Optimal, Distributed Decision-Making: The Case of No Communication

Stavros Georgiades
University of Cyprus, Nicosia, Cyprus

Marios Mavronicolas
University of Cyprus, Nicosia, Cyprus

Paul Spirakis
University of Patras & Computer Technology Institute, Patras, Greece

(April 24, 2000)

We are honored to dedicate this work to the memory of *Professor Gian-Carlo Rota* (1932–1999).

A preliminary version of this work appears in the *Proceedings of the 12th International Symposium on Fundamentals of Computation Theory,* G. Ciobanu and G. Paun eds., pp. 293–303, Vol. 1684, Lecture Notes in Computer Science, Springer-Verlag, Iasi, Romania, August/September 1999.

This work has been supported in part by the European Union's ESPRIT Long Term Research Projects ALCOM-IT (contract # 20244) and GEPPCOM (contract # 9072), and by the Greek Ministry of Education.

Part of the work of M. Mavronicolas was done while at Department of Computer Science and Engineering, University of Connecticut, Storrs, CT, and while at AT&T Labs – Research, Florham Park, NJ, as a visitor to the DIMACS Special Year on Networks, organized by the DIMACS Center for Discrete Mathematics and Theoretical Computer Science, Piscataway, NJ.

Authors' addresses: S. Georgiades, Department of Computer Science, University of Cyprus, P. O. Box 1537, Nicosia, CY-1678, Cyprus; M. Mavronicolas, Department of Computer Science, University of Cyprus, P. O. Box 1537, Nicosia, CY-1678, Cyprus; P. Spirakis, Department of Computer Engineering and Informatics, University of Patras, Rion, 265 00 Patras, Greece, & Computer Technology Institute, P. O. Box 1122, 261 10 Patras, Greece.

Abstract. We present a combinatorial framework for the study of a natural class of *distributed optimization* problems that involve *decision-making* by a collection of n *distributed agents* in the presence of incomplete information; such problems were originally considered in a *load balancing* setting by Papadimitriou and Yannakakis (*Proceedings of the 10th Annual ACM Symposium on Principles of Distributed Computing,* pp. 61–64, August 1991). Within our framework, we are able to settle completely the case where *no communication* is allowed anong the agents. For that case, for any given *decision protocol,* our framework allows to obtain a combinatorial inclusion-exclusion expression for the probability that no "overflow" occurs, called the *winning probability,* in terms of the *volume* of some simple combinatorial polytope.

Within our general framework, we offer a complete resolution to the special cases of *oblivious algorithms,* for which agents do not "look at" their inputs, and *non-oblivious algorithms,* for which they do, of the general optimization problem. In either case, we derive optimality conditions in the form of combinatorial polynomial equations. For oblivious algorithms, we explicitly solve these equations to show that the optimal algorithm is simple and *uniform,* in the sense that agents need not "know" n. Most interestingly, we show that optimal non-oblivious algorithms must be *non-uniform:* we demonstrate that the optimality conditions admit different solutions for particular, different "small" values of n; however, these solutions improve in terms of the winning probability over the optimal, oblivious algorithm. Our results demonstrate an interesting trade-off between the amount of knowledge used by agents and uniformity for optimal, distributed decision-making with no communication.

Categories and Subject Descriptors: C.2.1 [**Computer-Communication Networks**]: Network Architecture and Design– *distributed networks*; C.2.3 [**Computer-Communication Networks**]: Network Operations– *network management*; C.2.4 [**Computer-Communication Networks**]: Distributed Systems– *distributed applications*; C.4 [**Performance of Systems**]: *performance attributes*; F.1.1 [**Computation by Abstract Devices**]: Models of Computation– *relations among models*; F.1.2 [**Computation by Abstract Devices**]: Modes of Computation– *probabilistic computation*; F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Nonnumerical Algorithms and Problems– *computations on discrete structures, sequencing and scheduling*; G.2.1 [**Discrete Mathematics**]: Combinatorics– *combinatorial algorithms, counting problems*;

General Terms: Algorithms, Performance, Theory

Additional Key Words and Phrases: distributed computing, decision-making, oblivious and non-oblivious protocols, uniform and non-uniform protocols, optimality, principle of inclusion-exclusion.

Contents

1 Introduction

In a *distributed optimization problem,* each of n *distributed agents* receives a private *input,* communicates possibly with other agents to learn about their own inputs, and decides, based on this possibly partial knowledge, on an *output*; the task is to maximize a common objective function. Such problems were originally introduced by Papadimitriou and Yannakakis [11], in an effort to understand the crucial economic value of *information* [1] as a computational resource in a distributed system (see, also, [2, 5, 10, 12]). Intuitively, the more information available to agents, the better decisions they make, but naturally the more expensive the solution becomes due to the need for increased communication. Such natural trade-offs between communication cost and the quality of decision-making have been studied in the contexts of *communication complexity* [9] and *concurrency control* [8] as well.

Papadimitriou and Yannakakis [11] examined the special case of such distributed optimization problems where there are just *three* agents. More specifically, Papadimitriou and Yannakakis focused on a natural *load balancing* problem (see, e.g., [4, 7, 13], where each agent is presented with an input, and must decide on a binary output, representing one of two available "bins," each of capacity one; the input is assumed to be distributed uniformly in the unit interval $[0, 1]$. The load balancing property is modeled by requiring that no "overflow" occurs, namely that inputs dropped into each "bin" not exceed together its capacity. Papadimitriou and Yannakakis [11] pursued a comprehensive study of how the best possible probability, over the distribution of inputs, of "no overflow" depends on the amount of communication available to the agents. For each possible communication pattern, Papadimitriou and Yannakakis [11] discovered the corresponding optimal decision protocol to be unexpectedly sophisticated. The proof techniques of Papadimitriou and Yannakakis [11] were surprisingly complex, even for this seemingly simplest case, combining tools from nonlinear optimization with geometric and combinatorial arguments; these techniques have not been hoped to be conveniently extendible to instances of even this particular load balancing problem whose size exceeds three.

In this work, we introduce a novel combinatorial framework in order to enhance the study of general instances of distributed optimization problems of the kind considered by Papadimitriou and Yannakakis [11]. More specifically, we proceed to the general case of n agents, with each still receiving an input uniformly distributed over $[0, 1]$ and having to choose one out of two "bins"; however, in order to render the problem interesting, we make the technical assumption that the capacity of each "bin" is equal to δ, for some real number δ possibly greater than one, so as to compensate for the increase in the number of players. Papadimitriou and Yannakakis [11] focused on a specific kind of decision protocols by which each agent chooses a "bin" by comparing a "weighted average" of the inputs it "sees" against some "threshold" value; in contrast, our framework allows for the consideration of *general* decision protocols by which each agent decides by using *any* (computable) function of the inputs it "sees".

Our starting point is a combinatorial result that provides an explicit *inclusion-exclusion* formula [17, Section 2.1] for calculating the *volume* of any particular geometric polytope, in any given dimension, of some speficic form (Proposition 2.2). Roughly speaking, such polytopes are the intersection of a simplex in the positive quadrant with an orthogonal parallelepiped. An immediate implication of this result are inclusion-exclusion formulas for calculating the (conditional) probability of "no overflow" for a *single* "bin," as a function of the capacity δ and the number of inputs that are dropped into the "bin" (Lemmas 2.4 and 2.7).

In this work, we focus on the case where there is no communication among the agents, which we completely settle for the case of general n. Since communication comes at a cost, which it would be desirable to avoid, it is both natural and interesting to choose the case of no communication as an initial "testbed". We consider both *oblivious algorithms,* where players do not "look" at their inputs, and *non-oblivious algorithms,* where they do. For each case, we are interested in optimal algorithms.

We first consider oblivious algorithms. Our first major result is a combinatorial expression in the form of an inclusion-exclusion formula for the probability that "no overflow" occurs for either of the "bins" (Theorem 4.1). This formula incorporates a suitable inclusion-exclusion summation, over all possible input vectors, of the probabilities, induced by any particular decision algorithm, on the space of all possible decision vectors, as a function of the corresponding input vector. The coefficients of these probabilities in the summation are independent of any specific parameters of the algorithm, while they do depend on the input vector. A first implication of this expression is the reduction of the general problem of computing the probability that "no overflow" occurs to the problem of computing, given a particular decision algorithm, the probability distribution of the binary output vectors it yields. Most significantly, this expression contributes a methodology for the design of *optimal* decision algorithms "compatible" with any specific pattern of communication, and not just for the case of no communication that we particularly examine: one simply renders only those parameters of the decision algorithm that correspond to the possible communications, and computes values for these parameters that maximize the combinatorial expression as a function of these parameters. This is done by solving a certain system of *optimality conditions* (Corollary 4.2).

