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Radiocoloring in planar graphs: Complexity and approximations☆

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Abstract

The Frequency Assignment Problem (FAP) in radio networks is the problem of assigning frequencies to transmitters, by exploiting frequency reuse while keeping signal interference to acceptable levels. The FAP is usually modelled by variations of the graph coloring problem. A Radiocoloring (RC) of a graph $G(V, E)$ is an assignment function $\Phi : V \rightarrow \mathbb{N}$ such that $|\Phi(u) - \Phi(v)| \geqslant 2$, when u, v are neighbors in G, and $|\Phi(u) - \Phi(v)| \geqslant 1$ when the distance of u, v in G is two. The number of discrete frequencies and the range of frequencies used are called order and span, respectively. The optimization versions of the Radiocoloring Problem (RCP) are to minimize the span or the order. In this paper we prove that the radiocoloring problem for general graphs is hard to approximate (unless NP = ZPP) within a factor of $n^{1/2-\varepsilon}$ (for any $\varepsilon > 0$), where n is the number of vertices of the graph. However, when restricted to some special cases of graphs, the problem becomes easier. We prove that *the min span RCP is* NP-*complete for planar graphs*. Next, we provide an $O(n\Delta)$ time algorithm ($|V| = n$) which obtains a radiocoloring of a planar graph G that *approximates the minimum order within a ratio which tends to* 2 (where Δ the maximum degree of G). Finally, we provide a *fully polynomial randomized approximation scheme* (fpras) for the *number of valid radiocolorings of a planar graph G* with λ colors, in the case where $\lambda \geqslant 4\Delta + 50$.

Keywords: Mobile computing; Radio communication; Coloring; Computational complexity; Approximations; Planar graphs; Rapid mixing; Coupling

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1. Introduction, previous work and our results

The Frequency Assignment Problem (FAP) in radio networks is a well-studied, interesting problem, aiming at assigning frequencies to transmitters exploiting frequency reuse while keeping signal interference to acceptable levels. The interference between transmitters are modeled by an interference graph $G(V, E)$, where $V(|V| = n)$ corresponds to the set of transmitters and E represents distance constraints (e.g. if two neighbor nodes in G get the same or close frequencies then this causes unacceptable levels of interference). In most real life cases the network topology formed has some special properties, e.g. G is a lattice network or a planar graph. Planar graphs are mainly the object of study in this work.

The FAP is usually modeled by variations of the graph coloring problem. The set of colors represents the available frequencies. In addition, each color in a particular assignment gets an integer value which has to satisfy certain inequalities compared to the values of colors of nearby nodes in G (frequency-distance constraints). The FAP has been considered in, e.g. [9–10,18]. Despite the important work done in either lattices or general networks, almost nothing has been reported for *planar* interference graphs, with the exception of [3,20].

In the sequel, we denote by $D(u, v)$ the distance of u, v in G. A discrete version of FAP is the k-coloring problem:

Definition 1 (*k-coloring problem, Hale [12]*). Given a graph $G(V, E)$ find a function $\phi : V \to \{1, \ldots, \infty\}$ such that $\forall u, v \in V, x \in \{0, 1, \ldots, k\}$: if $D(u, v) \geqslant k - x + 1$ then $|\phi_u - \phi_v| = x$. This function is called a k-coloring of G. Let $|\phi(V)| = \lambda$. Then λ is the *number of colors* that ϕ actually uses (it is usually called *order* of G under ϕ). The number $\nu = \max_{v \in V} \phi(v) - \min_{u \in V} \phi(u) + 1$ is usually called the *span* of G under ϕ.

Note that the case $k = 1$ corresponds to the well-known problem of vertex graph coloring. Thus, k-coloring problem (with k as an input) is NP-complete. Here we study the case of k-coloring problem where $k = 2$, called the Radiocoloring problem.

Definition 2 (*Radiocoloring problem*). Given a graph $G(V, E)$ find a function $\Phi : V \to N^*$ such that $|\Phi(u) - \Phi(v)| \geqslant 2$ if $D(u, v) = 1$ and $|\Phi(u) - \Phi(v)| \geqslant 1$ if $D(u, v) = 2$. The least possible number λ (order) needed to radiocolor G is denoted by $X_{\text{order}}(G)$. The least possible number $\nu = \max_{v \in V} \Phi(v) - \min_{u \in V} \Phi(u) + 1$ (span) needed for the radiocoloring of G is denoted as $X_{\text{span}}(G)$.

Real networks reserve bandwidth (range of frequencies) rather than distinct frequencies. In this case, an assignment seeks to use as small range of frequencies as possible. It is sometimes desirable to use as few distinct frequencies of a given bandwidth (span) as possible, since the unused frequencies are available for other use. However, there are cases where the primary objective is to minimize the number of frequencies used and the span is a secondary objective, since we do not want to reserve unnecessary large span. These optimization versions of the Radiocoloring Problem (RCP) are the main objects of study in this work and are defined as follows.

Definition 3 (*Min span RCP*). The optimization version of the RCP that tries to minimize the span. The optimal span is called X_{span}.

Definition 4 (*Min span order RCP*). The optimization version of the RCP that tries to find from all minimum span assignments, one that uses as few colors as possible. The order of such an assignment is called X'_{order}.

Definition 5 (*Min order RCP*). The optimization version of the RCP that tries to minimize the order. The optimal order is called X_{order}.

Definition 6 (*Min order span RCP*). The optimization version of the RCP that tries to find, from all minimum order assignments, one that uses a minimum span. The span of such an assignment is called X'_{span}.

It easy to see that $X_{\text{order}} \leqslant X'_{\text{order}}$ and $X_{\text{span}} \leqslant X'_{\text{span}}$. Also, it holds that $X_{\text{order}} \leqslant X_{\text{span}}$. Another variation of FAP is related to the square of a graph G, which is defined as follows:

Definition 7. Given a graph $G(V, E)$, G^2 is the graph having the same vertex set V and an edge set $E' : \{u, v\} \in E'$ iff $D(u, v) \leqslant 2$ in G.

The related variation of FAP is to color the square of a graph G, G^2, with the minimum number of colors, denoted as $X(G^2)$.

Observe that for any graph G, $X_{\text{order}}(G)$ is the same as the (vertex) chromatic number of G^2, i.e. $X_{\text{order}}(G) = X(G^2)$.

To see this assume to the contrary that $X(G^2) < X_{\text{order}}(G)$. Then, from an optimal coloring of G^2, we can obtain a radiocoloring of G with $X(G^2)$ colors by doubling the assigned color of each node. In this way we get a new radiocoloring assignment of G with less than $X_{\text{order}}(G)$ colors, which contradicts the definition of $X_{\text{order}}(G)$. Assume now that $X(G^2) > X_{\text{order}}(G)$. From an optimal min order radiocoloring we can easily get a coloring of G^2 assigning to each node the same color as in the radiocoloring assignment. Such an assignment is valid for the coloring of G^2 since both distance one and two constraints hold in any feasible radiocoloring. Thus, we find a new coloring G^2 with less than $X(G^2)$ colors, which contradicts the definition of $X(G^2)$. Concluding, $X(G^2) = X_{\text{order}}(G)$.

However, notice that although the number of colors used in a minimal coloring of G^2 and a min order span radiocoloring is the same, the set of colors in the two solutions may not be the same. To see this recall the previous argument showing that from an optimal coloring of G^2 we can obtain an optimal min order radiocoloring by doubling the assigned color to each node.

Observe also that $X(G^2) \leqslant X_{\text{span}} \leqslant 2X(G^2)$. It is obvious that $X(G^2) \leqslant X_{\text{span}}$. Furthermore, notice that from a valid coloring of G^2 we can always obtain a valid radiocoloring of G by multiplying the assigned color of every vertex by two. The resulting radiocoloring has span $2X(G^2)$.

In [10,9] it has been proved that the problem of min span RCP is NP-complete, even for graphs of diameter 2. The reductions use highly non-planar graphs. In [19] it is proved that the problem of coloring the square of a general graph is NP-complete.

In [3] a similar problem for *planar* graphs has been considered. This is the *hidden terminal interference avoidance* (*HTIA*) problem, which requests to color a planar graph G so that vertices at distance *exactly* 2 get different colors. In [3] this problem is shown to be NP-complete.

However, the above-mentioned result does not imply the NP-hardness of the min span order RCP which is proved here to be NP-complete. This so because HTIA is a different problem; in HTIA it is allowed to color neighbors in G with the same color while in RCP the colors of neighbor vertices should be at frequency distance at least two apart. Thus, the minimum number of colors as well as the span needed for HTIA can vary arbitrarily from $X_{\text{order}}(G)$ and $X_{\text{span}}(G)$. To see this consider e.g. the t-size clique graph K_t. In HTIA this can be colored with only one color. In our case (RCP) we need t colors and span of size $2t$ for K_t. In addition, the reduction used by [3], heavily exploits the fact that neighbors in G *get the same color* in the component substitution part of the reduction. Consequently, the reduction in [3] considers a different problem and it cannot be easily modified to produce an NP-hardness proof of RCP.

Note more specifically that, any polynomial time decision procedure for RCP does not imply a decision procedure for HTIA in the case of “No” answers. Also, any polynomial time decision procedure for HTIA does not give a decision for RCP in the case of “Yes” answers. In fact, the minimum number of colors needed for HTIA is the chromatic number of $G^2 - G$. To our knowledge, the relation between $X(G^2)$ and $X(G^2 - G)$ for a planar G has not been investigated.

