Direct Routing: Algorithms and Complexity

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Abstract. *Direct routing* is the special case of *bufferless routing* where N packets, once injected into the network, must be routed along specific paths to their destinations without conflicts. We give a general treatment of three facets of direct routing:

(i) *Algorithms.* We present a polynomial time *greedy* algorithm for arbitrary direct routing problems which is worst-case optimal, i.e., there exist instances for which no direct routing algorithm is better than the greedy. We apply variants of this algorithm to commonly used network topologies. In particular, we obtain *near-optimal* routing time for the *tree* and *d-dimensional mesh*, given arbitrary sources and destinations; for the *butterfly* and the *hypercube*, the same result holds for random destinations.

(ii) *Complexity.* By a reduction from Vertex Coloring, we show that Direct Routing is inapproximable, unless P=NP.

(iii) *Lower Bounds for Buffering.* We show that certain direct routing problems cannot be solved efficiently; to solve these problems, *any* routing algorithm needs buffers. We give non-trivial lower bounds on such buffering requirements for general routing algorithms.

1 Introduction

Direct routing is the special case of *bufferless routing* where N packets are routed along specific paths from *source* to *destination* so that they do not conflict with each other, i.e., once injected, the packets proceed "directly" to their destination without being delayed (buffered) at intermediate nodes. Since direct routing is bufferless, packets spend the minimum possible time in the network given the paths they must follow – this is appealing in power/resource constrained environments (for example optical networks or sensor networks). From the point of view of quality of service, it is often desirable to provide a guarantee on the delivery time after injection, for example in streaming applications like audio and video. Direct routing can provide such guarantees.

The task of a direct routing algorithm is to compute the injection times of the packets so as to minimize the *routing time*, which is the time at which the last packet is delivered to its destination. Algorithms for direct routing are inherently offline in order to guarantee no conflicts. We give a general analysis of three aspects of direct routing, namely efficient algorithms for direct routing; the computational complexity of direct routing; and, the connection between direct routing and buffering.

Algorithms for direct routing. We give a general and efficient (polynomial time) algorithm. We modify this algorithm for specific routing problems on commonly used network topologies to obtain more optimal routing times; thus, in many cases, efficient routing can be achieved without the use of buffers.

Arbitrary: We give a simple *greedy* algorithm which considers packets in some order, assigning the first available injection time to each packet. This algorithm is worst-case optimal: there exist instances of direct routing for which no direct routing algorithm achieves better routing time.

Tree: For arbitrary sources and destinations on arbitrary trees, we give a direct routing algorithm with routing time $rt = O(rt^*)$ where rt^* is the minimum possible routing time achievable by *any* routing algorithm (direct or not).

d-Dimensional Mesh: For arbitrary sources and destinations on a d-dimensional mesh with n nodes, we give a direct routing algorithm with routing time $rt = O(d^2 \cdot \log^2 n \cdot rt^*)$ with high probability (w.h.p.).

Butterfly and Hypercube: For permutation and random destination problems with one packet per node, we obtain routing time $rt = O(rt^*)$ w.h.p. for the butterfly with n inputs and the n-node hypercube.

Computational complexity of direct routing. By a reduction from vertex coloring, we show that direct routing is NP-complete. The reduction is *gap-preserving*, so direct routing is as hard to approximate as coloring.

Lower bounds for buffering. There exist direct routing problems which cannot be efficienty solved; such problems require buffering. We show that for *any buffered algorithm* there exist routing instances, for which packets are buffered $\Omega(N^{4/3})$ times in order to achieve near-optimal routing time.

Next, we discuss related work, followed by preliminaries and main results.

Related Work

The only previous known work on direct routing is for permutation problems on trees [1, 2], where the authors obtain routing time $O(n)$ for any tree with n nodes, which is worst-case optimal. Our algorithm for trees is every-case optimal for arbitrary routing problems. Cypher et al. [3] study an online version of direct routing in which a worm (packet of length L) can be re-transmitted if it is dropped (they also allow the links to have bandwidth $B \geq 1$). For the case corresponding to our work ($L = B = 1$), they give an algorithm with routing time $O((C + \log n) \cdot D)$. We give an off line algorithm with routing time $O(C \cdot D)$, show that this is worst case optimal, and that it is NP-hard to give a good approximation to the optimal direct routing time. We also obtain near-optimal routing time (with respect to buffered routing) for many interesting networks, for example the Mesh.