We demonstrate that our methodology for designing optimal algorithms for distributed decision-making is both effective and useful by applying it to the special case of no communication that we consider. We manage to settle down completely this case for oblivious algorithms. We exploit the underlying "symmetry" with respect to different agents in order to simplify the optimality conditions (by observing that all parameters satisfying them must be equal). This simplification reveals a beautiful combinatorial structure; more specifically, we discover that each optimality condition eventually amounts to zeroing a particular "symmetric" polynomial of a single variable. In turn, we explicitly solve these conditions to show that the best possible oblivious algorithm for the case of no communication is the very simple one by which each agent uses $1/2$ as its "threshold" value; given that the optimal (non-oblivious) algorithms presented by Papadimitriou and Yannakakis for the special case where $n = 3$ are somehow unexpectedly sophisticated, it is perhaps surprising that such simple oblivious algorithm is indeed optimal for *all* values of n.

We next turn to non-oblivious algorithms, still for the case of no communication. In that case, we demonstrate that the optimality conditions do not admit a "constant" solution.

Through a more sophisticated analysis, we are able to compute more complex expressions for the optimality conditions, which still allow exploitation of "symmetry". We consider the particular instances of the optimality conditions where $n = 3$ and $\delta = 1$ (considered by Papadimitriou and Yannakakis [11]), and $n = 4$ and $\delta = 4/3$. We discover that the optimal algorithms are different in each of these cases. However, they achieve larger winning probabilities than their oblivious counterparts. This shows that the improved performance of non-oblivious algorithms comes at the cost of sacrificing *uniformity*.

We believe that our work opens up the way for the design and analysis of algorithms for general instances of the problem of distributed decision-making in the presence of incomplete information. We envision that algorithms that are more complex, general communication patterns, and more realistic assumptions on the distribution of inputs, can all be treated in our combinatorial framework to yield optimal algorithms for distributed decision-making for these cases as well.

The rest of this paper is organized as follows. Section 2 presents a framework for distributed decision-making. Formal definitions for the model and the problem are included in Section 3. Oblivious and non-oblivious algorithms are treated in Sections 4 and 5, respectively. We conclude, in Section 6, with a discussion of our results and some open problems.

2 Combinatorial Framework

In this section, we introduce a combinatorial framework for our study of distributed decision-making. Section 2.1 offers some geometrical preliminaries; built on their top are two probabilistic lemmas presented in Section 2.2. Finally, Section 2.3 summarizes some notation.

2.1 Geometrical Preliminaries

Throughout, we use $\Re^+$ to denote the set of non-negative real numbers. A *polyhedron* is the solution set of a finite system of linear inequalities (cf. [3, Section 6.2]). A polyhedron is *bounded* if it contains no infinite half-line. A polyhedron of this type is called a *polytope*. We will be interested in computing volumes of polytopes that have some specific form. For any polytope $\mathbf{\Pi}$, denote $\mathsf{Vol}(\mathbf{\Pi})$ the *volume* of $\mathbf{\Pi}$.

Fix any integer $m \geq 2$. Consider any pair of (real) vectors $\overline{\sigma} = \langle \sigma_1, \sigma_2, \ldots, \sigma_m \rangle^{\mathrm{T}}$, and $\overline{\pi} = \langle \pi_1, \pi_2, \ldots, \pi_m \rangle^{\mathrm{T}}$, where for any l, $1 \leq l \leq m$, $0 < \sigma_l, \pi_l < \infty$. These define the m-dimensional, orthogonal simplex

$$\mathbf{\Sigma}^{(m)}(\overline{\sigma}) \quad = \quad \{\langle x_1, x_2, \ldots, x_m \rangle^{\mathrm{T}} \in \Re^+ \ \mid \ \sum_{l=1}^{m} \frac{x_l}{\sigma_l} \leq 1\}\,,$$

with orthogonal sides $\sigma_1, \sigma_2, \ldots, \sigma_m$, and the m-dimensional orthogonal parallelepiped

$$\mathbf{\Pi}^{(m)}(\overline{\pi}) \quad = \quad [0, \pi_1] \times [0, \pi_2] \times \ldots \times [0, \pi_m]\,,$$

with orthogonal sides $\pi_1, \pi_2, \ldots, \pi_m$, respectively.

Both $\mathbf{\Sigma}^{(m)}(\overline{\sigma})$ and $\mathbf{\Pi}^{(m)}(\overline{\pi})$ are polytopes. The following lemma recalls simple expressions for the volumes of $\mathbf{\Sigma}^{(m)}(\overline{\sigma})$ and $\mathbf{\Pi}^{(m)}(\overline{\pi})$.

Lemma 2.1 (Volumes of $\mathbf{\Sigma}^{(m)}(\overline{\sigma})$ and $\mathbf{\Pi}^{(m)}(\overline{\pi})$) *1.* $\mathsf{Vol}(\mathbf{\Sigma}^{(m)}(\overline{\sigma})) = \frac{1}{m!}\prod_{l=1}^{m}\sigma_l$;

2. $\mathsf{Vol}(\mathbf{\Pi}^{(m)}(\overline{\pi})) = \prod_{l=1}^{m}\pi_l$.

A cornerstone for our analysis is a particular polytope that has some specific form. Define the m-dimensional polytope

$$\mathbf{\Sigma\Pi}^{(m)}(\overline{\sigma}, \overline{\pi}) \quad = \quad \mathbf{\Sigma}^{(m)}(\overline{\sigma}) \cap \mathbf{\Pi}^{(m)}(\overline{\pi})\,;$$

thus, $\mathbf{\Sigma\Pi}^{(m)}(\overline{\sigma}, \overline{\pi})$ is the intersection of the m-dimensional orthogonal simplex $\mathbf{\Sigma}^{(m)}(\overline{\sigma})$ and the m-dimensional orthogonal parallelepiped $\mathbf{\Pi}^{(m)}(\overline{\pi})$, so that

$$\mathbf{\Sigma\Pi}^{(m)}(\overline{\sigma}, \overline{\pi}) \quad = \quad \{\langle x_1, x_2, \ldots, x_m\rangle^{\mathrm{T}} \in [0, \pi_1] \times [0, \pi_2] \times \ldots \times [0, \pi_m] \ \mid\ \sum_{l=1}^{m}\frac{x_l}{\sigma_l} \leq 1\}\,.$$

We provide an explicit *inclusion-exclusion* formula (see, e.g., [14], or [15, Chapter III] for a textbook discussion) for the volume of $\mathbf{\Sigma\Pi}^{(m)}(\overline{\sigma}, \overline{\pi})$.

Proposition 2.2 (Volume of $\mathbf{\Sigma\Pi}^{(m)}(\overline{\sigma}, \overline{\pi})$)

$$\mathsf{Vol}(\mathbf{\Sigma\Pi}^{(m)}(\overline{\sigma}, \overline{\pi})) \quad = \quad \frac{1}{m!}\prod_{l=1}^{m}\sigma_l \cdot \sum_{i=0}^{m}(-1)^i \sum_{\substack{\mathcal{I} \subseteq \{1, 2, \ldots, m\},\\ |\mathcal{I}| = i,\\ \sum_{l\in\mathcal{I}}\pi_l/\sigma_l < 1}} \left(1 - \sum_{l\in\mathcal{I}}\frac{\pi_l}{\sigma_l}\right)^m$$

Proof: Clearly,

$$\begin{aligned}
& \mathsf{Vol}(\mathbf{\Sigma\Pi}^{(m)}(\overline{\sigma}, \overline{\pi})) \\
= \ & \mathsf{Vol}(\{\langle x_1, x_2, \ldots, x_m\rangle^{\mathrm{T}} \in [0, \pi_1] \times [0, \pi_2] \times \ldots \times [0, \pi_m] \ \mid\ \sum_{l=1}^{m}\frac{x_l}{\sigma_l} \leq 1\}) \\
& \text{(by definition of } \mathbf{\Sigma\Pi}^{(m)}(\overline{\sigma}, \overline{\pi})) \\
= \ & \mathsf{Vol}(\{\langle x_1, x_2, \ldots, x_m\rangle^{\mathrm{T}} \in \Re^+ \ \mid\ \sum_{l=1}^{m}\frac{x_l}{\sigma_l} \leq 1\}) \\
& -\sum_{i=1}^{m}\mathsf{Vol}(\{\langle x_1, x_2, \ldots, x_m\rangle^{\mathrm{T}} \in \Re^+ \ \mid\ \sum_{l=1}^{m}\frac{x_l}{\sigma_i} \leq 1 \text{ and } x_i \notin [0, \pi_i]\})
\end{aligned}$$

$$+\sum_{1\le i<j\le m} \mathsf{Vol}(\{\langle x_1,x_2,\ldots,x_m\rangle^{\mathrm T}\in\Re^+ \ \mid\ \sum_{l=1}^{m}\frac{x_l}{\sigma_l}\le 1 \text{ and } \bigwedge_{l=i,j} x_l\notin[0,\pi_l]\})$$
$$-\ldots+\ldots$$
$$+(-1)^m\mathsf{Vol}(\{\langle x_1,x_2,\ldots,x_m\rangle^{\mathrm T}\in\Re^+ \ \mid\ \sum_{l=1}^{m}\frac{x_l}{\sigma_l}\le 1 \text{ and } \bigwedge_{1\le l\le m} x_l\notin[0,\pi_l]\})$$