Another variation of FAP for planar graphs, called *distance*-2-*coloring* is studied in [20]. This is the problem of coloring a given graph G with the minimum number of colors so that the vertices of distance *at most* two get different colors. Note that this problem is equivalent to coloring the square of the graph G, G^2. In the above work it is proved that the distance-2-coloring problem for planar graphs is NP-complete. As we show, this problem is different from the min span order RCP considered here. Thus, the NP-completeness proof in [20] certainly does not imply the NP-completeness of min span order RCP proved here. Additionally, the NP-completeness proof of [20] does not work for planar graphs of maximum degree $\Delta > 7$. Hence, their proof gives no information on the complexity of distance-2-coloring of planar graphs of maximum degree > 7. In contrast, our NP-completeness proof works for planar graphs of all maximum degrees. In [20] a 9-approximation algorithm for the distance-2-coloring of planar graphs is also provided.

In this paper, we are interested in *min span order*, *min order* and *min span* RCP of a *planar* graph G. We prove the following four basic results:

(a) We first show that the number of colors $X'_{\text{order}}(G)$ used in the *min span order RCP* of graph G is different from the chromatic number of the square of the graph, $X(G^2)$.

(b) We prove that the radiocoloring problem for general graphs is hard to approximate (unless NP $=$ ZPP, the class of problems with polynomial time zero-error randomized algorithms) within a factor of $n^{1/2-\varepsilon}$ (for any $\varepsilon > 0$), where n is the number of vertices of the graph. However, when restricted to some special cases of graphs, the problem becomes easier. We show that *the min span RCP* and *min span order RCP* are NP-*complete* for planar graphs. Note that few combinatorial problems remain hard for *planar* graphs and their proofs of hardness are not easy since they have to use planar gadgets which are difficult to find and understand [18]. As we argued above, this result is *not* implied by the known NP-completeness results of similar problems [3,20].

(c) We then present an $O(n\Delta)$ algorithm that *approximates* the minimum order of RCP, X_{order}, of a planar graph G *by a constant ratio which tends to* 2 as the maximum degree Δ of G increases.

Our algorithm is motivated by a constructive coloring theorem of Heuvel and McGuiness [13]. Their construction can lead (as we show) to an $O(n^2)$ technique assuming that a planar embedding of G is given. We improve the time complexity of the approximation, and we present a much more simple algorithm to verify and implement. Our algorithm does not need any planar embedding as input.

(d) Finally, we study the problem of *estimating the number of different radiocolorings* of a planar graph G. This is a #P-complete problem (as can be easily seen from our completeness reduction that can be done parsimonious). We employ here standard techniques of rapidly mixing Markov Chains and the *new method of coupling* for purposes of proving *rapid convergence* (see e.g. [14]) and we present *a fully polynomial randomized approximation scheme* for estimating the number of radiocolorings with λ colors for a planar graph G, when $\lambda \geqslant 4\Delta + 50$.

Very recently and independently, Agnarsson and Halldórsson in [2] presented approximations for the chromatic number of square and power graphs (G^k). Their method does not explicitly present an algorithm. A straightforward implementation is difficult and not efficient. Also, the performance ratio for planar graphs of general Δ obtained in [2] is 2, i.e. it is the same as the approximation ratio obtained by our algorithm.

We note that Bodlaender et al. [4] proved very recently and independently that the problem of min span radiocoloring, they call it λ-labeling, is NP-complete for planar graphs, using a reduction which is very similar to our reduction. In the same work the authors presented approximations for the best λ for some interesting families of graphs: outerplanar graphs, graphs of treewidth k, permutation and split graphs.

Another relevant work is that of Formann et al. [7], where the authors proved the chromatic number of the square of any planar graphs is at most $(13\Delta/7) + \Theta(\Delta^{2/3})$. However, this bound is bigger than the bound of Agnarsson et al. [2] and it does not improve the bound obtained by our algorithm since it holds only for graphs of quite large (> 749) maximum degree. Also their method is non-constructive.

A preliminary version of this work has appeared in the Proceedings of th 25th International Symposium on Mathematical Foundations of Computer Science (MFCS 2000) [8].

2. The difference between radiocoloring and distance-2-coloring in planar graphs

The distance-2-coloring problem, discussed above, is formally defined as follows:

Definition 8. The *Distance*-2-*coloring* of a graph G is the problem of coloring the vertices of the graph G with the minimum number of colors such that every pair of vertices that are located at distance at most two get different colors.

The following theorem states that the minimum order of min span order RCP of a graph G may be different (larger) from the order of distance-2-coloring problem (or the coloring of

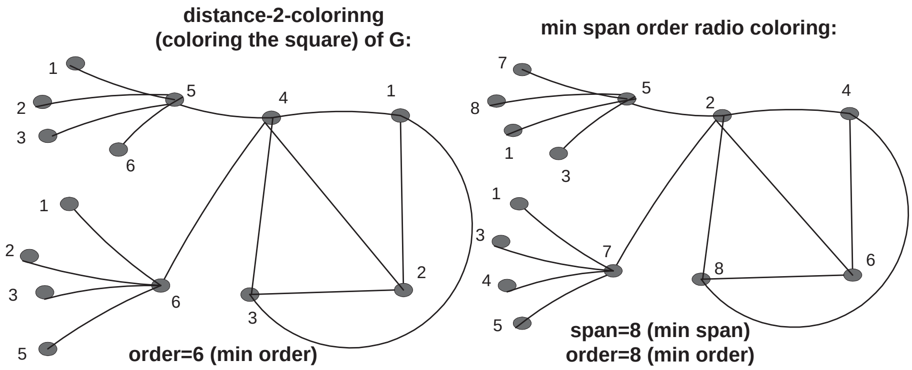

Fig. 1. An instance where the problem of min span order radiocoloring and the problem of distance-2-coloring have different orders.

G^2). Thus, the two problems are different. Hence-forth, the NP-completeness of distance-2-coloring problem does not imply the NP-completeness of min span order RCP proved here.

Theorem 9. *There is at least one instance* (*a graph G*) *where the minimum order of min span order RCP of G is different from the minimum order of distance*-2-*coloring of G* (*coloring the square of the graph*).

Proof. Consider the instance of the two problems appearing in Fig. 1. The vertices of the graph are named as shown in Fig. 1. Given a palette of colors (integers) *S*, used in an assignment, we call *endmost colors* the smallest and largest integer of the set *S*. We call the rest of the colors as *internal* colors. For example in the set $S = \{1, \dots, 8\}$ the colors 1, 8 are the two endmost colors of set *S*.

It is easy to see that the minimum number of colors (order) needed for the distance-2-coloring of *G* is 6 colors, while the minimum span of the min span order RCP of *G* is at least 7 (consider the colors needed to radiocolor vertex *new*1, the central vertex neighbor to it, and its radial vertices).

We assert that any optimal min span order radiocoloring assignment needs a span of size at least 8 and the order of such an assignment is also 8.

We distinguish three cases based on the colors of vertices *new*1, *new*2. Let x, y the colors of vertices *new*1, *new*2, respectively (note that $x \neq y$). Let also c the color of the central vertex. Note that $|c - x| \geqslant 2$ and $|c - y| \geqslant 2$.

(1) Both vertices *new*1, *new*2 get endmost colors. We prove that then, the four vertices of the clique formed are forced to take colors of span more than 8. This because the clique vertices should take a consecutive sequence of odd (or even) colors. In other case, they will leave more colors unused increasing the span more than 8. Thus, assume that the clique vertices take consecutive odds (evens). We will need four consecutive numbers, hence we will need a range of size 8. Also, we should use an endmost color. But, this is not possible, since we allocated the endmost colors to the vertices *new*1, *new*2.

Both vertices *new*1, *new*2 take internal colors. Then, *radial* neighbors of each of the two vertices, assume *new*1 (resp., *new*2) is avoided to take four colors $\{x-1, x, x+1, c\}$ ($\{y-1, y, y+1, c\}$) instead of three $x-1$ or $x+1, x, c(y-1$ or $y+1, y, c)$. Since, there are four radial vertices, we will need a span of size 8 for the coloring of vertex *new*1 (*new*2), its radial vertices, and the central vertex. Let S a palette of size 8.

We now prove that the order of such an assignment is also 8. Let $a=\{a_1, a_2, a_3, a_4\}$ the set of colors of the radial vertices of vertex *new*1. Then, $a = S-\{x-1, x, x+1, c\}$. Note that $|S-\{x-1, x, x+1, c\}| = 4$ in this case. Respectively, let $b=\{b_1, b_2, b_3, b_4\}$ the set of colors of the radial vertices of vertex *new*2. Then, $b = S-\{y-1, y, y+1, c\}$. Again, $|S-\{x-1, x, x+1, c\}| = 4$. We distinguish three cases for the numbers x, y.

- We consider first the case where x, y are consecutive internals. Then, we get that $a \cup b \geqslant 5$. Also note that in this case $x \notin b$ and $y \notin a$ because x, y are consecutive integers. Thus, the set of colors used to color vertices *new*1, *new*2, their radial and the central vertex has size $|\{x \cup y \cup (a \cup b) \cup c\}| = 8$, i.e. we get an order equal to 8.
- Now, consider the case where x, y are not consecutive internals and differ by at least 3. Then, it can be easily seen that $a \cup b \geqslant 7$. Also note that in this case it might be that $x \in b$ or $y \in a$. Thus, the set of colors used to color vertices *new*1, *new*2, their radial and the central vertex has size $|\{x \cup y \cup (a \cup b) \cup c\}| \geqslant 8$, i.e. we get an order at least 8.
- Now, the only case left is the case where x, y are not consecutive internals and differ by exactly 2. Then, it can be easily seen that $a \cup b \geqslant 6$. Also note that in this case it might be that $x \in b$ or $y \in a$. Thus, the set of colors used to color vertices *new*1, *new*2, their radial and the central vertex has size $|\{x \cup y \cup (a \cup b) \cup c\}| \geqslant 7$. However, using similar arguments as the case (1), we conclude that then the four vertices of the clique formed are force to take colors of span more than 8.