A dual to direct routing is time constrained routing where the task is to schedule as many packets as possible within a given time frame [4]. In these papers, the authors show that the time constrained version of the problem is NP-complete, and also study approximation algorithms on linear networks, trees and meshes. They also discuss how much buffering could help in this setting.

Other models of bufferless routing in which packets do not follow specific paths are *matching routing* [5] and *hot-potato routing* [6–8]. In [6], the authors also study lower bounds for near-greedy hot-potato algorithms on the mesh. Optimal routing for given paths on arbitrary networks has been studied extensively in the context of store-and-forward algorithms, [9–11].

2 Preliminaries

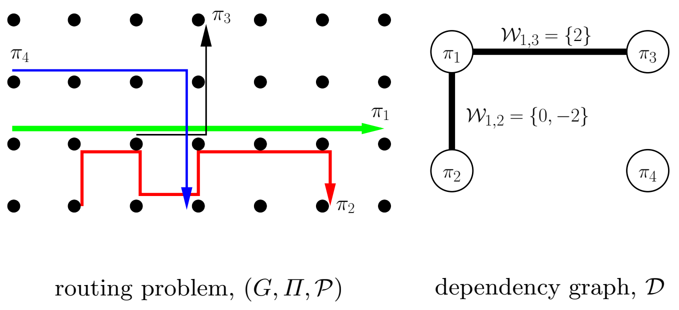
Problem Definition. We are given a graph $G = (V, E)$ with $n \geq 1$ nodes, and a set of packets $\Pi = \{\pi_i\}_{i=1}^N$. Each packet π_i is to be routed from its source, $\mathrm{s}(\pi_i) \in V$, to its destination, $\delta(\pi_i) \in V$, along a pre-specified path p_i. The nodes in the graph are synchronous: time is discrete and all nodes take steps simultaneously. At each time step, at most one packet can follow a link in each direction (thus, at most two packets can follow a link at the same time, one packet at each direction).

A path p is a sequence of vertices $p = (v_1, v_2, \ldots, v_k)$. Two paths p_1 and p_2 *collide* if they share an edge in the same direction. We also say that their respective packets π_1 and π_2 collide. Two packets *conflict* if they are routed in such a way that they appear in the same node at the same time, *and* the next edge in their paths is the same. The length of a path p, denoted $|p|$, is the number of edges in the path. For any edge $e = (v_i, v_j) \in p$, let $d_p(e)$ denote the length of path $(v_1, \ldots, v_i, v_j)$. The *distance* between two nodes, is the length of a shortest path that connects the two nodes.

A *direct routing problem* has the following components. Input: $(G, \Pi, \mathcal{P})$, where G is a graph, and the packets $\Pi = \{\pi_i\}_{i=1}^N$ have respective paths $\mathcal{P} = \{p_i\}_{i=1}^N$. Output: The injection times $\mathcal{T} = \{\tau_i\}_{i=1}^N$, denoted a *routing schedule* for the routing problem. Validity: If each packet π_i is injected at its corresponding time τ_i into its source s_i, then it will follow a conflict-free path to its destination where it will be absorbed at time $t_i = \tau_i + |p_i|$.

For a routing problem $(G, \Pi, \mathcal{P})$, the *routing time* $rt(G, \Pi, \mathcal{P})$ is the maximum time at which a packet gets absorbed at its destination, $rt(G, \Pi, \mathcal{P}) = \max_i\{\tau_i + |p_i|\}$. The *offline time*, $ol(G, \Pi, \mathcal{P})$ is the number of operations used to compute the routing schedule $\mathcal{T}$. We measure the efficiency of a direct routing algorithm with respect to the *congestion* C (the maximum number of packets that use an edge) and the *dilation* D (the maximum length of any path). A well known lower bound on the routing time of any routing algorithm (direct or not) is given by $\Omega(C + D)$.

