right)\cdot f(2\gamma)+w\cdot\frac{\gamma}{2}\\ &=2\,\mathrm{OPT}-\left(q-\frac{k}{2}\right)\cdot\left(\frac{\gamma}{2}-\frac{1}{2}\right)\leq 2\,\mathrm{OPT}\,.\end{aligned}$$

If $p=\lfloor\gamma\rfloor$ is odd, then

$$\mathrm{OPT}=k\cdot f(\gamma)+\left(q-\frac{k}{2}\right)\cdot\left(\frac{\gamma}{2}+\frac{1}{4}\right)=\left(\frac{k}{4}+v\right)\cdot f(2\gamma)+w\cdot\frac{\gamma}{2}+\frac{q-\frac{k}{2}}{4}\,,$$

so clearly, $\mathrm{SC}(\boldsymbol{s})\leq 2\,\mathrm{OPT}$. □

Theorem 6. *The bounds in Theorem 5 are tight.*

Proof. Let $k\in\mathbb{N}$ even and $n=\gamma\cdot k$, where $2\gamma\in\mathbb{N}$. Consider a profile $\boldsymbol{s}$ with $\ell=\frac{k}{2}$ and $d_1=\cdots=d_\ell=2\gamma$. Clearly, a standard (and thus optimal) profile $\boldsymbol{s}'$ has $\ell'=k$ and $d_1'=\cdots=d_k'=\gamma$. Then $\mathrm{SC}(\boldsymbol{s}')=\mathrm{OPT}=k\cdot f(\gamma)$.

If γ is even or $\gamma\notin\mathbb{N}$, then $\mathrm{SC}(\boldsymbol{s})=\ell\cdot f(2\gamma)=\ell\cdot 4f(\gamma)=2k\cdot f(\gamma)=2\,\mathrm{OPT}$. On the other hand, if γ is odd, then $\mathrm{SC}(\boldsymbol{s})=\ell\cdot f(2\gamma)=\ell\cdot(4f(\gamma)+1)=2k\cdot(f(\gamma)+\frac{1}{4})=(2+\frac{1}{2\cdot f(\gamma)})\cdot\mathrm{OPT}$. To see the last equality, recall that $\frac{k}{2}=\frac{\mathrm{OPT}}{2f(\gamma)}$. For the case $\gamma=3$ this means $\mathrm{SC}(\boldsymbol{s})=\frac{9}{4}\cdot\mathrm{OPT}$. □

Theorem 7. *Consider the Voronoi game $\mathcal{C}(n,k)$. Up to rotation, the following holds:*

1. *If $\frac{n}{2}<k\leq\frac{2}{3}n$, then the best Nash equilibrium has social cost $\mathrm{OPT}=n-k$, whereas the worst Nash equilibrium has social cost $\lfloor\frac{2n}{3}\rfloor\leq 2\,\mathrm{OPT}$.*
2. *If $k=n$, then the best Nash equilibrium has social cost 0. If n is even, then the only other Nash equilibrium has social cost $\frac{n}{2}$. Otherwise, there is no other Nash equilibrium.*

Proof. 1. Theorem 1 and Lemma 1 imply that $\gamma \in \{1, \frac{3}{2}\}$. By way of contradiction, assume first that $\gamma = 1$. Then it must hold for all $i \in \mathbb{Z}_\ell$ that $c_i = 1$ and $d_i \leq 2$. Due to (S4), we have $d_{i-1} + d_i \leq 3$; so $v_i = \frac{d_{i-1}+d_i}{2} \leq \frac{3}{2}$. This is a contradiction because $\sum_{i=1}^{k} v_i \leq 1 + (k-1) \cdot \frac{3}{2} \leq n - \frac{1}{2}$.
   Consequently, it most hold that $\gamma = \frac{3}{2}$. Since for all $i \in \mathbb{Z}_\ell : d_i \leq 2\gamma = 3$, it follows that $\mathrm{SC}(\boldsymbol{s}) = n - \ell$. In the worst case, $\ell = \lceil \frac{n}{2\gamma} \rceil = \lceil \frac{n}{3} \rceil$.
2. Clearly, $\gamma = 1$. Due to (S3) and Lemma 1, every Nash equilibrium must satisfy either $\forall i \in \mathbb{Z}_\ell : c_i = 2$ or $\forall i \in \mathbb{Z}_\ell : c_i = 1$. The claim follows. □