(2) One of the two vertices *new*1, *new*2 get an endmost color. Using similar arguments as the case (1), we conclude that the four vertices of the clique formed are forced to take colors of span more than 8.

We conclude that any radiocoloring assignment either uses a span of size 8 and an order also equal to 8 or a span of size more than 8, i.e. the assignment is sub-optimal. There is a radiocoloring assignment of span 8, as illustrated in Fig. 1. By the above analysis we conclude that any optimal assignment has order equal to 8. □

3. The inapproximability of radiocoloring for general graphs

In this section we prove that the radiocoloring problem is hard to approximate for general graphs.

Theorem 10. *The min order RCP for general graphs is hard to approximate* (*unless* NP = ZPP, *the class of problems with polynomial time zero-error randomized algorithms*) *within a factor of* $n^{1/2-\varepsilon}$ (*for any* $\varepsilon > 0$), *where n is the number of vertices of the graph*.

Proof. We reduce min order RCP from the COLORING Problem. Since we are concerned only in the order of a radiocoloring, the problem is equivalent to the distance-2-coloring

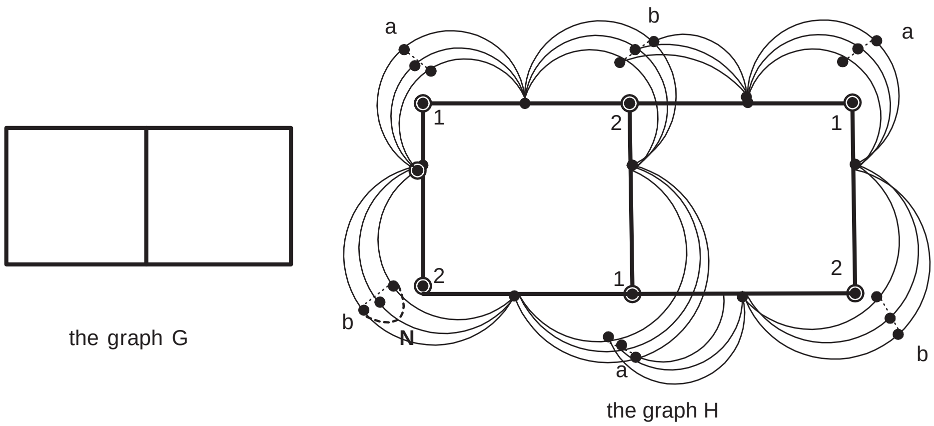

Fig. 2. Graph H obtained by G.

(d2c) problem. We use the term distance-2-coloring or d2c when referring to a min order radiocoloring assignment, for terminology convenience purposes.

We start with an arbitrary graph $G(V, E)$ with $|V| = N$. Let $n = \mathrm{O}(N^2 + N)$. We construct a graph H of $\mathrm{O}(n)$ vertices as follows:

Vertex set of H:

(1) For each vertex u_i of G we add a new vertex u_i in H, which we call *existing vertex*.
(2) For each edge of $E(G)$ we add in H a new vertex, called *intermediate vertex* and denoted u_{ij}, where u_i, u_j are the end vertices constituting the edge.
(3) Finally, for each vertex u_i of G, we add N new vertices, called *auxiliary vertices* and denoted as $y_{ij} : 1 \leqslant i, j \leqslant N$.

Edge set of H:

(1) For each intermediate vertex u_{ij}, obtained by end vertices u_i, u_j, we add the edges (u_i, u_{ij}), (u_{ij}, u_j).
(2) We connect each auxiliary vertex y_{ij} with all neighbor intermediate vertices of the existing vertex u_i from which the auxiliary vertex is obtained. Formally, the derived graph H can be described as follows:

$$V(H) = \{u_i : 1 \leqslant i \leqslant N\} \cup \{u_{ij} : (u_i, u_j) \in E(G)\} \cup \{y_{i,j} : 1 \leqslant i \leqslant N, 1 \leqslant j \leqslant N\}.$$

$$E(H) = \{(u_i, u_{ij}), (u_{ij}, u_j) : (u_i, u_j) \in E(G)\} \cup \{(u_{ij}, y_{i,j}) : (u_i, u_j) \in E(G)\}$$

An example of the graph H derived by a graph G is presented in the Fig. 2.

Observe that *if G is k-colorable then H is* $((k+1)N + \Delta + 1)$-*distance*-2-*colorable*. Such a coloring can be obtained as follows:

First, k-color each set $\{y_{1j}, y_{2j}, \ldots, y_{Nj}\}$, where $1 \leqslant j \leqslant N$. To show that the radiocoloring is valid, for any j consider the corresponding set $\{y_{1j}, y_{2j}, \ldots, y_{Nj}\}$. Its distance-one constraints in G are in H distance-two constraints. For each auxiliary vertex in this set, its coloring in G is equivalent to its distance-2-coloring in H. Therefore, k colors are enough for each such set to be distance-2-colored. Since we have N such sets, we need $k \cdot N$ colors for their coloring. Next, color the existing vertices with k additional colors (this is valid, based on similar arguments as above) and color the intermediate vertices with $\Delta + 1$ additional colors (valid, since it is equivalent to an edge coloring of G).

Summing up, we used, for the coloring of the auxiliary, existing and intermediate vertices, i.e. the graph H $(k+1)N + \Delta + 1$ colors.

On the other hand, from a distance-2-coloring of H we can get a coloring of G by the following procedure:

For each existing vertex $u_i \in V(G)$ we select one color from the set of N distinct colors of the set $\{y_{i1}, y_{i2}, \ldots, y_{iN}\}$ and color the vertex u_i with this color. This color result to a valid coloring of G as proved here: A neighbor of u_i, a vertex u_j will also take one color of its N $y_{j1}, y_{j2}, \ldots, y_{jN}$ auxiliary vertices. Since all $y_{j1}, y_{j2}, \ldots, y_{jN}$ vertices are distance-two neighbors with vertices $y_{i1}, y_{i2}, \ldots, y_{iN}$ in H, they all get different colors. Hence, the resulting coloring of vertices u_i, u_j is a valid coloring of G.

Thus, when H is distance-2-colorable with qN colors, where $q = N^{\Omega(1)}$, G is $O(q)$-colorable.

We know that it is NP-hard to determine if G needs at most $O(N^{\varepsilon})$ or at least $\Omega(N^{1-\varepsilon})$ colors to be colored [6]. Thus, it is also NP-hard to determine whether the optimal distance-2-coloring of a given graph (H in our case) with $O(n)$ vertices needs at most $O(N^{\varepsilon}N)$ or at least $\Omega(N^{1-\varepsilon}N)$ colors, i.e. the inapproximability ratio of distance-2-coloring (of a graph H) is

$$\frac{\Omega(N^{1-\varepsilon}N)}{O(N^{\varepsilon}N)} \geqslant \frac{\Omega(N^{2-\varepsilon})}{O(N^{1+\varepsilon})} \geqslant \frac{\Omega(n^{1-\varepsilon/2})}{O(n^{1/2+\varepsilon/2})} \geqslant \Omega(n^{1/2-\varepsilon}). \qquad \square$$

4. The NP-completeness of the RCP for planar graphs

In the previous section we proved that the radiocoloring problem for general graphs is hard to approximate within a factor of $n^{1/2-\varepsilon}$ (for any $\varepsilon > 0$), where n is the number of vertices of the graph. However, the problem, when restricted to some special cases of graphs, such as planar graphs, becomes, as we prove, easier.

In this section, we show that the decision version of min span RCP remains NP-complete for planar graphs. This version asks given a planar graph G and an integer B, to decide whether there exists a valid radiocoloring for G of span no more than B. Therefore, the optimization version of min span RCP remains NP-hard for planar graphs.

In the sequel the degree of vertex v in a graph G is denoted by $d_G(v)$ and when there is no confusion simply as $d(v)$. We also denote the subtraction operation between sets as $\setminus$.

Theorem 11. *The following decision problem is NP-complete*:
Input: *A planar graph $G(V, E)$ and an integer B.*
Question: *Does there exist a radiocoloring for G with span no more than B*?

Proof. It can be easily shown that the decision version of min span RCP, where we seek to decide whether a radiocoloring assignment with span v exists, is in NP (guess the assignment

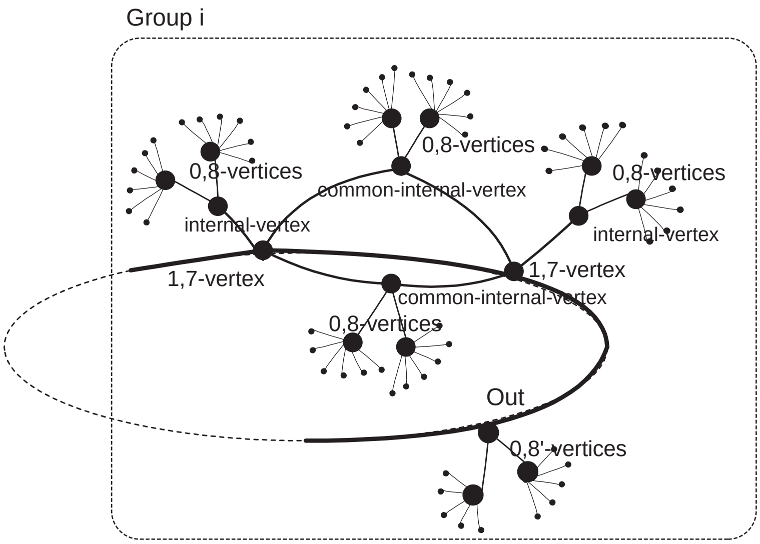

Fig. 3. The *Group* component.

and check in polynomial time the local constraints). To prove the theorem, we transform the PLANAR-3-COLORING problem to min span RCP. The PLANAR-3-COLORING problem is, given a planar graph $G(V, E)$, to determine whether the vertices of G can be colored with three colors, such that no adjacent vertices get the same color.