Dependency Graphs. Consider a routing problem $(G, \Pi, \mathcal{P})$. The *dependency graph* $\mathcal{D}$ of the routing problem is a graph in which each packet $\pi_i \in \Pi$ is a node. We will use π_i to refer to the corresponding node in $\mathcal{D}$. There is an edge between two packets in $\mathcal{D}$ if their paths collide. An edge (π_i, π_j) with $i < j$ in $\mathcal{D}$ has an associated set of weights $\mathcal{W}_{i,j}$: $w \in \mathcal{W}_{i,j}$ if and only if π_i and π_j collide on some edge e for which $d_{p_i}(e) - d_{p_j}(e) = w$. Thus, in a valid direct routing schedule with injection times τ_i, τ_j for π_i, π_j, it must be that $\tau_j - \tau_i \notin \mathcal{W}_{i,j}$. An illustration of a direct routing problem and its corresponding dependency graph are shown in Figure 1.



π_3
π_4
π_1
π_2
routing problem, $(G, \Pi, \mathcal{P})$
$\mathcal{W}_{1,3} = \{2\}$
$\mathcal{W}_{1,2} = \{0, -2\}$
π_1
π_3
π_2
π_4
dependency graph, $\mathcal{D}$

Fig. 1. An example direct routing problem and its dependency graph.

We say that two packets are *synchronized*, if the packets are adjacent in $\mathcal{D}$ with some edge e and 0 is in the weight set of e. A clique $\mathcal{K}$ in $\mathcal{D}$ is *synchronized* if all the packets in $\mathcal{K}$ are synchronized, i.e., if 0 is in the weight set of every edge in $\mathcal{K}$. No pair in a synchronized clique can have the same injection time, as otherwise they would conflict. Thus, the size of the maximum synchronized clique in $\mathcal{D}$ gives a lower bound on the routing time:

Lemma 1 (Lower Bound on Routing Time). *Let $\mathcal{K}$ be a maximum synchronized clique in the dependency graph $\mathcal{D}$. Then, for any routing algorithm, $rt(G, \Pi, \mathcal{P}) \geq |\mathcal{K}|$*

We define the *weight degree of an edge e* in $\mathcal{D}$, denoted $W(e)$, as the size of the edge's weight set. We define the *weight degree of a node π* in $\mathcal{D}$, denoted $W(\pi)$, as the sum of the weight degrees of all edges incident with π. We define the *weight of the dependency graph, $W(\mathcal{D})$,* as the sum of the weight degrees of all its edges, $W(\mathcal{D}) = \sum_{e \in E(\mathcal{D})} W(e)$. For the example in Figure 1, $W(D) = 3$.

3 Algorithms for Direct Routing

Here we consider algorithms for direct routing. The algorithms we consider are variations of the following greedy algorithm, which we apply to the tree, the mesh, the butterfly and the hypercube.

```
1: // **Greedy direct routing algorithm:**
2: // **Input:** routing problem $(G, \Pi, \mathcal{P})$ with $N$ packets $\Pi = \{\pi_i\}_{i=1}^N$.
3: // **Output:** Set of injection times $\mathcal{T} = \{\tau_i\}_{i=1}^N$.
4: Let $\pi_1, \ldots, \pi_N$ be any specific, arbitrarily chosen ordering of the packets.
5: **for** $i = 1$ *to* $N$ **do**
6:     Greedily assign the first available injection time $\tau_i$ to packet $\pi_i \in \Pi$, so
       that it does not conflict with any packet already assigned an injection
       time.
7: **end for**
```

The greedy direct routing algorithm is really a family of algorithms, one for each specific ordering of the packets. It is easy to show by induction, that no packet π_j conflicts with any packet π_i with $i < j$, and thus the greedy algorithm produces a valid routing schedule. The routing time for the greedy algorithm will be denoted $rt_{Gr}(G, \Pi, \mathcal{P})$. Consider the dependency graph $\mathcal{D}$ for the routing problem $(G, \Pi, \mathcal{P})$. We can show that $\tau_i \leq W(\pi_i)$, where $W(\pi_i)$ is the weight degree of packet π_i, which implies:

Lemma 2. $rt_{Gr}(G, \Pi, \mathcal{P}) \leq \max_i\{W(\pi_i) + |p_i|\}$.

We now give an upper bound on the routing time of the greedy algorithm. Since the congestion is C and $|p_i| \leq D \; \forall i$, a packet collides with other packets at most $(C-1) \cdot D$ times. Thus, $W(\pi_i) \leq (C-1) \cdot D$, $\forall i$. Therefore, using Lemma 2 we obtain:

Theorem 1 (Greedy Routing Time Bound). $rt_{Gr}(G, \Pi, \mathcal{P}) \leq C \cdot D$.