(by the principle of inclusion-exclusion)

$$= \frac{1}{m!}\prod_{l=1}^{m}\sigma_l$$
$$-\sum_{i=1}^{m}\mathsf{Vol}(\{\langle x_1,x_2,\ldots,x_m\rangle^{\mathrm T}\in\Re^+ \ \mid\ \sum_{l=1}^{m}\frac{x_l}{\sigma_i}\le 1 \text{ and } x_i\notin[0,\pi_i]\})$$
$$+\sum_{1\le i<j\le m} \mathsf{Vol}(\{\langle x_1,x_2,\ldots,x_m\rangle^{\mathrm T}\in\Re^+ \ \mid\ \sum_{l=1}^{m}\frac{x_l}{\sigma_l}\le 1 \text{ and } \bigwedge_{l=i,j} x_l\notin[0,\pi_l]\})$$
$$-\ldots+\ldots$$
$$+(-1)^m\mathsf{Vol}(\{\langle x_1,x_2,\ldots,x_m\rangle^{\mathrm T}\in\Re^+ \ \mid\ \sum_{l=1}^{m}\frac{x_l}{\sigma_l}\le 1 \text{ and } \bigwedge_{1\le l\le m} x_l\notin[0,\pi_l]\})$$

(by Lemma 2.1(2))

$$= \frac{1}{m!}\prod_{l=1}^{m}\sigma_l+\sum_{i=1}^{+}(-1)^i\cdot$$
$$\sum_{\substack{\mathcal{I}\subseteq\{1,2,\ldots,m\},\\ |\mathcal{I}|=i}} \mathsf{Vol}(\{\langle x_1,x_2,\ldots,x_m\rangle^{\mathrm T}\in\Re^+ \ \mid\ \sum_{l=1}^{m}\frac{x_l}{\sigma_l}\le 1 \text{ and } \bigwedge_{l\in\mathcal{I}} x_l\notin[0,\pi_l]\})\,.$$

We continue to calculate the polytope volumes involved in the last summation.

Lemma 2.3 *For any non-empty set $\mathcal{I}\subseteq\{1,2,\ldots,m\}$*

$$\mathsf{Vol}(\{\langle x_1,x_2,\ldots,x_m\rangle^{\mathrm T}\in\Re^+ \ \mid\ \sum_{l=1}^{m}\frac{x_l}{\sigma_l}\le 1 \textit{ and } \bigwedge_{l\in\mathcal{I}} x_l\notin[0,\pi_l]\})$$
$$= \frac{1}{m!}\prod_{l=1}^{m}\sigma_l\cdot\left(1-\sum_{l\in\mathcal{I}}\frac{\pi_l}{\sigma_l}\right)^m,$$

if $1>\sum_{l\in\mathcal{I}}\pi_l/\sigma_l$, and 0 otherwise.

Proof: Clearly, the hyperplanes $\sum_{l\in\mathcal{I}}x_l/\sigma_l=1$ and $x_l=\pi_l$, $l\in\mathcal{I}$, intersect if and only if $\sum_{l\in\mathcal{I}}\pi_l/\sigma_l=1$. Since the polytope

$$\{\langle x_1,x_2,\ldots,x_m\rangle^{\mathrm T}\in\Re^+ \ \mid\ \sum_{l=1}^{m}\frac{x_l}{\sigma_l}\le 1 \text{ and } \bigwedge_{l\in\mathcal{I}} x_l\notin[0,\pi_l]\}$$

is the intersection of the half-spaces $\sum_{l\in\mathcal{I}} x_l/\sigma_l \leq 1$, $x_l \geq \pi_l$ for $l \in \mathcal{I}$ and $x_l \geq 0$ for $1 \leq l \leq m$, it follows that this polytope is non-null if and only if $\sum_{l\in\mathcal{I}} \pi_l/\sigma_l < 1$.

In that case, this polytope is an orthogonal simplex; its orthonormal faces lie on the hyperplanes $x_l = \pi_l$ for $l \in \mathcal{I}$ and $x_l = 0$ for $l \notin \mathcal{I}$, while its non-orthonormal one lies on the hyperplane $\sum_{l\in\mathcal{I}} x_l/\sigma_l = 1$. Thus, this polytope is similar to the original orthogonal simplex

$$\{\langle x_1, x_2, \ldots, x_m\rangle^{\mathrm{T}} \in \Re^+ \;\mid\; \sum_{l=1}^{m} \frac{x_l}{\sigma_l} \leq 1\},$$

and the *similarity ratio* is equal to $(1 - \sum_{l\in\mathcal{I}} \pi_l \sigma_l^{-1})^m$. Thus,

$$\begin{aligned}
&\mathsf{Vol}(\{\langle x_1, x_2, \ldots, x_m\rangle^{\mathrm{T}} \in \Re^+ \;\mid\; \sum_{l=1}^{m} \frac{x_l}{\sigma_l} \leq 1 \text{ and } \bigwedge_{l\in\mathcal{I}} x_l \notin [0, \pi_l]\}) \\
= &\left(1 - \sum_{l\in\mathcal{I}} \frac{\pi_l}{\sigma_l}\right)^m \cdot \mathsf{Vol}(\Sigma^{(m)}(\overline{\sigma})) \\
= &\frac{1}{m!} \prod_{l=1}^{m} \sigma_l \cdot \left(1 - \sum_{l\in\mathcal{I}} \frac{\pi_l}{\sigma_l}\right)^m \\
&\text{(by Lemma 2.1(1))},
\end{aligned}$$

as needed. ■

Thus, by Lemma 2.3,

$$\begin{aligned}
&\mathsf{Vol}(\mathbf{\Sigma\Pi}^{(m)}(\overline{\sigma}, \overline{\pi})) \\
= &\frac{1}{m!} \prod_{l=1}^{m} \sigma_l + \sum_{i=1}^{m} (-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{1,2,\ldots,m\}, \\ |\mathcal{I}| = i, \\ \sum_{l\in\mathcal{I}} \pi_l/\sigma_l < 1}} \frac{1}{m!} \prod_{l=1}^{m} \sigma_l \cdot \left(1 - \sum_{l\in\mathcal{I}} \frac{\pi_l}{\sigma_l}\right)^m \\
= &\frac{1}{m!} \prod_{l=1}^{m} \sigma_l + \frac{1}{m!} \prod_{l=1}^{m} \sigma_l \cdot \sum_{i=1}^{m} (-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{1,2,\ldots,m\}, \\ |\mathcal{I}| = i, \\ \sum_{l\in\mathcal{I}} \pi_l/\sigma_l < 1}} \left(1 - \sum_{l\in\mathcal{I}} \frac{\pi_l}{\sigma_l}\right)^m \\
= &\frac{1}{m!} \prod_{l=1}^{m} \sigma_l \cdot \left(1 + \sum_{i=1}^{m} (-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{1,2,\ldots,m\}, \\ |\mathcal{I}| = i, \\ \sum_{l\in\mathcal{I}} \pi_l/\sigma_l < 1}} \left(1 - \sum_{l\in\mathcal{I}} \frac{\pi_l}{\sigma_l}\right)^m \right)
\end{aligned}$$

$$= \frac{1}{m!} \prod_{l=1}^{m} \sigma_l \cdot \sum_{i=0}^{m} (-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{1,2,\ldots,m\},\\ |\mathcal{I}| = i,\\ \sum_{l\in\mathcal{I}} \pi_l/\sigma_l < 1}} \left(1 - \sum_{l\in\mathcal{I}} \frac{\pi_l}{\sigma_l}\right)^m ,$$

as needed. ∎

2.2 Probabilistic Tools

In this section, we present two consequences of Proposition 2.2. Each of these two claims determines the probability that a certain sum of independent, uniformly distributed random variables does not exceed a given threshold value; the probability is a function of the threshold value and the distribution intervals of the random variables. The proofs of these claims exploit the reduction of probability to a volume ratio that is possible for uniform random variables and appeal to Proposition 2.2. (Both of these claims will be used in our later proofs.)

We first recall some basic notions from probability theory (see, e.g., [15, 16]). For a (continuous) random variable x, denote $\mathsf{F}_x(t)$ the *cumulative distribution function* of x; that is, $\mathsf{F}_x(t) = \mathbf{P}(x \leq t)$ is the probability of the event $x \leq t$. The *density function* of the random variable X is given by $\mathsf{f}_x(t) = d\mathsf{F}_x(t)/dt$.