We denote y modulo x as $(y)MOD(x)$. From the planar graph $G(V, E)$, we construct a new graph $G'(V', E')$ using the component replacement technique.

The construction uses a component called *Group*, see Fig. 3, constructed as follows:

- Add one vertex called *out* vertex.
- Add two vertices called 1,7-*vertices* and connect one of them to the out-vertex and to each other, as shown in Fig. 3. We call the 1,7-vertex connected to the out-vertex as *first* 1,7-vertex and the other as *second*.
- Add two vertices called *common internal* and connect them to the 1,7-vertices.
- Add one new neighbor to each of the 1,7-vertex, called *internal*.
- For each 1,7, common internal, internal vertex add two new neighbors called 0,8-*vertices*. The two vertices added for each such vertex are also called *a pair of* 0,8-*vertices*.
- For each 0,8-vertex add six new neighbors, called *radial*.
- Add two new neighbors called 0,8′-*vertices* to the out-vertex. These two vertices added to the vertex are also called *a pair of* 0,8′-*vertices*.
- For each 0,8′-vertex add five new neighbors, called also *radial*.

The construction replaces every vertex v of degree $d(v)$ in the initial graph G with a component, called a ‘cycle node’. The cycle node obtained by a vertex of degree $d(v)$ is said to be ‘a cycle node of size $d(v)$’ and is constructed as follows:

- Add $d(v)$ copies of the subgraph *Group* shown in Fig. 3. Call the ith such group as $Group_i$.

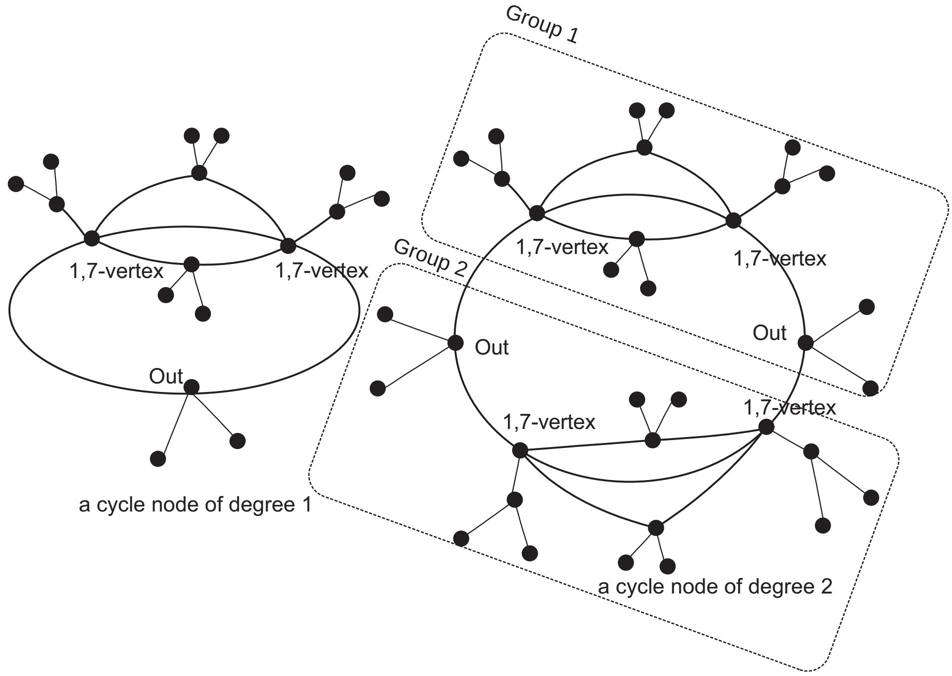

Fig. 4. The cycle nodes of size 1 and 2 in abbreviation (the radial vertices attached to the 0,8 and 0,8′-vertices are not shown).

- Connect consecutive groups as follows: Connect the second 1,7-vertex of $Group_i$ to the out-vertex of $Group_{(i)MOD(d(v))+1}$, for $i = 1 : d(v)$.

For example, the cycle nodes of size 1 and 2 are illustrated in Fig. 4 in abbreviation (the radial vertices attached to the 0,8 and 0,8′-vertices are not shown).

Now the graph $G'(V', E')$ is defined as follows:

(1) Replace each vertex v of degree $d(v)$ in the graph G with a cycle node of size $d(v)$.
(2) For each vertex v of the graph G, number the edges incident to v in increasing clockwise order.
(3) For every edge of the initial graph $e = (u, v)$ connecting u and v, let u_e be the number of edge e given by vertex u and let v_e be the number of the edge e given by vertex v.
Then, take one of the 0,8′-vertices of the u_eth group of the cycle node of vertex u and one of the 0,8′-vertices of the v_eth group of the cycle node of vertex v and collapse them to a single vertex named also as 0,8′-vertex. Do the same for the second 0,8′-vertex of u and the second 0,8′-vertex of v.

An example of a graph G and the new graph G' obtained is shown in Fig. 5 (depicted in a compact way). It can easily be seen that the new graph G' is a planar graph. We next prove two lemmas showing that G' can be radiocolored using a span of size at most 9 if and only if the initial graph G is 3-colorable.

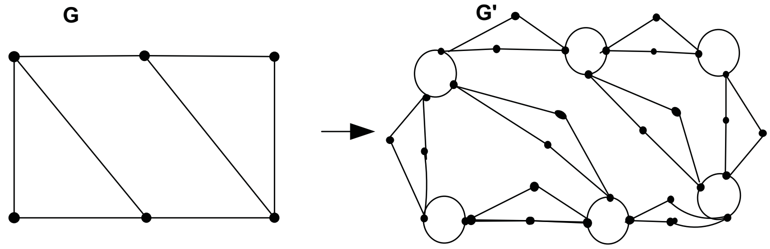

Fig. 5. The G' obtained by the graph G in abbreviation.

Lemma 12. *If* $\chi(G) \leqslant 3$ *then* $X_{\text{span}}(G') \leqslant 9$.

Proof. Consider a 3-coloring of the initial graph G, using colors $\{1, 2, 3\}$. Let the following radiocoloring assignment on the graph G' using a palette $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ of size 9:

(i) For each vertex u of the graph G colored $i, i \in \{1, 2, 3\}$, color all out-vertices of the cycle node of vertex u in G', with color $i + 2$ (i.e. a color from set $\{3, 4, 5\}$).
For each $u \in V$, color each group $Group_i$, $1 \leqslant i \leqslant d(u)$ of the cycle node of u as follows:

(ii) Color the first 1,7-vertex of the group with the color 1 and second with color 7.

(iii) Color 0,8-vertices and 0,8′-vertices with colors 0,8.

(iv) Assuming that the out-vertices of the group are colored $i, i \in \{3, 4, 5\}$, color the common-internal vertices of the group with colors of set $\{3, 4, 5\} \backslash \{i\}$.

(v) Color the internal-vertex neighbor to the 1,7-vertex colored 1, with color 6 and the internal-vertex neighbor to the 1,7-vertex colored 7, with color 1.

(vi) Consider a 0,8-vertex and the neighbor to it (common)-internal vertex colored $i, i \in \{2, 3, 4, 5, 6\}$. If the 0,8-vertex is colored 0, color the six uncolor neighbors with colors $\{2, 3, 4, 5, 6, 7, 8\} \backslash \{i\}$. If the 0,8-vertex is colored 8, color the six uncolor neighbors with colors $\{0, 1, 2, 3, 4, 5, 6\} \backslash \{i\}$.

(vii) Consider a 0,8′-vertex and the neighbors to it out-vertices u, v colored $i, j, i, j \in \{3, 4, 5\}$. If the 0,8-vertex is colored 0, color the five uncolor neighbors with colors $\{2, 3, 4, 5, 6, 7, 8\} \backslash \{i, j\}$. If the 0,8-vertex is colored 8, color the five uncolor neighbors with colors $\{0, 1, 2, 3, 4, 5, 6\} \backslash \{i, j\}$.

(viii) Color the radial vertices of 0,8 and 0,8′-vertices with the unused colors from set S.

Claim 13. *The suggested radiocoloring assignment is valid.*

Proof. The following hold for the suggested radiocoloring assignment:
Considering any cycle node, we have the following observations:

- Note first that, internal, common-internal, out-vertices are colored using colors from set $\{2, 3, 4, 5, 6\}$. Note also that common-internal, out-vertices are colored using colors $\{3, 4, 5\}$.
- Radial vertices neighbors to a 0,8-vertex: These are six vertices. They are neighbors to a (common)-internal vertex colored using a color $i, i \in \{2, 3, 4, 5, 6\}$. The vertices are

also neighbors to a 0,8-vertex colored 0 or 8. Thus, if the 0,8-vertex is colored 0, they can be colored with the six colors of set $\{2, 3, 4, 5, 6\}\backslash\{i\} \cup \{7, 8\}$, else (the 0,8-vertex takes color 8) with the six colors of set $\{2, 3, 4, 5, 6\}\backslash\{i\} \cup \{0, 1\}$.