The general $O(C \cdot D)$ bound on the routing time of the greedy algorithm is worst-case optimal, within constant factors, since from Theorem 9, there exist worst-case routing problems with $\Omega(C \cdot D)$ routing time. In the next sections, we will show how the greedy algorithm can do better for particular routing problems by using a more careful choice of the order in which packets are considered.

Now we discuss the offline time of the greedy algorithm. Each time an edge on a packets path is used by some other packet, the greedy algorithm will need to desynchronize these packets if necessary. This will occur at most $C \cdot D$ times for a packet, hence, the offline computation time of the greedy algorithm is $ol_{Gr}(G, \Pi, \mathcal{P}) = O(N \cdot C \cdot D)$, which is polynomial. This bound is tight since, in the worst case, each packet may have $C \cdot D$ collisions with other packets.

3.1 Trees

Consider the routing problem $(T, \Pi, \mathcal{P})$, in which T is a tree with n nodes, and all the paths in $\mathcal{P}$ are shortest paths. Shortest paths are optimal on trees given sources and destinations because any paths must contain the shortest path. Thus, $rt^* = \Omega(C + D)$, where rt^* is the minimum routing time for the given sources and destinations using *any* routing algorithm. The greedy algorithm with a particular order in which the packets are considered gives an asymptotically optimal schedule.

Let r be an arbitrary node of T. Let d_i be the closest distance that $\pi_i's$ path comes to r. The direct routing algorithm can now be simply stated as the greedy algorithm with the packets considered in sorted order, according to the distance d_i, with $d_{i_1} \leq d_{i_2} \leq \cdots \leq d_{i_N}$.

Theorem 2. *Let $(T, \Pi, \mathcal{P})$ be any routing problem on the tree T. Then the routing time of the greedy algorithm using the distance-ordered packets is $rt(T, \Pi, \mathcal{P}) \leq 2C + D - 2$.*

Proof. We show that every injection time satisfies $\tau_i \leq 2C - 2$. When packet π_i with distance d_i is considered, let v_i be the closest node to r on its path. All packets that are already assigned times that could possibly conflict with π_i are those that use the two edges in π_i's path incident with v_i, hence there are at most $2C - 2$ such packets. Since π_i is assigned the smallest available injection time, it must therefore be assigned a time in $[0, 2C - 2]$.

3.2 *d*-Dimensional Mesh

A d-dimensional mesh network $M = M(m_1, m_2, \ldots, m_d)$ is a multi-dimensional grid of nodes with side length m_i in dimension i. The number of nodes is $n = \prod_{i=1}^{d} m_i$, and define $m = \sum_{i=1}^{d} m_i$. Every node is connected to up to $2d$ of its neighbors on the grid. Theorem 1 implies that the greedy routing algorithm achieves asymptotically optimal worst case routing time in the mesh. We discuss some important special cases where the situation is considerably better. In particular, we give a variation of the greedy direct routing algorithm which is analyzed in terms of the number of times that the packet paths “bend” on the mesh. We then apply this algorithm to the 2-dimensional mesh in order to obtain optimal permutation routing, and the d-dimensional mesh, in order to obtain near-optimal routing, given arbitrary sources and destinations.

Multi-bend Paths. Here, we give a variation of the greedy direct routing algorithm which we analyze in terms of the number of times a packet bends in the network. Consider a routing problem $(G, \Pi, \mathcal{P})$. We first give an upper bound on the weight degree $W(\mathcal{D})$ of dependency graph $\mathcal{D}$ in terms of bends of the paths. We then use the weight degree bound in order to obtain an upper bound on the routing time of the algorithm.