Lemma 2.4 *Assume that for each i, $1 \leq i \leq m$, x_i is uniformly distributed over $[0, \pi_i]$. Then, for any parameter $t > 0$,*

$$\mathsf{F}_{\sum_{i=1}^m x_i}(t) = \frac{1}{m! \prod_{l=1}^m \pi_l} \cdot \sum_{i=0}^{m} (-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{1,2,\ldots,m\},\\ |\mathcal{I}| = i,\\ \sum_{l\in\mathcal{I}} \pi_l < t}} \left(t - \sum_{l\in\mathcal{I}} \pi_l\right)^m .$$

Proof: Since each random variable x_i, $1 \leq i \leq m$, is uniformly distributed over $[0, \pi_i]$, the probability $\mathbf{P}(\sum_{i=1}^m x_i \leq t)$ is the ratio of the volume of the polytope (actually, its portion falling in the product domain) corresponding to the inequality $\sum_{i=1}^m x_i \leq t$ to the volume of the product domain $[0, \pi_1] \times [0, \pi_2] \times \ldots \times [0, \pi_m]$ of the variables $x_1, x_2, \ldots, x_m$. Thus,

$$\begin{aligned}
\mathsf{F}_{\sum_{l=1}^m x_l}(t) &= \mathbf{P}(\sum_{l=1}^m x_l \leq t) \\
&= \frac{\mathsf{Vol}(\{\langle x_1, x_2, \ldots, x_m\rangle^{\mathrm{T}} \in [0,\pi_1] \times [0,\pi_2] \times \ldots \times [0,\pi_m] \mid \sum_{l=1}^m x_l \leq t\})}{\mathsf{Vol}(\{\langle x_1, x_2, \ldots, x_m\rangle \in [0,\pi_1] \times [0,\pi_2] \times \ldots \times [0,\pi_m]\})} \\
&= \frac{\mathsf{Vol}(\mathbf{\Sigma\Pi}^{(m)}(t\,\overline{1}^{(m)}, \overline{\pi}))}{\mathsf{Vol}(\mathbf{\Pi}^{(m)}(\overline{\pi}))}
\end{aligned}$$

(by definitions of $\mathbf{\Sigma\Pi}^{(m)}(\overline{\sigma}, \overline{\pi})$ and $\mathbf{\Pi}^{(m)}(\overline{\sigma}, \overline{\pi})$)

$$= \frac{1}{\prod_{l=1}^{m} \pi_l} \cdot \frac{1}{m!} \prod_{l=1}^{m} t \cdot \sum_{i=0}^{m} (-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{1, 2, \ldots, m\}, \\ |\mathcal{I}| = i, \\ \sum_{l \in \mathcal{I}} \pi_l / t < 1}} \left(1 - \sum_{l \in \mathcal{I}} \frac{\pi_l}{t}\right)^m$$

(by Lemma 2.1(2) and Lemma 2.2)

$$= \frac{1}{m! \prod_{l=1}^{m} \pi_l} \cdot t^m \cdot \sum_{i=0}^{m} (-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{1, 2, \ldots, m\}, \\ |\mathcal{I}| = i, \\ \sum_{l \in \mathcal{I}} \pi_l < t}} \frac{1}{t^m} \cdot \left(t - \sum_{l \in \mathcal{I}} \pi_l\right)^m$$

$$= \frac{1}{m! \prod_{l=1}^{m} \pi_l} \cdot \sum_{i=0}^{m} (-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{1, 2, \ldots, m\}, \\ |\mathcal{I}| = i, \\ \sum_{l \in \mathcal{I}} \pi_l < t}} \left(t - \sum_{l \in \mathcal{I}} \pi_l\right)^m,$$

as needed. ■

Lemma 2.4 allows for the computation of the density function of m independent, uniformly distributed random variables.

Lemma 2.5 *Assume that for each i, $1 \leq i \leq m$, x_i is uniformly distributed over $[0, \pi_i]$. Then, for any parameter $t > 0$,*

$$\mathsf{f}_{\sum_{i=1}^{m} x_i}(t) = \frac{1}{(m-1)! \prod_{l=1}^{m} \pi_l} \cdot \sum_{i=0}^{m} (-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{1, 2, \ldots, m\}, \\ |\mathcal{I}| = i, \\ t > \sum_{l \in \mathcal{I}} \pi_l}} \left(t - \sum_{l \in \mathcal{I}} \pi_l\right)^{m-1}.$$

Proof: Clearly,

$$\mathsf{f}_{\sum_{i=1}^{m} x_i}(t) = \frac{d\mathsf{F}_{\sum_{i=1}^{m} x_i}(t)}{dt}$$

$$= \frac{d}{dt} \left[\frac{1}{m! \prod_{l=1}^{m} \pi_l} \cdot \sum_{i=0}^{m} (-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{1, 2, \ldots, m\}, \\ |\mathcal{I}| = i, \\ t > \sum_{l \in \mathcal{I}} \pi_l}} \left(t - \sum_{l \in \mathcal{I}} \pi_l\right)^m \right]$$

$$= \frac{1}{m! \prod_{l=1}^{m} \pi_l} \cdot \sum_{i=0}^{m} (-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{1,2,\ldots,m\}, \\ |\mathcal{I}| = i, \\ t > \sum_{l \in \mathcal{I}} \pi_l}} m \left(t - \sum_{l \in \mathcal{I}} \pi_l \right)^{m-1}$$

$$= \frac{1}{(m-1)! \prod_{l=1}^{m} \pi_l} \cdot \sum_{i=0}^{m} (-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{1,2,\ldots,m\}, \\ |\mathcal{I}| = i, \\ t > \sum_{l \in \mathcal{I}} \pi_l}} \left(t - \sum_{l \in \mathcal{I}} \pi_l \right)^{m-1} ,$$

as needed. ■

Lemma 2.5 is of independent interest since it provides an answer to a Research Problem of Rota [16, Research Problem 10, p. xviii & 4/15/98.14, p. 314]:

> "Find a nice formula for the density of n independent, uniformly distributed random variables."

We continue with an immediate implication of Lemma 2.4 that concerns the special case where for each i, $1 \leq i \leq n$, $\pi_i = 1$.

Corollary 2.6 *Assume that for each i, $1 \leq i \leq m$, x_i is uniformly distributed over* $[0,1]$*. Then, for any parameter $t > 0$,*

$$\mathsf{F}_{\sum_{i=1}^{m} x_i}(t) = \frac{1}{m!} \cdot \sum_{\substack{0 \leq i \leq m \\ i < t}} (-1)^i \cdot \binom{m}{i} (t-i)^m .$$

Proof: By Lemma 2.4,

$$\mathsf{F}_{\sum_{i=1}^{m} x_i}(t) = \frac{1}{m! \prod_{l=1}^{m} 1} \cdot \sum_{i=0}^{m} (-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{1,2,\ldots,m\}, \\ |\mathcal{I}| = i, \\ \sum_{l \in \mathcal{I}} 1 < t}} \left(t - \sum_{l \in \mathcal{I}} 1 \right)^{m} ,$$

$$= \frac{1}{m!} \cdot \sum_{i=0}^{m} (-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{1,2,\ldots,m\}, \\ |\mathcal{I}| = i, \\ |\mathcal{I}| < t}} (t - |\mathcal{I}|)^m ,$$

$$= \frac{1}{m!} \cdot \sum_{i=0}^{m} (-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{1,2,\ldots,m\}, \\ |\mathcal{I}| = i, \\ i < t}} (t-i)^m \ ,$$

$$= \frac{1}{m!} \cdot \sum_{\substack{0 \leq i \leq m \\ i < t}} (-1)^i \cdot \binom{m}{i} (t-i)^m \ ,$$

as needed. ■

We continue to show:

Lemma 2.7 *Assume that for each* i*,* $1 \leq i \leq m$*,* x_i *is uniformly distributed over* $[\pi_i, 1]$*. Then, for any parameter* $t > 0$*,*

$$\begin{aligned} & \mathsf{F}_{\sum_{i=1}^m x_i}(t) \\ = \ & 1 - \frac{1}{m! \prod_{l=1}^m (1-\pi_l)} \cdot \sum_{i=0}^{m} (-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{1,2,\ldots,m\}, \\ |\mathcal{I}| = i, \\ |\mathcal{I}| < m - t + \sum_{l \in \mathcal{I}} \pi_l}} \left(m - t - |\mathcal{I}| + \sum_{l \in \mathcal{I}} \pi_l \right)^m . \end{aligned}$$

Proof: Clearly, for each i, $1 \leq i \leq n$, the random variable $x'_i = 1 - x_i$ is uniformly distributed over $[0, 1-\pi_i]$. Thus,

$$\begin{aligned} & \mathsf{F}_{\sum_{i=1}^m x_i}(t) \\ & \mathbf{P}(\sum_{i=1}^m x_i \leq t) \\ = \ & \mathbf{P}(-\sum_{i=1}^m x_i \geq -t) \\ = \ & \mathbf{P}(m - \sum_{i=1}^m x_i \geq m - t) \\ = \ & \mathbf{P}(\sum_{i=1}^m (1 - x_i) \geq m - t) \\ = \ & 1 - \mathbf{P}(\sum_{i=1}^m (1 - x_i) \leq m - t) \\ = \ & 1 - \mathsf{F}_{\sum_{i=1}^m (1-x_i)}(m - t) \end{aligned}$$

$$= 1 - \frac{1}{m! \prod_{l=1}^{m}(1-\pi_l)} \cdot \sum_{i=0}^{m}(-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{1,2,\ldots,m\},\\ |\mathcal{I}| = i,\\ \sum_{l\in\mathcal{I}}(1-\pi_l) < m-t}} \left(m - t - \sum_{l\in\mathcal{I}}(1-\pi_l) \right)^m$$

(by Lemma 2.2)

$$= 1 - \frac{1}{m! \prod_{l=1}^{m}(1-\pi_l)} \cdot \sum_{i=0}^{m}(-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{1,2,\ldots,m\},\\ |\mathcal{I}| = i,\\ \sum_{l\in\mathcal{I}} 1 - \sum_{l\in\mathcal{I}}\pi_l < m-t}} \left(m - t - \sum_{l\in\mathcal{I}} 1 + \sum_{l\in\mathcal{I}} \pi_l \right)^m$$

$$= 1 - \frac{1}{m! \prod_{l=1}^{m}(1-\pi_l)} \cdot \sum_{i=0}^{m}(-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{1,2,\ldots,m\},\\ |\mathcal{I}| = i,\\ |\mathcal{I}| < m-t+\sum_{l\in\mathcal{I}}\pi_l}} \left(m - t - |\mathcal{I}| + \sum_{l\in\mathcal{I}} \pi_l \right)^m ,$$

as needed. ■

2.3 Notation

Throughout, for any bit $b \in \{0,1\}$ and real number $\alpha \in [0,1]$, denote $\bar{b}$ the complement of b, and $\alpha^{(b)}$ to be α if $b = 1$, and $1 - \alpha$ if $b = 0$. For any binary vector $\mathbf{b}$, denote $|\mathbf{b}|$ the number of entries of $\mathbf{b}$ that are 1.