- Radial vertices neighbors to a 0,8′-vertex: These are five vertices. They are neighbors to two out-vertices of neighbor cycle nodes colored $i, j, i \neq j, i, j \in \{2, 3, 4, 5, 6\}$. The vertices are also neighbors to a 0,8-vertex colored 0 or 8. Thus, if the 0,8-vertex is colored 0, they can be colored with the five colors of set $\{2, 3, 4, 5, 6\}\backslash\{i, j\} \cup \{7, 8\}$, else (the 0,8-vertex takes color 8) with the five colors of set $\{2, 3, 4, 5, 6\}\backslash\{i, j\} \cup \{0, 1\}$.
- 1,7-vertices: Each of them is connected to four vertices (internal, common-internal, out) colored using colors of set $\{2, 3, 4, 5, 6\}$. The vertex is at distance one from the other 1,7-vertex of the group and at distance two from one 1,7-vertex of the next group. Recall that all 1,7-vertices of the cycle node are colored by alternating between colors 1, 7. Also the vertex is at distance two from at least one pair of 0,8-vertices. So, one of the colors 1, 7 is available for each such vertex.
- Common-internal vertices: Each of them is at distance two from the two internal vertices colored $\{2, 6\}$ and the out-vertices of the same cycle node colored $i, i \in \{3, 4, 5\}$. It is at distance one from the 1,7-vertices of the group, hence it cannot take colors 1,2,6,7. Also the vertex is at distance two from at least a pair 0,8-vertices. Hence, in the set $\{3, 4, 5\}\backslash\{i\}$ there are two colors free for the two common-internal vertices.
- Internal vertices: Each of them is at distance two from three vertices (common-internal, out) colored using the colors of set $\{3, 4, 5\}$. Also, the vertex is at distance two from at least one pair of 0,8-vertices. It is at distance one from one of 1,7-vertex, hence it cannot take colors $\{0, 1, 2\}$ or $\{6, 7, 8\}$ and at distance two from the other 1,7-vertex. Hence, one of the colors 6 or 2 is available for each such vertex.
- Out-vertices: Each of them is at distance two from the two common-internal vertices colored using two colors of set $\{3, 4, 5\}$. The vertex is also at distance two from two internal vertices, one of its group and the other of the next group, colored 2 and 6. Also, the vertex is at distance one from a pair of 0,8′-vertices colored 0,8 and at distance one from two 1,7-vertices colored 1,7. Hence, one of the colors of set $\{3, 4, 5\}$ is available for each such vertex.

Now, consider any two out-vertices connecting two neighbor cycle nodes u, v. Since they take the corresponding colors as the vertices u, v in the 3-coloring of G, there is no conflict between any two of them.

Thus, the suggested radiocoloring assignment is valid. □

Lemma 14. *If* $X_{\text{span}}(G') \leqslant 9$ *then* $\chi(G) \leqslant 3$.

Proof. Consider any radiocoloring assignment of size 9 of G'. Then, we get that,

Claim 15. *Each pair of* 0,8-*vertices or* 0,8′-*vertices, neighbors to a vertex are colored* 0,8.

Proof. Each 0,8-vertex, 0,8′-vertex has even neighbors. In a set S of colors of range 9, if the vertex takes a color other than 0 or 8, then there will not be enough colors for its neighbors. Considering a pair of 0,8-vertices neighbors to a vertex, they take colors 0,8. □

Claim 16. *The common-internal, internal, out-vertices of any* $Group_i$, $1 \leqslant i \leqslant d(v)$ *of a cycle node* v *are colored using colors from set* $\{2, 3, 4, 5, 6\}$.

Proof. Each such vertex is connected to at least one pair of 0,8-vertices colored 0,8. Thus, they can take one of the colors from set $S \setminus \{0, 1, 7, 8\} = \{2, 3, 4, 5, 6\}$. □

Claim 17. *Each pair of* 1,7*-vertices of any* $Group_i$, $1 \leqslant i \leqslant d(v)$, *of a cycle node* v *are colored* 1, 7.

Proof. Each 1,7-vertex, has four neighbors ((common)-internal, out) colored using four colors from set $\{2, 3, 4, 5, 6\}$ by claim 16. Moreover, the vertex is at distance two from at least one pair of 0,8-vertices. If the vertex takes a color other than 1 or 7, then there will not be enough colors for its neighbors from the set S. Since, the two 1,7-vertices of a $Group_i$ are at distance one apart they take colors 1,7. □

Claim 18. *Any out-vertex or common-internal vertex of any* $Group_i$, $1 \leqslant i \leqslant d(v)$, *of a cycle node* v *is colored using one color from the set* $\{3, 4, 5\}$.

Proof. Any out-vertex (common-internal vertex) is at distance one from a pair of 0,8′-vertices (0,8-vertices). The vertex is at distance one from two 1,7-vertices at distance two (one) apart each other. Thus, the vertex cannot take colors 0, 1, 2, 6, 7, 8. Thus, it can take one of the colors 3, 4, 5. □

Claim 19. *Any internal vertex of any* $Group_i$, $1 \leqslant i \leqslant d(v)$ *of a cycle node* v *is colored using either* 2 *or* 6.

Proof. Any internal vertex is at distance one from a pair of 0,8-vertices. The vertex is at distance one from a 1,7-vertex and at distance two from the other 1,7-vertex of the group. Also, the vertex is at distance two from the two common-internal vertices and the out-vertex colored $\{3, 4, 5\}$ (by Claim 18). Thus, it can take the color 2 or 6, depending on the color of the 1,7-vertex neighbor to it. □

Claim 20. *For any cycle node* v, *assuming that one out-vertex is colored* i, $i \in \{3, 4, 5\}$, *then all out-vertices of the cycle node of* v *are colored* i.

Proof. Consider the next out-vertex of the cycle node of v. By Claim 18, the vertex is at distance one from the two common-internal vertices colored $\{3, 4, 5\} \setminus \{i\}$. Also, it is at distance one from two 1,7-vertices colored 1,7, and from a pair of 0,8′-vertices colored 0,8. Hence, it cannot take colors $\{0, 1, 2, 6, 7, 8\} \cup \{3, 4, 5\} \setminus \{i\}$. Thus the only color available for it is i. The argument holds for all consecutive out-vertices of the cycle node.

We now compute a 3-coloring of G as follows: Assign to each vertex u of the graph G the color that any out-vertex of the cycle node corresponding to it takes in G'. We argue that this is a valid 3-coloring of G. First note that, by Claim 18 we know that the computed assignment on G uses only 3 colors. Moreover, recall also that, by Claim 20, all out-vertices of a cycle node get the same color i, $i \in \{3, 4, 5\}$. Thus, for each cycle node u of G', all of its out-vertices (colored with the same color) ‘see’ the colors of the corresponding out-vertices of all neighbor cycle nodes of u. This is equivalent to the colors that the corresponding to u vertex in G ‘see’ by all of its neighbors. Hence, since the out-vertices of G' have no conflicts with their neighbor out-vertices, there is no conflict with the colors of any vertex in G and its neighbors. Thus, if G' can be radiocolored with a span of size 9, then there is a 3-coloring of G. □

(End of proof of Theorem 11). □

Corollary 21. *The following decision problem is* NP-*complete*:

Input: *A planar graph* $G(V, E)$ *and integers* $B_1, B_2, B_1 \geqslant B_2$.

Question: *Does there exist a radiocoloring for G with span no more than* B_1 *and order no more than* B_2?

5. A constant ratio approximation algorithm for min order RCP

We provide here an approximation algorithm for min order RCP for planar graphs by modifying the constructive proof of the theorem presented by Heuvel and McGuiness in [13]. Our algorithm is easier to verify with respect to correctness than what the proof given in [13] suggests. It also has better time complexity (i.e. $O(n\Delta)$) compared to the (implicit) algorithm in [13] which needs time $O(n^2)$. The improvement was achieved by performing the heavy part of the computation of the algorithm only in some instances of G instead of all as in [13]. This enables less checking and computations in the algorithm. Also, the behavior of our algorithm is very simple and more time efficient for graphs of small maximum degree. Finally, the algorithm provided here needs no planar embedding of G, as opposed to the algorithm implied in [13].

Very recently and independently, Agnarsson and Halldórsson in [2] presented approximations for the chromatic number of square and power graphs (G^k). Their method does not explicitly present an algorithm. A straightforward implementation is difficult and not efficient. Also, the approximation ratio for planar graphs of general Δ obtained is also 2.

The main theorem of Heuvel and McGuiness [13] states that a planar graph G can be radiocolored with at most $2\Delta+25$ colors. More specifically, the authors consider the problem of $L_{-}(p, q)$-Labeling, which is defined as follows:

Definition 22 ($L_{-}(p, q)$-*Labeling*). Find an assignment $L: V \longrightarrow \{0, 1, \ldots, v\}$, called $L_{-}(p, q)$-Labeling, which satisfies $|L(u)-L(v)| \geqslant p$ if $D(u, v) = 1$ and $|L(u)-L(v)| \geqslant q$ if $D(u, v) = 2$.

Definition 23. The minimum number v for which an $L_{-}(p, q)$-labeling exists is denoted by $\lambda(G; p, q)$ and is called p, q-span of G.