For any subset of packets $\Pi' \subseteq \Pi$, let $\mathcal{D}_{\Pi'}$ denote the subgraph of $\mathcal{D}$ induced by the set of packets Π'. (Note that $\mathcal{D} = \mathcal{D}_{\Pi}$.) Consider the path p of a packet π. Let's assume that $p = (\ldots, v_i, v, v_j, \ldots)$, such that the edges (v_i, v) and (v, v_j) are in different dimensions. We say that the path of packet π *bends* at node v, and that v is an *internal bending node*. We define the source and destination nodes of a packet π to be *external bending nodes*. The *segment* $p' = (v_i, \ldots, v_j)$ of a path p, is a subpath of p in which only v_i and v_j are bending nodes. Consider two packets π_1 and π_2 whose respective paths p_1 and p_2 collide at some edge e. Let p'_1 and p'_2 be the two respective segments of p_1 and p_2 which contain e. Let p' be the longest subpath of p'_1 and p'_2 which is common to p'_1 and p'_2; clearly e is an edge in p'. Let's assume that $p' = (v_i, \ldots, v_j)$. It must be that v_i is a bending node of one of the two packets, and the same is true of v_j. Further, none of the other nodes in p' are bending nodes of either of the two packets. We refer to such a path p' as a *common subpath.* Note there could be many common subpaths for the packets π_1 and π_2, if they meet multiple times on their paths.

Since p_1 and p_2 collide on e, the edge $h = (\pi_1, \pi_2)$ will be present in the dependency graph $\mathcal{D}$ with some weight $w \in \mathcal{W}_{1,2}$ representing this collision. Weight w suffices to represent the collision of the two packets on the entire subpath p'. Therefore, a common subpath contributes at most one to the weight-number of $\mathcal{D}$. Let $A_{\mathcal{P}}$ denote the number of common subpaths. We have that $W(\mathcal{D}) \leq A_{\mathcal{P}}$. Therefore, in order to find an upper bound on $W(\mathcal{D})$, we only need to find an upper bound on the number of common subpaths.

For each common subpath, one of the packets must bend at the beginning and one at end nodes of the subpath. Thus, a packet contributes to the number of total subpaths only when it bends. Consider a packet π which bends at a node v. Let e_1 and e_2 be the two edges of the path of π adjacent to v. On e_1 the packet may meet with at most $C-1$ other packets. Thus, e_1 contributes at most $C-1$ to the number of common subpaths. Similarly, e_2 contributes at most $C-1$ to the number of common subpaths. Thus, each internal bend contributes at most $2C-2$ to the number of common subpaths, and each external bend $C-1$. Therefore, for the set of packets Π', where the maximum number of internal bends is b, $A_{\mathcal{P}} \leq 2(b+1)|\Pi'|(C-1)$. Hence, it follows:

Lemma 3. *For any subset $\Pi' \subseteq \Pi$, $W(\mathcal{D}_{\Pi'}) \leq 2(b+1)|\Pi'|(C-1)$, where b is the maximum number of internal bending nodes of any path in Π'.*

Since the sum of the node weight degrees is $2W(\mathcal{D})$, we have that the average node weight degree of the dependency graph for *any* subset of the packets is upper bounded by $4(b+1)(C-1)$. We say that a graph $\mathcal{D}$ is K-*amortized* if the average weight degree for every subgraph is at most K. K-amortized graphs are similar to balanced graphs [12]. Thus $\mathcal{D}$ is $4(b+1)C$-amortized. A *generalized coloring* of a graph with weights on each edge is a coloring in which the difference between the colors of adjacend nodes cannot equal a weight. K-Amortized graphs admit generalized colorings with $K+1$ colors. This is the content of the next lemma.

Lemma 4 (Efficient Coloring of Amortized Graphs). *Let $\mathcal{D}$ be a K-amortized graph. Then $\mathcal{D}$ has a valid $K+1$ generalized coloring.*

A generalized coloring of the dependency graph gives a valid injection schedule with maximum injection time one less than the largest color, since with such an injection schedule no pair of packets is sycnhronized. Lemma 4 implies that the dependency graph $\mathcal{D}$ has a valid $4(b+1)(C-1)+1$ generalized coloring. Lemma 4 essentially determines the order in which the greedy algorithm considers the packets so as to ensure the desired routing time. Hence, we get the following result:

Theorem 3 (Multi-bend Direct Routing Time). *Let $(M, \Pi, \mathcal{P})$ be a direct routing problem on a mesh M with congestion C and dilation D. Suppose that* *each packet has at most b internal bends. Then there is a direct routing schedule with routing time $rt \leq 4(b+1)(C-1)+D$.*

Permutation Routing on the 2-Dimensional Mesh. Consider a $\sqrt{n} \times \sqrt{n}$ mesh. In a permutation routing problem every node is the source and destination of exaclty one packet. We solve permutation routing problems by using paths with one internal bend. Let e be a column edge in the up direction. Since at most $\sqrt{n}$ packet originate and have destination at each row, the congestion at each edge in the row is at most $O(\sqrt{n})$. Similarly for edges in rows. Applying Theorem 3, and the fact that $D = O(\sqrt{n})$, we then get that $rt = O(\sqrt{n})$, which is worst case optimal for permutation routing on the mesh.