5 Conclusion

Similar in spirit to Hotelling's famous "Stability in Competition" [8], Voronoi games provide a very simple scenario of competitive sellers: Modeling the market by some metric measurable space and assuming market shares to be proportional to the size of a seller's Voronoi area in the space, which position maximizes a seller's market share? In this work, we looked at Voronoi games from the stability angle by a comprehensive examination of their Nash equilibria. As a first step for a thorough understanding of Voronoi games, we assumed that the metric measurable space is merely a (discrete) cycle graph. Even for these very simple graphs, the analysis turned out to be non-trivial; with much of the complexity owed to the discrete nature of graphs and parity issues. While we consider now Voronoi games on cycle graphs to be fully understood – by giving an exact characterization of all Nash equilibria, an existence criterion and exact prices of anarchy and stability – a generalization to less restrictive classes of graphs remains open.

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A Appendix

A.1 Simple property of auxiliary function f in Section 4

Let $x \in \mathbb{R}_{\geq 0}$ and define $\alpha := \lfloor x \rfloor$, $\beta := x - \alpha$. So,

$$f(x) = f(\alpha+\beta) = (1-\beta)\cdot f(\alpha) + \beta \cdot f(\alpha+1) \geq (1-\beta)\cdot \frac{\alpha^2}{4} + \beta \cdot \frac{(\alpha+1)^2}{4} - \frac{1}{4}$$
$$= \frac{\alpha^2 - \beta\alpha^2 + \beta\alpha^2 + 2\beta\alpha + \beta - 1}{4} > \frac{\alpha^2 + 2\beta\alpha + \beta^2 - 1}{4} = \frac{x^2-1}{4} .$$

A.2 Mistakes in Lemma 2 of Dürr and Thang (2007)

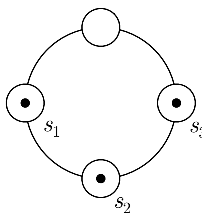
Conditions (S1) and (S2) were also the first two conditions of the characterization in [6]. In this paper, we included new proofs, since the (extended) version of [6] contained mistakes:

(S1) The new utility (denoted v' in our and u' in their proof) was wrong.
(S2) Their proof assumes that $2\gamma \in \mathbb{N}$. This was not shown.

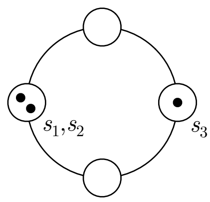
The third condition in [6], “if $c_j = 1$ and $d_{j-1} = d_j = 2\gamma$ then $c_{j-1} = c_{j+1} = 2$”, is wrong. Its proof was based on the incorrect assumption that $\gamma \in \mathbb{N}$ so that 2γ is even.

While the fourth and last condition in [6], “if $c_{j-1} = 2$, $c_j = 1$, $c_{j+1} = 1$ then d_{j-1} is odd” (and a corresponding one for c_{j-1} and c_{j+1} swapped), is indeed necessary for a Nash equilibrium, their four conditions are not sufficient: Consider the Voronoi game and the strategy profile depicted in Figure 2(a). It satisfies the following:

1. Each chosen node θ_j is used by $c_j = 1$ players.
2. The minimum payoff γ among all players is $\gamma = 1$. The distance d_j between any two chosen nodes is at most $d_j \leq 2\gamma = 2$.
3. No two consecutive distances d_{j-1}, d_j between any two chosen nodes satisfy $d_{j-1} = d_j = 2\gamma = 2$.
4. Trivially, no chosen node θ_j is used by $c_j = 2$ players.



(a) Player 2 can improve by copying player 1's strategy.



(b) His new utility is strictly greater than 1.

Fig. 1. The profile on the left is not a Nash equilibrium.

Hence, all conditions of Lemma 2 in [6] are fulfilled. Yet, player 2 may improve his utility from 1 to $(\frac{1}{3} + \frac{1}{2} + \frac{1}{3}) = \frac{7}{6} > 1$ by copying the strategy of player 1, as shown in Figure 2(b).