In other words, when the two vertices are at distance one apart, they should take colors (integers) that differ by at least p, and when they are located at distance two apart, they should take colors that differ by at least q. Note that $L_{-}(p, q)$-labeling is a generalization of radiocoloring since $L_{-}(p, q)$-labeling is equal to radiocoloring when $p = 2$ and $q = 1$. The main theorem of [13] is the following:

Theorem 24 (*Heuvel and McGuiness [13]*). *If G is a planar graph with maximum degree Δ and p, q are positive integers with $p \geqslant q$, then $\lambda(G;\, p, q) \leqslant (4q-2)\Delta + 10p + 38q - 23$.*

By setting $p = q = 1$ and using the observation $\lambda(G;\, 1, 1) = \chi(G^2)$, where $\chi(G^2)$ is the chromatic number of the graph G^2 (defined in the Introduction section), we get immediately, as also [13] notices, that:

Corollary 25 (*Heuvel and McGuiness [13]*). *If G is a planar graph with maximum degree Δ then $\chi(G^2) \leqslant 2\Delta + 25$.*

The theorem of [13] is proved using two lemmata. For an edge $e \in E(G)$, let $t(e)$ the number of triangular faces containing edge e and for a vertex $v \in V(G)$, let $t(v)$ be the number of triangular faces containing v, in the maximal planar graph of G. The first of the two lemmata, used to prove the theorem for the case where $\Delta(G) \geqslant 12$, is the following:

Lemma 26 (*Heuvel and McGuiness [13]*). *Let G be a simple planar graph. Then there exists a vertex v with k neighbors $v_1, v_2, \ldots, v_k$ with $d(v_1) \leqslant \cdots \leqslant d(v_k)$ such that one of the following is true*:
(i) $k \leqslant 2$;
(ii) $k = 3$ *with* $d(v_1) \leqslant 11$;
(iii) $k = 4$ *with* $d(v_1) \leqslant 7$ *and* $d(v_2) \leqslant 11$;
(iv) $k = 5$ *with* $d(v_1) \leqslant 6$, $d(v_2) \leqslant 7$, *and* $d(v_3) \leqslant 11$.

The second lemma, used to prove the theorem for the case where $\Delta(G) \leqslant 11$, is quite similar.

Lemma 27 (*Heuvel and McGuiness [13]*). *Let G be a simple planar graph with maximum degree Δ. Then there exists a vertex v with k neighbors $v_1, v_2, \ldots, v_k$ with $d(v_1) \leqslant \cdots \leqslant d(v_k)$ such that one of the following is true*:
(i) $k \leqslant 2$;
(ii) $k = 3$ *with* $d(v_1) \leqslant 5$;
(iii) $k = 3$ *with* $t(vv_i) \geqslant 1$ *for some i*;
(iv) $k = 4$ *with* $d(v_1) \leqslant 4$;
(v) $k = 4$ *with* $t(vv_i) = 2$ *for some i*;
(vi) $k = 5$ *with* $d(v_i) \leqslant 4$ *and* $t(vv_i) \geqslant 1$ *for some i*;
(vii) $k = 5$ *with* $d(v_i) \leqslant 5$ *and* $t(vv_i) = 2$ *for some i*;
(viii) $k = 5$ *with* $d(v_1) \leqslant 7$ *and* $t(vv_i) \geqslant 1$ *for all i*;
(ix) $k = 5$ *with* $d(v_1) \leqslant 5$, $d(v_2) \leqslant 7$, *and for each i with* $t(vv_i) = 0$ *it holds that* $d(v_i) \leqslant 5$.

These two lemmata give the so-called *unavoidable configurations* of G. The following operations apply to G: For an edge $e \in E$ let G/e denote the graph obtained from G by contracting e. For a vertex $v \in V$ let $G * v$ denote the graph obtained by deleting v and for each $u \in N(v)$ adding an edge between u and u^- and between u and u^+ (if these edges do not exist in G already). The notation $N(v)$ denotes the neighbors of v. The notation u^-, with

$u^- \in N(v)$, denotes the edge $vu-$ which directly precedes edge vu (moving clockwise), and u^+, with $u^+ \in N(v)$, refers to the edge $vu+$ which directly succeeds edge vu (moving clockwise).

The two lemmata are used to define the graph H, a vertex $v \in V(G)$ and an edge $e \in E(G)$ using the following rules:

- If $\Delta \geqslant 12$, then let v be as described in Lemma 26, and set $e = vv_1$ and $H = G/e$.
- If $6 \leqslant \Delta \leqslant 11$ and one of 27 (i), (ii), or (iv) holds, then let v be as described, and set $e = vv_1$ and $H = G/e$.
- If $6 \leqslant \Delta \leqslant 11$ and Lemma 27 (iii) holds, then let v be as described, set $e = vv_i$ with $t(vv_i) \geqslant 1$, and set $H = G/e$.
- If $6 \leqslant \Delta \leqslant 11$ and Lemma 27 (v) holds, then let v be as described, set $e = vv_i$ with $t(vv_i) = 2$ and set $H = G/e$.
- If $6 \leqslant \Delta \leqslant 11$ and Lemma 27 (vi) holds, then let v be as described, set $e = vv_i$ with $d(v_i) \leqslant 4$ and $t(vv_i) \geqslant 1$, and set $H = G/e$.
- If $6 \leqslant \Delta \leqslant 11$ and Lemma 27 (vii) holds, then let v be as described, set $e = vv_i$ with $d(v_i) \leqslant 5$ and $t(vv_i) = 2$, and set $H = G/e$.
- If $6 \leqslant \Delta \leqslant 11$ and Lemma 27 (viii) holds, then let v be as described and set $H = G * v$.
- If $6 \leqslant \Delta \leqslant 11$ and Lemma 27 (ix) holds, then let v be as described and set $H = G * v$.

The main idea of theorem of [13] is to define H to be $H = G/e$ or $H = G * v$, with $e = vv_1$ and $d(v) \leqslant 5$, depending on which case of the two Lemmata holds, so that always $\Delta(H) \leqslant \Delta$. Using these observations it is proved, by induction, that the minimum (p, q)-span needed for the $L_-(p, q)$-labeling of H is $\lambda(H; p, q) \leqslant (4q - 2)\Delta + 10p + 38q - 23$.

From H we can easily return to G as follows. If $H = G/e$ then let v' the new vertex created from the contraction of edge e. In this case, in G we set $v_1 = v'$ (this is a valid assumption since $d(v_1) \leqslant d(v')$) and color v_1 with the color of v'. Now we only need to color vertex v (for both cases of $H = G/e$ or $H = G * v$). From the way v was chosen, it can be easily seen that there is always one color free for the vertex in the set of colors of span $\leqslant (4q - 2)\Delta + 10p + 38q - 23$ as concluded for H.

For the case of radiocoloring of a planar graph G, we can use $p = 1$ and $q = 1$ for the order. Thus, the above theorem states that we need at most $2\Delta + 25$ colors.

5.1. The algorithm

We will use only Lemma 26 and the operation G/e in order to provide a much more simple and more efficient algorithm than what implied in [14]. We provide below a high-level description of our algorithm.

Algorithm Radiocoloring(G)

[I] Sort the vertices of the graph G by their degree.

[II] If $\Delta \leqslant 12$ then follow Procedure (1) below:
Procedure (1): Compute graph G^2. Consider the next vertex of the order. Delete v from G^2 to get $G^{2,}$. Now recursively color $G^{2,}$ with 145 colors. The number of colors that v has to avoid is at most $\Delta^2 = 144$. Thus, in a set of 145 colors, there is one free color for v.

[III] If $\Delta > 12$ then

(1) Find a vertex v and a neighbor v_1 of it, as described in Lemma 26, and set $e = vv_1$.

(2) Form $G' = G/e$ ($G' = (V', E')$ with $|V'| = n - 1$, while $|V| = n$) and denote the new vertex in G' obtained by the contraction of edge e as v'.

Modify the sorted degrees of G by deleting v, v_1, and inserting v' at the appropriate place, and also modify the possible affected degrees of the neighbors of both v and v_1.

(3) $\Phi(G') = \text{Radiocoloring}(G')$

(4) Extend $\Phi(G')$ to a valid radiocoloring of G:

(a) Set $v_1 = v'$ and give to v_1 the color of v'.

(b) Color v with one of the colors used in the radiocoloring Φ of G'.

5.2. Analysis of the algorithm

5.2.1. Correctness

Notice first that Procedure [1] implies a radiocoloring of G with $X = 145$ colors: Assign frequencies $1, 3, \ldots, 2X - 1$ to the obtained color classes of G.

Proposition 28. *The algorithm* Radiocoloring(G) *outputs a valid radiocoloring for G using no more than* $\max\{66, 2\Delta + 25\}$ *colors.*

Proof. By induction assume that, the recursive step 3 in [III] outputs a radiocoloring of G using at most $\max\{66, 2\Delta + 25\}$ colors. Note that $\Delta(G') = \Delta(G)$, because of the way v and $e = vv_1$ are chosen.

At step 4, the radiocoloring $\Phi(G')$ of G' is extended to a valid radiocoloring of G, using no more colors than those used in the previous step. This extension procedure is valid as explained here: At step (a) the vertex v_1 of G takes the color of the vertex v' of G'. This assignment is valid since v_1 has only a subset of the neighbors of v' at distance one and two.

Also, at step (b), the vertex v of G is colored with one of the colors used in the radiocoloring $\Phi(G')$ of G'. These colors are enough for v to get a valid color. The correctness of this claim is explained below.