3 Model

Our model is based on the one of Papadimitriou and Yannakakis [11].

3.1 Distributed Decision-Making

We consider a collection of n distributed entities $P_1, P_2, \ldots, P_n$, called *players,* where $n \geq 2$; n is the *size* of the distributed system. Each player P_i receives an *input* x_i, which is the value of a random variable distributed uniformly over $[0,1]$; denote $\mathbf{x} = \langle x_1, x_2, \ldots, x_n\rangle^{\mathrm{T}}$ the *input vector.* Associated with each player P_i is a (local) *decision-making algorithm* A_i, that may be either deterministic or randomized, and "maps" the input x_i of player P_i and the inputs of other players that are "known" to player P_i to P_i's *output* y_i. (A player P_j's input x_j, $j \neq i$, is not "known" to player P_i if A_i is independent of x_j.) A *distributed decision-making algorithm* is a collection $\mathsf{A} = \langle \mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n\rangle$ of (local) decision algorithms, one for each player.

Formally, a *deterministic decision-making algorithm* is a function $\mathsf{A}_i : [0,1]^n \to \{0,1\}$, that maps the input vector $\mathbf{x}$ to P_i's (boolean) *output* $y_i = \mathsf{A}_i(\mathbf{x})$; denote

$$\mathbf{y}_{\mathsf{A}}(\mathbf{x}) \quad = \quad \langle \mathsf{A}_1(x_1), \mathsf{A}_2(x_2), \ldots, \mathsf{A}_n(x_n) \rangle^{\mathrm{T}}$$

the *output vector of $\mathcal{A}$ on input vector* $\mathbf{x}$. A *randomized decision-making algorithm* is a function A_i which assigns, for each input vector $\mathbf{x}$, a probability distribution on $\{0,1\}$. We consider that $\mathsf{A}_i(\mathbf{x})$ is the probability that player P_i decides on 0.

For each $b \in \{0,1\}$, define

$$\Sigma_b \quad = \quad \sum_{i:\mathsf{A}_i(\mathbf{x})=b} x_i\,;$$

thus, Σ_b is the sum of the inputs of the players that decide on b. Thus, Σ_b is a random variable induced by the distribution of the inputs (and the coin tosses of the algorithm if the algorithm is randomized). For each parameter $t > 0$, we are interested in the event that neither Σ_0 nor Σ_1 exceeds t; denote

$$\mathbf{P}_{\mathsf{A}}(t) \quad = \quad \mathbf{P}(\Sigma_0 \leq t \text{ and } \Sigma_1 \leq t)$$

the probability of this event, taken over all input vectors $\mathbf{x}$ (and coin tosses of the algorithm A in case A is randomized). Call $\mathbf{P}_{\mathsf{A}}(t)$ the *winning probability* of algorithm A. We wish to maximize $\mathbf{P}_{\mathsf{A}}(t)$ over all algorithms A; any maximizing algorithm will be called an *optimal* algorithm.

A set of algorithms is *optimally uniform,* or *uniform* for short, if it includes a particular algorithm that is optimal for all values of the size n.

3.2 The No Communication Case

We focus on the case where there is *no communication* among the players. We model this by assuming that for each i, $1 \leq i \leq n$, $\mathsf{A}_i = \mathsf{A}_i(x_i)$; thus, A_i does not depend on the input of any player other than P_i. From this point on, all of our discussion will refer to the no communication case.

We distinguish between oblivious and non-oblivious, distributed decision-making algorithms. A distributed decision-making algorithm is *oblivious* if for each player P_i, A_i does not depend on P_i's input x_i. Thus, an oblivious algorithm is a collection $\langle \mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n \rangle$ of probability distributions on $\{0,1\}$. We identify an oblivious algorithm A with a probability vector $\overline{\alpha}$ such that for each i, $\alpha_i = \mathbf{P}(y_i = 0)$. Thus, for any vector $\mathbf{b} \in \{0,1\}^n$, $\mathbf{P}(\mathbf{y}_{\mathsf{A}} = \mathbf{b}) = \prod_{i=1}^{n} \alpha_i^{(b_i)}$.

A distributed decision-making algorithm is *non-oblivious* if it is not oblivious. A (deterministic) non-oblivious algorithm is *single-threshold* if for each player P_i, A_i is a single-threshold function; that is,

$$\mathsf{A}_i(x_i) \quad = \quad \begin{cases} 0, & x_i \leq a_i \\ 1, & x_i > a_i \end{cases} \quad ,$$

where $0 \leq a_i < \infty$.

4 Oblivious Algorithms

In this section, we present our results for oblivious algorithms. A combinatorial expression for the winning probability of any oblivious algorithm and corresponding optimality conditions are provided in Section 4.1; Section 4.2 uses these conditions to derive the optimal oblivious algorithm.

4.1 The Winning Probability and Optimality Conditions

We show:

Theorem 4.1 *Assume that* A *is any oblivious algorithm. Then,*

$$\begin{aligned} & \mathbf{P}_{\mathsf{A}}(t) \\ = & \sum\nolimits_{\mathbf{b}\in\{0,1\}^n} \frac{1}{|\mathbf{b}|!} \cdot \sum_{\substack{0 \leq i \leq |\mathbf{b}| \\ i < t}} (-1)^i \cdot \binom{|\mathbf{b}|}{i} (t-i)^{|\mathbf{b}|} \cdot \\ & \frac{1}{(n-|\mathbf{b}|)!} \cdot \sum_{\substack{0 \leq i \leq n-|\mathbf{b}| \\ i < t}} (-1)^i \cdot \binom{n-|\mathbf{b}|}{i} (t-i)^{n-|\mathbf{b}|} \cdot \\ & \prod_{i=1}^{n} \alpha_i^{(b_i)} . \end{aligned}$$

Proof: Clearly,

$$\begin{aligned} \mathbf{P}_{\mathsf{A}}(t) &= \mathbf{P}(\Sigma_0 \leq t \text{ and } \Sigma_1 \leq t) \\ &= \sum_{\mathbf{b}\in\{0,1\}^n} \mathbf{P}(\Sigma_0 \leq t \text{ and } \Sigma_1 \leq t \mid \mathbf{y}_{\mathsf{A}}(\mathbf{x}) = \mathbf{b}) \cdot \mathbf{P}_{\mathsf{A}}(\mathbf{y}_{\mathsf{A}}(\mathbf{x}) = \mathbf{b}) \\ & \quad \text{(by the Conditional Law of Alternatives)}. \end{aligned}$$

Since A is a randomized oblivious algorithm, $\mathbf{y}_{\mathsf{A}}(\mathbf{x})$ does not depend on $\mathbf{x}$, so that $\mathbf{P}_{\mathsf{A}}(\mathbf{y}_{\mathsf{A}}(\mathbf{x}) = \mathbf{b}) = \mathbf{P}_{\mathsf{A}}(\mathbf{y}_{\mathsf{A}} = \mathbf{b}) = \prod_{i=1}^n \alpha_i^{(b_i)}$. Thus,

$$\begin{aligned} \mathbf{P}_{\mathsf{A}}(t) &= \sum_{\mathbf{b}\in\{0,1\}^n} \mathbf{P}(\Sigma_0 \leq t \text{ and } \Sigma_1 \leq t \mid \mathbf{y}_{\mathsf{A}} = \mathbf{b}) \cdot \mathbf{P}_{\mathsf{A}}(\mathbf{y}_{\mathcal{A}} = \mathbf{b}) \\ &= \sum_{\mathbf{b}\in\{0,1\}^n} \mathbf{P}(\sum_{i=1}^n b_i x_i \leq t \text{ and } \sum_{i=1}^n \overline{b}_i x_i \leq t) \cdot \mathbf{P}_{\mathsf{A}}(\mathbf{y}_{\mathsf{A}} = \mathbf{b}) . \end{aligned}$$