Near Optimal Direct Routing on the Mesh. Maggs *et al.* [13, Section 3] give a strategy to select paths in the mesh M for a routing problem with arbitrary sources and destinations. The congestion achieved by the paths is within a logarithmic factor from the optimal, i.e., $C = O(dC^* \log n)$ w.h.p., where C^* is the minimum congestion possible for the given sources and destinations. Following the construction in [13], it can be shown that the packet paths are constructed from $O(\log n)$ shortest paths between random nodes in the mesh. Hence, the number of bends b that a packet makes is $b = O(d \log n)$, and $D = O(m \log n)$, where m is the sum of the side lengths. We can thus use Theorem 3 to obtain a direct routing schedule with the following properties:

Theorem 4. *For any routing problem (M, Π) with given sources and destinations, there exists a direct routing schedule with routing time $rt = O(d^2 C^* \log^2 n + m \log n)$, w.h.p..*

Let D^* denote the maximum length of the shortest paths between sources and destinations for the packets in Π. D^* is the minimum possible dilation. Let rt^* denote the optimal routing time (direct or not). For any set of paths, $C + D = \Omega(C^* + D^*)$, and so the optimal routing time rt^* is also $\Omega(C^* + D^*)$. If $D^* = \Omega(m/(d^2 \log n))$, then $rt^* = \Omega(C^* + m/(d^2 \log n))$, so Theorem 4 implies:

Corollary 1. *If $D^* = \Omega(m/(d^2 \log n))$, then there exists a direct routing schedule with $rt = O(rt^* d^2 \log^2 n)$, w.h.p..*

3.3 Butterfly and Hypercube

We consider the n-input butterfly network B, where $n = 2^k$, [14]. There is a unique path from an input node to an output node of length $\lg n + 1$. Assume that every input node is the source of one packet and the destinations are randomly chosen.

For packet π_i, we consider the Bernoulli random variables x_j which are one if packet π_j collides with π_i. Then the degree of π_i in the dependency graph, $X_i = \sum_j x_j$. We show that $E[X_i] = \frac{1}{4}(\lg n - 1)$, and since the x_j are independent, we use the Chernoff bound to get a concentration result on X_i. Thus, we show that w.h.p, $\max_i X_i = O(\lg n)$. Since the injection time assigned by the greedy algorithm to any packet is at most its degree in the dependency graph, we show that the routing time of the greedy algorithm for a random destination problem on the butterfly satisfies $\mathbf{P}\left[rt_{Gr}(B, \Pi, \mathcal{P}) \leq \frac{5}{2}\lg n\right] > 1 - 2\sqrt{2}n^{-\frac{1}{2}}$. (the details are given in an appendix).

Valiant [15, 16] proposed permutation routing on butterfly-like networks by connecting two butterflies, with the outputs of one as the inputs to the other. The permutation is routed by first sending the packets to random destinations on the outputs of the first butterfly. This approach avoids hot-spots and converts the permutation problem to two random destinations problems. Thus, we can apply the result for random destinations twice to obtain the following theorem:

Theorem 5. *For permutation routing on the double-butterfly with random intermediate destinations, the routing time of the greedy algorithm satisfies* $\mathbf{P}\left[rt_{Gr} \leq 5\lg n\right] > 1 - 4\sqrt{2}n^{-\frac{1}{2}}$.

A similar analysis holds for the hypercube network (see appendix).

Theorem 6. *For a permutation routing using random intermediate destinations and bit-fixing paths, the routing time of the greedy algorithm satisfies* $\mathbf{P}\left[rt_{Gr} < 14\lg n\right] > 1 - 1/(16n)$.