For any vertex $v \in V(G)$, the number of vertices at distance two from v is equal to $\sum_{u \in N(v)} d(u) - d(v) - 2t(v)$. By the way v was chosen, it holds that $d(v) \leqslant 5$ and the above sum gives that there are at most $2\Delta + 19$ vertices at distance two from v. In total, the number of distance one and two neighbors of the vertex is $5 + (2\Delta + 19) = 2\Delta + 24$. Assuming that a palette of $2\Delta + 25$ colors is given, there is always one color free for v.

Thus, algorithm Radiocoloring(G) gives a valid radiocoloring to G using no more than $\max\{66, 2\Delta + 25\}$ colors. □

5.2.2. Time efficiency and approximation ratio

Lemma 29. *Our algorithm approximates* $X_{\text{order}}(G)$ *by a constant factor of at most* $\max\{2+\frac{25}{\Delta}, \frac{66}{\Delta}\}$.

Proof. Obviously, $X_{\text{order}}(G) > \Delta(G)$. By Proposition 28, our algorithm uses at most $\max\{66, 2\Delta + 25\}$ colors, i.e.

$$1 < \frac{X_{\text{order}}(G)}{\Delta} \leqslant \max\left\{\frac{66}{\Delta}, 2 + \frac{25}{\Delta}\right\}. \quad \Box \tag{1}$$

Lemma 30. *Our algorithm runs in* $O(n\Delta)$ *sequential time.*

Proof. Step [I] takes $O(n \log n)$ time and Step [II] takes $O(n)$ time. Let S be the set of neighbors of both v, v_1. Each implementation of [III].1, 2 needs time $d(v) + d(v_1) + \sum_{x \in S} d(x)$ in order to perform the operation G/e and $O(\log n)$ time to modify the sorted degree list. The total time spent recursively is then just $O(\sum_{v \in V} d(v) \cdot \log n) = O(n \log n)$. Each implementation of [III].4 needs $O(\Delta)$ time at most and this step is executed at most n times. Thus, the total time for all executions of [III].4 is $O(n\Delta)$. This dominates the total execution time. $\Box$

A more sophisticated and efficient implementation of the contraction operation can be found in [16].

6. An FPRAS for the number of radiocolorings of a planar graph

6.1. Sampling and counting

Let G be a *planar* graph of maximum degree $\Delta = \Delta(G)$ on vertex set $V = \{0, 1, \ldots, n-1\}$ and C be a set of λ colors. Let $\Phi: V \to C$ be a (proper) radiocoloring assignment of the vertices of G. Such a radiocoloring always exists if $\lambda \geqslant 2\Delta + 25$ and can be found by our $O(n\Delta)$ time algorithm of the previous section.

Consider the Markov Chain (X_t) whose state space $R = R_\lambda(G)$ is the set of all radiocolorings of G with λ colors and whose transition probabilities from state (radiocoloring) X_t are modelled by:

1. Choose a vertex $v \in V$ and a color $c \in C$ uniformly at random (u.a.r.)
2. Recolor vertex v with color c. If the resulting coloring X' is a *valid* radiocoloring assignment then let $X_{t+1} = X'$ else $X_{t+1} = X_t$.

The procedure above is similar to the “Glauber Dynamics” of an antiferromagnetic Potts model at zero temperature, and was used in [14] to estimate the number of proper colorings of any low degree graph with k colors.

The Markov Chain (X_t), which we refer to in the sequel as $M(G, \lambda)$, is *ergodic* (as we showed below), provided $\lambda \geqslant 2\Delta + 26$, in which case its stationary distribution is *uniform* over R. We show here that $M(G, \lambda)$ is *rapidly mixing*, i.e. converges, in time polynomial in n, to a close approximation of the stationary distribution, provided that $\lambda \geqslant 2(2\Delta + 25)$.

This can be used to get a fully polynomial randomized approximation scheme (fpras) for the number of radiocolorings of a planar graph G with λ colors, in the case where $\lambda \geqslant 4\Delta + 50$.

6.2. Some definitions and measures

For $t \in N$ let $P^t : R^2 \to [0, 1]$ denote the t-step transition probabilities of the Markov Chain $M(G, \lambda)$ so that $P^t(x, y) = \Pr\{X_t = y | X_0 = x\}, \forall x, y \in R$. It is easy to verify that $M(G, \lambda)$ is (a) *irreducible* and (b) *aperiodic*. The irreducibility of $M(G, \lambda)$ follows from the observation that any radiocoloring x may be transformed to any other radiocoloring y by sequentially assigning new colors to the vertices V in ascending sequence; before assigning a new color c to vertex v it is necessary to recolor all vertices $u > v$ that have color c. If we assume that $\lambda \geqslant 2\Delta + 26$ colors are given, removing the color c from this set, we are left with $\geqslant 2\Delta + 25$ for the coloring of the rest of the graph. The algorithm presented in previous section shows that the remaining graph can by radiocolored with a set of colors of this size. Hence, color c can be assigned to v.

Aperiodicity follows from the fact that the loop probabilities are $P(x, x) \neq 0, \forall x \in R$.

Thus, the finite Markov Chain $M(G, \lambda)$ is *ergodic*, i.e. it has a stationary distribution $\pi: R \to [0, 1]$ such that $\lim_{t \to \infty} P^t(x, y) = \pi(y), \forall x, y \in R$. Now if $\pi' : R \to [0, 1]$ is any function satisfying "local balance", i.e. $\pi'(x)P(x, y) = \pi'(y)P(y, x)$ then if $\sum_{x \in R} \pi'(x) = 1$ it follows that π' is indeed the stationary distribution. In our case $P(y, x) = P(x, y)$, thus the stationary distribution of $M(G, \lambda)$ is *uniform*.

The efficiency of any approach like this to sample radiocolorings crucially depends on the rate of convergence of $M(G, \lambda)$ to stationarity. There are various ways to define closeness to stationarity but all are essentially equivalent in this case and we will use the "variation distance" at time t with respect to initial vertex x

$$\delta_x(t) = \max_{S \subseteq R} |P^t(x, S) - \pi(S)| = \frac{1}{2} \sum_{y \in R} |P^t(x, y) - \pi(y)|,$$

where $P^t(x, S) = \sum_{y \in S} P^t(x, y)$ and $\pi(S) = \sum_{x \in S} \pi(x)$.

Note that this is a *uniform bound* over all events $S \subseteq R$ of the difference of probabilities of event S under the stationary and t-step distributions.

The *rate of convergence to stationarity* from initial vertex x is

$$\tau_x(\varepsilon) = \min\{t : \delta_x(t') \leqslant \varepsilon, \forall t' \geqslant t\}.$$

We also give the following definition:

Definition 31. A randomized approximation scheme for radiocolorings with λ colors of a planar graph G is a probabilistic algorithm that takes as input the graph G and an error bound $\varepsilon > 0$ and outputs a number Y (a random variable) such that

$$\Pr\{(1 - \varepsilon)|R_\lambda(G) \leqslant Y \leqslant (1 + \varepsilon)\,|R_\lambda(G)|\} \geqslant \tfrac{3}{4}.$$

Such a scheme is said to be *fully polynomial* if it runs in time polynomial in n and ε^{-1}. We abbreviate such schemes to *fpras*.

6.3. Rapid mixing

As indicated by the (by now standard) techniques for showing rapid mixing by *coupling* [14,15], our strategy here is to construct a coupling for $M = M(G, \lambda)$, i.e. a stochastic process (X_t, Y_t) on $R \times R$ such that each of the processes (X_t), (Y_t), considered in isolation, is a faithful copy of M. We will arrange a joint probability space for (X_t), (Y_t) so that, far from being independent, the two processes tend to *couple* so that $X_t = Y_t$ for t large enough. If coupling can occur rapidly (independently of the initial states X_0, Y_0), we can infer that M is rapidly mixing, because the variation distance of M from the stationary distribution is bounded above by the probability that (X_t) and (Y_t) *have not coupled* by time t.

The key result we use here is the *Coupling Lemma* (see [11, Chapter 4] by Jerrum), which apparently makes its first explicit appearance in the work of Aldous [1, Lemma 3.6] (see also Diaconis [5, Chapter 4, Lemma 5]).

Lemma 32. *Suppose that M is a countable, ergodic Markov chain with transition probabilities* $P(\cdot, \cdot)$ *and let* $((X_t, Y_t), t \in \mathbb{N})$ *be a coupling of M. Suppose further that* $t : (0, 1] \to \mathbb{N}$ *is a function such that* $\Pr(X_{t(\varepsilon)} \neq Y_{t(\varepsilon)}) \leqslant \varepsilon,\ \forall \varepsilon \in (0, 1]$, *uniformly over the choice of initial state* (X_0, Y_0). *Then the mixing time* $\tau(\varepsilon)$ *of M is bounded above by* $t(\varepsilon)$.

The transition $(X_t, Y_t) \to (X_{t+1}, Y_{t+1})$ in the coupling is defined by the following experiment:
1. Select $v \in V$ uniformly at random (u.a.r.).
2. Compute a permutation $g(G, X_t, Y_t)$ of C according to a procedure to be explained.
3. Choose a color $c \in C$ u.a.r.
4. In the radiocoloring X_t (respectively Y_t) recolor vertex v with color c (respectively $g(c)$) to get a new radiocoloring X' (respectively Y').
5. If X' (respectively Y') is a (valid) radiocoloring then $X_{t+1} = X'$ (respectively $Y_{t+1} = Y'$), else let $X_{t+1} = X_t$ (respectively $Y_{t+1} = Y_t$).

Note that, whatever procedure is used to select the permutation g, the distribution of $g(c)$ is *uniform*, thus (X_t) and (Y_t) are both faithful copies of M.