Notice that the sums $\sum_{i=1}^{n} b_i x_i$ and $\sum_{i=1}^{n} \overline{b_i} x_i$ are independent random variables since the random variables x_i, $1 \leq i \leq n$ are independent and none of them appears in both $\sum_{i=1}^{n} b_i x_i$ and $\sum_{i=1}^{n} \overline{b_i} x_i$. Thus,

$$\mathbf{P}(\sum_{i=1}^{n} b_i x_i \leq t \text{ and } \sum_{i=1}^{n} \overline{b}_i x_i \leq t) = \mathbf{P}(\sum_{i=1}^{n} b_i x_i \leq t) \cdot \mathbf{P}(\sum_{i=1}^{n} \overline{b}_i x_i \leq t)$$

so that

$$\mathbf{P}_{\mathsf{A}}(t) = \sum_{\mathbf{b} \in \{0,1\}^n} \mathbf{P}(\sum_{i=1}^{n} b_i x_i \leq t) \cdot \mathbf{P}(\sum_{i=1}^{n} \overline{b}_i x_i \leq t) \cdot \mathbf{P}_{\mathsf{A}}(\mathbf{y}_{\mathsf{A}} = \mathbf{b}).$$

Since all variables x_i, $1 \leq i \leq n$, are identically distributed, for any vector $\mathbf{b}$, the sums $\sum_{i=1}^{n} b_i x_i$ and $\sum_{i=1}^{n} \overline{b_i} x_i$ are identically distributed with the sums $\sum_{i=1}^{|\mathbf{b}|} x_i$ and $\sum_{i=1}^{n-|\mathbf{b}|} x_i$, respectively. It follows that

$$\begin{aligned}
&\mathbf{P}_{\mathsf{A}}(t) \\
= &\textstyle\sum_{\mathbf{b} \in \{0,1\}^n} \quad \mathbf{P}(\sum_{i=1}^{|\mathbf{b}|} x_i \leq t) \cdot \mathbf{P}(\sum_{i=1}^{n-|\mathbf{b}|} x_i \leq t) \cdot \mathbf{P}_{\mathsf{A}}(\mathbf{y}_{\mathsf{A}} = \mathbf{b}) \\
= &\textstyle\sum_{\mathbf{b} \in \{0,1\}^n} \quad \frac{1}{|\mathbf{b}|!} \cdot \sum_{\substack{0 \leq i \leq |\mathbf{b}| \\ i < t}} (-1)^i \cdot \binom{|\mathbf{b}|}{i} (t-i)^{|\mathbf{b}|} \cdot \\
&\qquad \frac{1}{(n-|\mathbf{b}|)!} \cdot \sum_{\substack{0 \leq i \leq n-|\mathbf{b}| \\ i < t}} (-1)^i \cdot \binom{n-|\mathbf{b}|}{i} (t-i)^{n-|\mathbf{b}|} \cdot \\
&\qquad \prod_{i=1}^{n} \alpha_i^{(b_i)} \\
&\text{(by Corollary 2.6)},
\end{aligned}$$

as needed. ∎

Notice that the winning probability is a function of the probability vector α of an algorithm A. Thus, an optimal algorithm corresponds to a probability vector that maximizes the winning probability. Clearly, all partial derivatives with respect to the vector's entries must vanish at an extreme point (maximum or minimum) of winning probability. Hence, Theorem 4.1 immediately implies necessary conditions for any optimal protocol.

Corollary 4.2 (Optimality conditions for oblivious algorithms) *Assume that $\mathcal{A}$ is an optimal, randomized oblivious algorithm. Then, for any index k, $1 \leq k \leq n$,*

$$
\begin{aligned}
& 0 \\
= & \sum_{\mathbf{b}\in\{0,1\}^n} \frac{1}{|\mathbf{b}|!} \cdot \sum_{\substack{0 \leq i \leq |\mathbf{b}| \\ i < t}} (-1)^i \cdot \binom{|\mathbf{b}|}{i} (t-i)^{|\mathbf{b}|} \cdot \\
& \frac{1}{(n-|\mathbf{b}|)!} \cdot \sum_{\substack{0 \leq i \leq n-|\mathbf{b}| \\ i < t}} (-1)^i \cdot \binom{n-|\mathbf{b}|}{i} (t-i)^{n-|\mathbf{b}|} \cdot \\
& \frac{\partial}{\partial \alpha_k} \alpha_k^{(b_k)} \cdot \prod_{i=1, i\neq k}^{n} \alpha_i^{(b_i)} .
\end{aligned}
$$

We remark that the (necessary) conditions for optimal oblivious algorithms determined in Corollary 4.2 amount to a system of n multilinear equations in the probability vector $\overline{\alpha}$. In Section 4.2, we will explicitly solve this system and show that the solution indeed determines an optimal algorithm.

4.2 Uniformity

We show that the optimal winning probability is achieved by the very simple uniform algorithm by which each player preassigns equal probability (1/2) to each of its choices.

Theorem 4.3 *Assume that* A *is an optimal oblivious algorithm. Then,* $\alpha = \langle 1/2, 1/2, \ldots, 1/2\rangle^{\mathrm{T}}$ *and*

$$
\begin{aligned}
& \mathbf{P}_{\mathsf{A}}(t) \\
= & \frac{1}{2^n} \sum_{\mathbf{b} \in \{0,1\}^n} \frac{1}{|\mathbf{b}|!} \cdot \sum_{\substack{0 \leq i \leq |\mathbf{b}| \\ i < t}} (-1)^i \cdot \binom{|\mathbf{b}|}{i} (t-i)^{|\mathbf{b}|} \cdot \\
& \frac{1}{(n-|\mathbf{b}|)!} \cdot \sum_{\substack{0 \leq i \leq n-|\mathbf{b}| \\ i < t}} (-1)^i \cdot \binom{n-|\mathbf{b}|}{i} (t-i)^{n-|\mathbf{b}|} .
\end{aligned}
$$

Proof: Take any optimal algorithm A. Fix any index j, $1 \leq j \leq n$. By Corollary 4.2,

$$0$$

$$= \sum_{\mathbf{b}\in\{0,1\}^n} \frac{1}{|\mathbf{b}|!} \cdot \sum_{\substack{0 \le i \le |\mathbf{b}| \\ i < t}} (-1)^i \cdot \binom{|\mathbf{b}|}{i} (t-i)^{|\mathbf{b}|} \cdot$$

$$\frac{1}{(n-|\mathbf{b}|)!} \cdot \sum_{\substack{0 \le i \le n-|\mathbf{b}| \\ i < t}} (-1)^i \cdot \binom{n-|\mathbf{b}|}{i} (t-i)^{n-|\mathbf{b}|} \cdot$$

$$\frac{\partial}{\partial \alpha_j} \alpha_j^{(b_j)} \cdot \prod_{i=1, i\neq j}^{n} \alpha_i^{(b_i)} \,.$$

Clearly, $\partial \alpha_j^{(b_j)} / \partial \alpha_j = 1$ if $b_j = 1$ and -1 if $b_j = 0$. It follows that

$$\sum_{\substack{\mathbf{b} \in \{0,1\}^n \\ b_j = 1}} \phi_t(|\mathbf{b}|) \prod_{\substack{1 \le i \le n \\ i \neq j}} \alpha_i^{(b_i)} - \sum_{\substack{\mathbf{b} \in \{0,1\}^n \\ b_j = 0}} \phi_t(|\mathbf{b}|) \prod_{\substack{1 \le i \le n \\ i \neq j}} \alpha_i^{(b_i)} = 0\,,$$

where for any vector $\mathbf{b} \in \{0,1\}^n$ and (real) parameter $t > 0$,

$$\phi_t(|\mathbf{b}|) = \frac{1}{|\mathbf{b}|!} \cdot \sum_{\substack{0 \le i \le |\mathbf{b}| \\ i < t}} (-1)^i \cdot \binom{|\mathbf{b}|}{i} (t-i)^{|\mathbf{b}|} \cdot$$

$$\frac{1}{(n-|\mathbf{b}|)!} \cdot \sum_{\substack{0 \le i \le n-|\mathbf{b}| \\ i < t}} (-1)^i \cdot \binom{n-|\mathbf{b}|}{i} (t-i)^{n-|\mathbf{b}|} \,.$$

By definition of $\phi_t(|\mathbf{b}|)$, it immediately follows:

Lemma 4.4 *For any vector* $\mathbf{b} \in \{0,1\}^n$ *and (real) parameter* $t > 0$, $\phi_t(|\mathbf{b}|) = \phi_t(n - |\mathbf{b}|)$.