4 Computational Complexity of Direct Routing

In this section, we show that direct routing and approximate versions of it are NP-complete. First, we introduce the formal definition of the direct routing decision problem. In our reductions, we will use the wll known NP-complete problem Vertex Color, the vertex coloring problem [17], which asks whether a given graph G is κ-colorable. The chromatic number, $\chi(G)$ is the smallest κ for which G is κ-colorable. An algorithm *approximates* $\chi(G)$ *with approximation ratio* $q(G)$ if on any input G, the algorithm outputs $u(G)$ such that $\chi(G) \leq u(G)$ and $u(G)/\chi(G) \leq q(G)$. Typically, $q(G)$ is expressed only as a function of the number of vertices in G. It is known [18] that unless P=NP[†], there does not exist a polynomial time algorithm to approximate $\chi(G)$ with approximation ratio $N^{1/2-\epsilon}$ for any constant $\epsilon > 0$.

By polynomially reducing coloring to direct routing, we will obtain hardness and inapproximability results for direct routing. We now formally define a generalization of the direct routing decision problem which allows for collisions. We say that an injection schedule is a valid K-collision schedule if at most K collisions occur during the course of the routing (a collision is counted for every collision of every pair of packets on every edge).

Problem: Approximate Direct Route
Input: A direct routing problem $(G, \Pi, \mathcal{P})$ and integers $T, K \geq 0$,
Question: Does there exist a valid k–collision direct routing schedule $\mathcal{T}$ for some $k \leq K$ and with maximum injection time $\tau_{max} \leq T$?

[†] It is also known that if NP$\not\subseteq$ZPP then χ is inapproximable to within $N^{1-\epsilon}$, however we cannot use this result as it requires both upper and lower bounds.

The problem Direct Route is the restriction of Approximate Direct Route to instances where $K = 0$. Denoting the maximum injection time of a valid K-collision injection schedule by T, we define the *K-collision injection number* $\tau_K(G, \Pi, \mathcal{P})$ for a direct routing problem as the minimum value of T for which a valid K-collision schedule exists. We say that a schedule approximates $\tau_K(G, \Pi, \mathcal{P})$ with ratio q if it is a schedule with at most K collisions and the maximum injection time for this schedule approximates $\tau_K(G, \Pi, \mathcal{P})$ with approximation ratio q. We now show that direct routing is NP-hard.

Theorem 7 (Direct Route **is** NP**-Hard).** *There exists a polynomial time reduction from any instance (G, κ) of* Vertex Color *to an instance $(G', \Pi, \mathcal{P}, T = \kappa - 1)$ of* Direct Route.

Sketch of Proof. We will use the direct routing problem illustrated to the right, for which the dependency graph is a synchronized clique. Each path is associated to a level, which denotes the x-coordinate at which the path moves vertically up after making its final left turn. There is a path for every level in $[0, L]$, and the total number of packets is $N = L+1$. The level-i path for $i > 0$ begins at $(1-i, i-1)$ and ends at $(i, L+i)$, and is constructed as follows. Beginning at $(1-i, i-1)$, the path moves right till $(0, i-1)$, then alternating between up and right moves till it reaches level i at node $(i, 2i-1)$ (i alternating up and right moves), at which point the path moves up to $(i, L+i)$. Every packet is synchronized with every other packet, and meets every other packet exactly once.

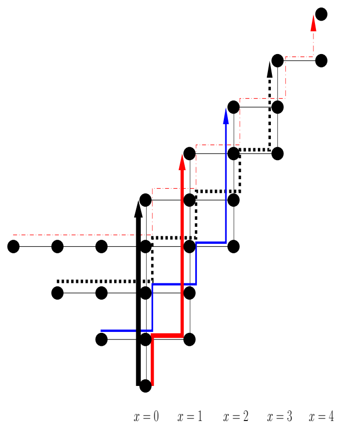


Fig. 2. A mesh routing problem.

Given an instance $I = (G, K)$ of Vertex Color, we reduce it in polynomial time to an instance $I' = (G', \Pi, \mathcal{P}, T = K - 1)$ of Direct Route. Each node in G corresponds to a packet in Π. The paths are initially as illustrated in the routing problem above with $L = N - 1$. The transformed problem will have dependency graph $\mathcal{D}$ that is isomorphic to G, thus a coloring of G will imply a schedule for $\mathcal{D}$ and vice versa.