We now remark that any set of vertices $F \subseteq V$ can have the same color in the graph G^2 only if they can have the same color in some radiocoloring of G. Thus, given a proper coloring of G^2 with λ' colors, we can construct a proper radiocoloring of G by giving the values (new colors) $1, 3, \ldots, 2\lambda' - 1$ in the color classes of G^2. Note that this transformation preserves the number of colors (but not the span).

Now let $A = A_t \subseteq V$ be the set of vertices on which the colorings of G^2 implied by X_t, Y_t agree and $D = D_t \subseteq V$ be the set on which they disagree. Let $d'(v)$ be the number of edges incident at v in G^2 that have one point in A and one in D. Clearly, if m' is the number of edges of G^2 spanning A, D, we get $\sum_{v \in A} d'(v) = \sum_{v \in D} d'(v) = m'$.

The procedure to compute $g(G, X_t, Y_t)$ is as follows:
(a) If $v \in D$ then g is the identity.
(b) If $v \in A$ then proceed as follows: Denote by N the set of neighbors of v in G^2. Define $C_x \subseteq C$ to be the set of all colors c, such that some vertex in N receives c in radiocoloring Y_t but no vertex in N receives c in radiocoloring Y_t. Let C_y be defined as C_x with the roles of X_t, Y_t interchanged. Observe $C_x \cap C_y = \emptyset$ and $|C_x|, |C_y| \leqslant d'(v)$.

Let, w.l.o.g., $|C_x| \leqslant |C_y|$. Choose any subset $C'_y \subseteq C_y$ with $|C'_y| \leqslant |C_x|$ and let $C_x = \{c_1, \ldots, c_r\}$, $C'_y = \{c'_1, \ldots, c'_r\}$ be enumerations of C_x, $C_{y'}$ coming from the orderings of X_t, Y_t. Finally, let g be the permutation $(c_1, c'_1), \ldots, (c_r, c'_r)$ which interchanges the color sets C_x, $C_{y'}$ and leaves all other colors fixed.

It is clear that $|D_{t+1}| - |D_t| \in \{-1, 0, 1\}$.

(i) Consider first the probability that $|D_{t+1}| = |D_t| + 1$. For this event to occur, the vertex v selected in step (1) of the procedure for g must lie in A and hence we follow (b). If the new radiocolorings are to disagree at vertex v then the color c selected in line (3) must be an element of C_y. But $|C_y| \leqslant d'(v)$ hence

$$\Pr\{|D_{t+1}| = |D_t| + 1\} \leqslant \frac{1}{n} \sum_{v \in A} \frac{d'(v)}{\lambda} = \frac{m'}{\lambda \cdot n} \tag{2}$$

(ii) Now consider the probability that $|D_{t+1}| = |D_t| - 1$. For this to occur, the vertex v must lie in D and hence the permutation g selected in line (2) is the identity. For X_{t+1}, Y_{t+1} to agree at v, it is enough that color c selected in step (3) is different from all the colors that X_t, Y_t imply for the neighbors of v in G^2. The number of colors c that satisfy this is (by our previous results) at least $\lambda - 2(2\Delta + 25) + d'(v)$. Hence

$$\begin{aligned} \Pr\{|D_{t+1}| = |D_t| - 1\} &\geqslant \frac{1}{n} \sum_{v \in D} \frac{\lambda - 2(2\Delta + 25) + d'(v)}{\lambda} \\ &\geqslant \frac{\lambda - 2(2\Delta + 25)}{\lambda n} |D| + \frac{m'}{\lambda n}. \end{aligned} \tag{3}$$

Define now

$$\alpha = \frac{\lambda - 2(2\Delta + 25)}{\lambda n} \quad \text{and} \quad \beta = \frac{m'}{\lambda n}.$$

So

$$\Pr\{|D_{t+1}| = |D_t| + 1\} \leqslant \beta$$

and $\Pr\{|D_{t+1}| = |D_t| - 1\} \geqslant \alpha |D_t| + \beta$. Given $\alpha > 0$, i.e. $\lambda > 2(2\Delta + 25)$, from Eqs. (2) and (3), we get

$$\begin{aligned} E(|D_{t+1}|) &\leqslant \beta(|D_t| + 1) + (\alpha|D_t| + \beta)(|D_t| - 1) + (1 - \alpha|D_t| - 2\beta)|D_t| \\ &= (1 - \alpha)|D_t|. \end{aligned}$$

Thus, from Bayes, we get $E(|D_{t+1}|) \leqslant (1 - \alpha)^t |D_0| \leqslant n(1 - \alpha)^t$ and since $|D_t|$ is a non-negative random variable, we get, by Markov inequality, that

$$\Pr\{D_t \neq 0\} \leqslant n(1 - \alpha)^t \leqslant n e^{-\alpha t}.$$

So, we note that, $\forall \varepsilon > 0$, $\Pr\{D_t \neq \emptyset\} \leqslant \varepsilon$ provided that $t \geqslant (1/\alpha) \ln(n/\varepsilon)$ thus proving:

Theorem 33. *Let G be a planar graph of maximum degree Δ on n vertices. Assuming $\lambda \geqslant 2(2\Delta + 25)$ the convergence time $\tau(\varepsilon)$ of the Markov Chain $M(G, \lambda)$ is bounded above by*

$$\tau_x(\varepsilon) \leqslant \frac{\lambda}{\lambda - 2(2\Delta + 25)} n \ln\left(\frac{n}{\varepsilon}\right)$$

regardless of the initial state x.

6.4. An fpras for radiocolorings with λ colors

The technique we employ is as in [14] and is fairly standard in the area. By using it we get the following theorem:

Theorem 34. *There is a fully polynomial randomized approximation scheme (fpars) for the number of radiocolorings of a planar graph G with λ colors, provided $\lambda > 2(2\Delta + 25)$, where Δ is the maximum degree of G.*

Proof. Recall that $R_\lambda(G)$ is the set of all radiocolorings of G with λ colors. Let m be the number of edges in G and let

$$G = G_m \supseteq G_{m-1} \supseteq \cdots \supseteq G_1 \supseteq G_0$$

be any sequence of graphs where G_{i-1} is obtained by G_i by removing a single edge. We can always erase an edge whose one node is of degree at most 5 in G_i. Clearly

$$|R_\lambda(G)| = \frac{|R_\lambda(G_m)|}{|R_\lambda(G_{m-1})|}\frac{|R_\lambda(G_{m-1})|}{|R_\lambda(G_{m-2})|}\cdots\frac{|R_\lambda(G_1)|}{|R_\lambda(G_0)|}\,|R_\lambda(G_0)|.$$

But $|R_\lambda(G_0)| = \lambda^n$ for all kinds of colorings. The standard strategy is to estimate the ratio

$$\rho_i = \frac{|R_\lambda(G_i)|}{|R_\lambda(G_{i-1})|}$$

for each i, $1 \leqslant i \leqslant m$.

Suppose that graphs G_i, G_{i-1} differ in the edge $\{u, v\}$ which is present in G_i but not in G_{i-1}. Clearly, $R_\lambda(G_i) \subseteq R_\lambda(G_{i-1})$. Any radiocoloring in $R_\lambda(G_{i-1}) \backslash R_\lambda(G_i)$ assigns either the same color to u, v or the color values of u, v differ by only 1. Let $\deg(v) \leqslant 5$ in G_i. So, we now have to recolor u with one of at least $\lambda - (2\Delta + 25)$, i.e. at least $2\Delta + 25$, colors (from Section 5 of this paper). Each radiocoloring of $R_\lambda(G_i)$ can be obtained in at most one way by our algorithm of the previous section as the result of such a perturbation. Thus,

$$\frac{1}{2} \leqslant \frac{2\Delta + 25}{2(\Delta + 1) + 25} \leqslant \rho_i < 1. \tag{4}$$

To avoid trivialities, assume $0 < \varepsilon \leqslant 1$, $n \geqslant 3$ and $\Delta > 2$. Let $Z_i \in \{0, 1\}$ be the random variable obtained by simulating the Markov Chain $M(G_{i-1}, \lambda)$ from any certain fixed initial state for

$$T = \frac{\lambda}{\lambda - 2(2\Delta + 25)} n \ln\left(\frac{4nm}{\varepsilon}\right)$$

steps and returning to 1 if the final state is a member of $R_\lambda(G_i)$ and 0 else.

Let $\mu_i = E(Z_i)$. By our theorem of rapid mixing, we have

$$\rho_i - \frac{\varepsilon}{4m} \leqslant \mu_i \leqslant \rho_i + \frac{\varepsilon}{4m}$$

and by Eq. (4), we get

$$\left(1 - \frac{\varepsilon}{2m}\right) \rho_i \leqslant \mu_i \leqslant \left(1 + \frac{\varepsilon}{2m}\right) \rho_i .$$

As our estimator for $|R_\lambda(G)|$ we use

$$Y = \lambda^n Z_1 Z_2 \cdots Z_m .$$

Note that $E(Y) = \lambda^n \mu_1 \mu_2 \cdots \mu_m$. But

$$\mathrm{Var}(Y) \leqslant \frac{\mathrm{Var}(Z_1 Z_2 \cdots Z_m)}{(\mu_1 \mu_2 \cdots \mu_m)^2} = \prod_{i=1}^{m} \left(1 + \frac{\mathrm{Var}(Z_i)}{\mu_i^2}\right) - 1.$$

By standard ways of working (as in [14]) one can easily show that Y satisfies the requirements for an fpras for the number of radiocolorings of graph G with λ colors $|R_\lambda(G)|$. □

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Further reading

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