We continue to show:

Lemma 4.5 $\alpha_1 = \alpha_2 = \ldots = \alpha_n$

Proof: Take any arbitrary indices j and k, $1 \le j, k \le n$ and $j \neq k$. Clearly,

$$\sum_{\substack{\mathbf{b} \in \{0,1\}^n \\ b_j = 1}} \phi_t(|\mathbf{b}|) \prod_{\substack{1 \le i \le n \\ i \neq j}} \alpha_i^{(b_i)} - \sum_{\substack{\mathbf{b} \in \{0,1\}^n \\ b_j = 0}} \phi_t(|\mathbf{b}|) \prod_{\substack{1 \le i \le n \\ i \neq j}} \alpha_i^{(b_i)} = 0$$

and

$$\sum_{\substack{\mathbf{b} \in \{0,1\}^n \\ b_k = 1}} \phi_t(|\mathbf{b}|) \prod_{\substack{1 \le i \le n \\ i \ne k}} \alpha_i^{(b_i)} - \sum_{\substack{\mathbf{b} \in \{0,1\}^n \\ b_k = 0}} \phi_t(|\mathbf{b}|) \prod_{\substack{1 \le i \le n \\ i \ne k}} \alpha_i^{(b_i)} \;=\; 0\,.$$

The first equation implies that

$$\begin{aligned} & \sum_{\substack{\mathbf{b} \in \{0,1\}^n \\ b_j = 1 \\ b_k = 1}} \alpha_k\, \phi_t(|\mathbf{b}|) \prod_{\substack{1 \le i \le n \\ i \ne j,k}} \alpha_i^{(b_i)} + \sum_{\substack{\mathbf{b} \in \{0,1\}^n \\ b_j = 1 \\ b_k = 0}} \overline{\alpha_k}\, \phi_t(|\mathbf{b}|) \prod_{\substack{1 \le i \le n \\ i \ne j,k}} \alpha_i^{(b_i)} \\ & - \sum_{\substack{\mathbf{b} \in \{0,1\}^n \\ b_j = 0 \\ b_k = 1}} \alpha_k\, \phi_t(|\mathbf{b}|) \prod_{\substack{1 \le i \le n \\ i \ne j,k}} \alpha_i^{(b_i)} - \sum_{\substack{\mathbf{b} \in \{0,1\}^n \\ b_j = 0 \\ b_k = 0}} \overline{\alpha_k}\, \phi_t(|\mathbf{b}|) \prod_{\substack{1 \le i \le n \\ i \ne j,k}} \alpha_i^{(b_i)} \\ = \;& 0\,. \end{aligned}$$

or

$$\begin{aligned} & \alpha_k \cdot \left(\sum_{\substack{\mathbf{b} \in \{0,1\}^n \\ b_j = 1 \\ b_k = 1}} \phi_t(|\mathbf{b}|) \prod_{\substack{1 \le i \le n \\ i \ne j,k}} \alpha_i^{(b_i)} - \sum_{\substack{\mathbf{b} \in \{0,1\}^n \\ b_j = 0 \\ b_k = 1}} \phi_t(|\mathbf{b}|) \prod_{\substack{1 \le i \le n \\ i \ne j,k}} \alpha_i^{(b_i)} \right) \\ & + \overline{\alpha_k} \cdot \left(\sum_{\substack{\mathbf{b} \in \{0,1\}^n \\ b_j = 1 \\ b_k = 0}} \phi_t(|\mathbf{b}|) \prod_{\substack{1 \le i \le n \\ i \ne j,k}} \alpha_i^{(b_i)} - \sum_{\substack{\mathbf{b} \in \{0,1\}^n \\ b_j = 0 \\ b_k = 0}} \phi_t(|\mathbf{b}|) \prod_{\substack{1 \le i \le n \\ i \ne j,k}} \alpha_i^{(b_i)} \right) \\ = \;& 0\,. \end{aligned}$$

Similarly, the second equation implies that

$$\alpha_j \cdot \left(\sum_{\substack{\mathbf{b} \in \{0,1\}^n \\ b_k = 1 \\ b_j = 1}} \phi_t(|\mathbf{b}|) \prod_{\substack{1 \le i \le n \\ i \ne k,j}} \alpha_i^{(b_i)} - \sum_{\substack{\mathbf{b} \in \{0,1\}^n \\ b_k = 0 \\ b_j = 1}} \phi_t(|\mathbf{b}|) \prod_{\substack{1 \le i \le n \\ i \ne k,j}} \alpha_i^{(b_i)} \right)$$

$$+\overline{\alpha_j} \cdot \left(\sum_{\substack{\mathbf{b} \in \{0,1\}^n \\ b_k = 1 \\ b_j = 0}} \phi_t(|\mathbf{b}|) \prod_{\substack{1 \le i \le n \\ i \ne k,j}} \alpha_i^{(b_i)} - \sum_{\substack{\mathbf{b} \in \{0,1\}^n \\ b_k = 0 \\ b_j = 0}} \phi_t(|\mathbf{b}|) \prod_{\substack{1 \le i \le n \\ i \ne k,j}} \alpha_i^{(b_i)} \right)$$

$$= \quad 0\,.$$

There is a one-to-one correspondence between vectors $\mathbf{b} \in \{0,1\}^n$ with $b_j = 0$ and $b_k = 1$ and vectors $\mathbf{b} \in \{0,1\}^n$ with $b_k = 0$ and $b_j = 1$: two such vectors $\mathbf{b}$ and $\mathbf{b}'$ correspond to each other if they have all other entries identical. Then, clearly, $|\mathbf{b}| = |\mathbf{b}'|$ and for each index i, $i \ne j,k$, $\alpha_i^{(b_i)} = \alpha_i^{(b'_i)}$. This implies that

$$\sum_{\substack{\mathbf{b} \in \{0,1\}^n \\ b_j = 0 \\ b_k = 1}} \phi_t(|\mathbf{b}|) \prod_{\substack{1 \le i \le n \\ i \ne j,k}} \alpha_i^{(b_i)} \quad = \quad \sum_{\substack{\mathbf{b} \in \{0,1\}^n \\ b_k = 0 \\ b_j = 1}} \phi_t(|\mathbf{b}|) \prod_{\substack{1 \le i \le n \\ i \ne k,j}} \alpha_i^{(b_i)}\,.$$

It follows that α_k and α_j satisfy identical linear equations (each involving the remaining variables α_i, $i \ne j,k$, as parameters) and, therefore, they are equal. Since the indices j and k were chosen arbitrarily, it follows that $\alpha_1 = \alpha_2 = \ldots = \alpha_n$, as needed. ■

Denote α the common value of the variables $\alpha_1, \alpha_2, \ldots, \alpha_n$. It follows that

$$\sum_{\substack{\mathbf{b} \in \{0,1\}^n \\ b_j = 1}} \phi_t(|\mathbf{b}|) \prod_{\substack{1 \le i \le n \\ i \ne j}} \alpha^{(b_i)} - \sum_{\substack{\mathbf{b} \in \{0,1\}^n \\ b_j = 0}} \phi_t(|\mathbf{b}|) \prod_{\substack{1 \le i \le n \\ i \ne j}} \alpha^{(b_i)} \quad = \quad 0\,,$$

For any vector $\mathbf{b}$ with $b_j = 1$, there are $|\mathbf{b}| - 1$ indices i, $i \ne j$, with $b_i = 1$ and $n - |\mathbf{b}|$ indices i with $b_i = 0$. Similarly, for any vector $\mathbf{b}$ with $b_j = 0$, there are $|\mathbf{b}|$ indices i with $b_i = 1$

and $n - |\mathbf{b}| - 1$ indices i, $i \neq j$, with $b_i = 0$. Thus,

$$\sum_{\substack{\mathbf{b} \in \{0,1\}^n \\ b_j = 1}} \phi_t(|\mathbf{b}|)\, \alpha^{|\mathbf{b}|-1}(1-\alpha)^{n-|\mathbf{b}|} - \sum_{\substack{\mathbf{b} \in \{0,1\}^n \\ b_j = 0}} \phi_t(|\mathbf{b}|)\, \alpha^{|\mathbf{b}|}(1-\alpha)^{n-|\mathbf{b}|-1} = 0\,.$$

Dividing through by $(1-\alpha)^{n-1}$ yileds that

$$\sum_{\substack{\mathbf{b} \in \{0,1\}^n \\ b_j = 1}} \phi_t(|\mathbf{b}|) \left(\frac{\alpha}{1-\alpha}\right)^{|\mathbf{b}|-1} - \sum_{\substack{\mathbf{b} \in \{0,1\}^n \\ b_j = 0}} \phi_t(|\mathbf{b}|) \left(\frac{\alpha}{1-\alpha}\right)^{|\mathbf{b}|} = 0\,.$$

There are $\binom{n-1}{|\mathbf{b}|-1}$ vectors $\mathbf{b} \in \{0,1\}^n$ with $b_j = 1$; for any such vector, $1 \leq |\mathbf{b}| \leq n$. Similarly, there are $\binom{n-1}{|\mathbf{b}|}$ vectors $\mathbf{b} \in \{0,1\}^n$ with $b_j = 0$; for any such vector, $0 \leq |\mathbf{b}| \leq n-1$. It follows that

$$\sum_{|\mathbf{b}=1}^{n} \binom{n-1}{|\mathbf{b}|-1} \phi_t(|\mathbf{b}|) \left(\frac{\alpha}{1-\alpha}\right)^{|\mathbf{b}|-1} - \sum_{|\mathbf{b}|=0}^{n-1} \binom{n-1}{|\mathbf{b}|} \phi_t(|\mathbf{b}|) \left(\frac{\alpha}{1-\alpha}\right)^{|\mathbf{b}|} = 0\,.$$

We next show:

Lemma 4.6 $\alpha = 1/2$

Proof: Assume, by way of contradiction, that $\alpha \neq 1/2$. This implies that $\alpha/(\alpha - 1) \neq 1$.