If (u, v) is not an edge in G, then we remove that edge in the dependency graph by altering the path of u and v without affecting their relationship to any other paths; we do so via altering the edges of G' by making one of them to pass above the other, thus avoiding the conflict. After this construction, the resulting dependency graph is isomorphic to G.

Direct Route is in NP, as one can check the validity and routing time of a direct routing schedule, by traversing every pair of packets, so Direct Route is NP complete. Further, we see that the reduction is gap preserving with gap preserving parameter $\rho = 1$ [19].

Theorem 8 (Inapproximability of Collision Injection Number). *A polynomial time algorithm that approximates $\tau_K(G, \Pi, \mathcal{P})$ with ratio r for an arbitrary direct routing problem yields a polynomial time algorithm that approximates the chromatic number of an arbitrary graph with ratio $r + K + 1$. In particular, choosing $K = O(r)$ preserves the approximation ratio.*

For $K = O(N^{1/2-\epsilon})$, since χ is inapproximable with ratio $O(N^{1/2-\epsilon})$, we have

Corollary 2 (Inapproximability of Scheduling). *Unless* P=NP*, for $K = O(N^{1/2-\epsilon})$, there is no polynomial time algorithm to determine a valid K-collision direct routing schedule that approximates $\tau_K(G, \Pi, \mathcal{P})$ with ratio $O(N^{1/2-\epsilon})$ for any $\epsilon > 0$.*

5 Lower Bounds for Buffering

Here we consider the buffering requirements of any routing algorithm. We construct a "hard" routing problem for which *any* direct routing algorithm has routing time $rt = \Omega(C \cdot D) = \Omega(C + D)^2$, which is asymptotically worse than optimal. We then analyze the amount of buffering that would be required to attain near optimal routing time, which results in a lower bound on the amount of buffering needed by any store-and-forward algorithm.

Theorem 9 (Hard Routing Problem). *For every direct routing algorithm, there exist routing problems for which the routing time is $\Omega(C \cdot D) = \Omega((C+D)^2)$*

Proof. We construct a routing problem for which the dependency graph is a synchronized clique. The paths are as in Figure 2, and the description of the routing problem is in the proof of Theorem 7. The only difference is that c packets use each path. The congestion is $C = 2c$ and the dilation is $D = 3L$. Since every pair of packets is synchronized, Lemma 1 implies that $rt(G, \Pi, \mathcal{P}) \geq N$. Since $N = c(L + 1) = \frac{C}{2}(\frac{D}{3} + 1)$, $rt(G, \Pi, \mathcal{P}) = \Omega(C \cdot D)$. Choosing $c = \Theta(\sqrt{N})$ and $L = \Theta(\sqrt{N})$, we have that $C + D = \Theta(\sqrt{N})$ so $C + D = \Theta(\sqrt{C \cdot D})$.

Let problem A denote the routing problem in the proof of Theorem 9. We would like to determine how much buffering is necessary in order to decrease the routing time for routing problem A. Let T be the maximum injection time (so the routing time is bounded by $T + D$). We give a lower bound on the number of packets that need to be buffered at least once:

Lemma 5. *In routing problem A, if $T \leq \alpha$, then at least $N - \alpha$ packets are buffered at least once.*

Sketch of Proof. If β packets are not buffered at all, then they form a synchronized clique, hence $T \geq \beta$. Therefore $\alpha \geq \beta$, and since $N - \beta$ packets are buffered at least once, the proof is complete.

If the routing time is $O(C + D)$, then $\alpha = O(C + D)$. Choosing c and L to be $\Theta(N^{1/2})$, we have that $\alpha = O(N^{1/2})$, and so from Lemma 5, the number of packets buffered is $\Omega(N)$:

Corollary 3. *There exists a routing problem for which any algorithm will buffer $\Omega(N)$ packets at least once to achieve asymptotically optimal routing time.*

Repeating problem A appropriately, we can strengthen this corrollary to obtain

Theorem 10 (Buffering-Routing Time Tradeoff). *There exists a routing problem which, for any $\epsilon > 0$, requires $\Omega(N^{(4-2\epsilon)/3})$ buffering in order to obtain a routing time that is a factor $O(N^{\epsilon})$ from optimal.*

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