We have established that $\alpha/(\alpha-1)$ satisfies a polynomial equation of degree $n-1$. Consider any exponent r, $0 \leq r \leq n-1$. The coefficient of $(\alpha/(\alpha-1))^r$ is

$$\binom{n-1}{r} \phi_t(r+1) - \binom{n-1}{r} \phi_t(r) = \binom{n-1}{r} (\phi_t(r+1) - \phi_t(r))\,,$$

while the coefficient of $(\alpha/(\alpha-1))^{n-1-r}$ is

$$\binom{n-1}{n-1-r} \phi_t(n-r) - \binom{n-1}{n-1-r} \phi_t(n-1-r) =$$
$$\binom{n-1}{n-1-r} (\phi_t(n-r) - \phi_t(n-1-r)) = \binom{n-1}{r} (\phi_t(n-r) - \phi_t(n-1-r))$$

By Lemma 4.4, $\phi_t(n-r) = \phi_t(r)$ and $\phi_t(n-1-r) = \phi_t(r+1)$. It follows that the coefficient of $(\alpha/(\alpha-1))^r$ is equal to the negative of the coefficient of $(\alpha/(\alpha-1))^{n-1-r}$. In particular, when n is odd, the coefficient of $(\alpha/(\alpha-1))^{(n-1)/2}$ is equal to the negative of the coefficient of $(\alpha/(\alpha-1))^{n-1-(n-1)/2} = (\alpha/(\alpha-1))^{(n-1)/2}$, which implies that in that case the coefficient of $(\alpha/(\alpha-1))^{(n-1)/2}$ is equal to zero. It follows that

$$\sum_{0 \le r \le \frac{n-1}{2}} \binom{n-1}{r} (\phi_t(r+1) - \phi_t(r)) \left(\left(\frac{\alpha}{\alpha-1}\right)^r - \left(\frac{\alpha}{\alpha-1}\right)^{n-1-r} \right) \quad = \quad 0 .$$

We proceed by case analysis. Assume first that $\alpha/(\alpha-1) > 1$. Then, for any integer r such that $1 \le r \le (n-1)/2$, $r < n-1-r$, so that

$$\left(\frac{\alpha}{\alpha-1}\right)^r \quad < \quad \left(\frac{\alpha}{\alpha-1}\right)^{n-1-r} .$$

Since

$$\binom{n-1}{r} (\phi_t(r+1) - \phi_t(r)) \quad > \quad 0 ,$$

it follows that

$$\sum_{0 \le r \le \frac{n-1}{2}} \binom{n-1}{r} (\phi_t(r+1) - \phi_t(r)) \left(\left(\frac{\alpha}{\alpha-1}\right)^r - \left(\frac{\alpha}{\alpha-1}\right)^{n-1-r} \right) \quad < \quad 0 ,$$

a contradiction.

Assume now that $\alpha/(\alpha-1) < 1$. Then, for any integer r such that $1 \le r \le (n-1)/2$, $r < n-1-r$, so that

$$\left(\frac{\alpha}{\alpha-1}\right)^r \quad > \quad \left(\frac{\alpha}{\alpha-1}\right)^{n-1-r} .$$

Since

$$\binom{n-1}{r} (\phi_t(r+1) - \phi_t(r)) \quad > \quad 0 ,$$

it follows that

$$\sum_{0 \le r \le \frac{n-1}{2}} \binom{n-1}{r} (\phi_t(r+1) - \phi_t(r)) \left(\left(\frac{\alpha}{\alpha-1}\right)^r - \left(\frac{\alpha}{\alpha-1}\right)^{n-1-r} \right) \quad > \quad 0 ,$$

a contradiction. This completes our proof. ∎

By Lemmas 4.5 and 4.6, Theorem 4.1 implies that

$$\mathbf{P}_{\mathsf{A}}(t) = \frac{1}{2^n} \sum_{\mathbf{b} \in \{0,1\}^n} \frac{1}{|\mathbf{b}|!} \cdot \sum_{\substack{0 \leq i \leq |\mathbf{b}| \\ i < t}} (-1)^i \cdot \binom{|\mathbf{b}|}{i} (t-i)^{|\mathbf{b}|} \cdot \frac{1}{(n-|\mathbf{b}|)!} \cdot \sum_{\substack{0 \leq i \leq n-|\mathbf{b}| \\ i < t}} (-1)^i \cdot \binom{n-|\mathbf{b}|}{i} (t-i)^{n-|\mathbf{b}|} ,$$

as needed. ∎

Theorem 4.3 implies that the optimal winning probability of an oblivious algorithm can be computed exactly in exponential time.

5 Non-Oblivious Algorithms

In this section, we present our results for non-oblivious, single threshold algorithms. A combinatorial expression for the winning probability of any non-oblivious, single threshold algorithm, and corresponding optimality conditions, are provided in Section 5.1; Section 5.2 uses the optimality conditions to demonstrate non-uniformity for the optimal non-oblivious, single threshold algorithm.

5.1 The Winning Probability and Optimality Conditions

We show:

Theorem 5.1 *Assume that* A *is any non-oblivious, single threshold algorithm. Then,*

$$\mathbf{P}_{\mathsf{A}}(t) = \sum_{\mathbf{b} \in \{0,1\}^n} \left(\frac{1}{(n-|\mathbf{b}|)!} \cdot \sum_{0 \leq i \leq n-|\mathbf{b}|} (-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{i : b_i = 0\}, \\ |\mathcal{I}| = i, \\ \sum_{l \in \mathcal{I}} \alpha_l < t}} \left(t - \sum_{l \in \mathcal{I}} \alpha_l \right)^{n-|\mathbf{b}|} \right) .$$

$$\left(\prod_{l:b_l=1} (1-\alpha_l) - \frac{1}{|\mathbf{b}|!} \sum_{0 \leq i \leq |\mathbf{b}|} (-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{i : b_i = 1\}, \\ |\mathcal{I}| = i, \\ |\mathcal{I}| < |\mathbf{b}| - t + \sum_{l \in \mathcal{I}} \alpha_l}} \left(|\mathbf{b}| - t - |\mathcal{I}| + \sum_{l \in \mathcal{I}} \alpha_l \right)^{|\mathbf{b}|} \right).$$

Proof: Clearly,

$$\begin{aligned}
& \mathbf{P}_{\mathsf{A}}(t) \\
= \; & \mathbf{P}(\Sigma_0 \leq t \text{ and } \Sigma_1 \leq t) \\
= \; & \sum_{\mathbf{b} \in \{0,1\}^n} \mathbf{P}(\Sigma_0 \leq t \text{ and } \Sigma_1 \leq t \;\mid\; \mathbf{y}_{\mathsf{A}}(\mathbf{x}) = \mathbf{b}) \cdot \mathbf{P}_{\mathsf{A}}(\mathbf{y}_{\mathsf{A}}(\mathbf{x}) = \mathbf{b}) \\
& \text{(by the Conditional Law of Alternatives)} \\
= \; & \sum_{\mathbf{b} \in \{0,1\}^n} \mathbf{P}(\sum_{i:\mathsf{A}_i(\mathbf{x})=0} x_i \leq t \text{ and } \sum_{i:\mathsf{A}_i(\mathbf{x})=1} x_i \leq t \;\mid\; \mathbf{y}_{\mathsf{A}}(\mathbf{x}) = \mathbf{b}) \cdot \mathbf{P}_{\mathsf{A}}(\mathbf{y}_{\mathsf{A}}(\mathbf{x}) = \mathbf{b}) \\
& \text{(by definition of } \Sigma_0 \text{ and } \Sigma_1\text{)} \\
= \; & \sum_{\mathbf{b} \in \{0,1\}^n} \mathbf{P}(\sum_{i:b_i=0} x_i \leq t \text{ and } \sum_{i:b_i=1} x_i \leq t \;\mid\; \mathbf{y}_{\mathsf{A}}(\mathbf{x}) = \mathbf{b}) \cdot \mathbf{P}_{\mathsf{A}}(\mathbf{y}_{\mathsf{A}}(\mathbf{x}) = \mathbf{b}) \\
= \; & \sum_{\mathbf{b} \in \{0,1\}^n} \\
& \mathbf{P}(\sum_{i:b_i=0} x_i \leq t \text{ and } \sum_{i:b_i=1} x_i \leq t \;\mid\; \bigwedge_{i:b_i=0} x_i \in [0, \alpha_i] \text{ and } \bigwedge_{i:b_i=1} x_i \in [\alpha_i, 1]) \cdot \\
& \mathbf{P}_{\mathsf{A}}(\mathbf{y}_{\mathsf{A}}(\mathbf{x}) = \mathbf{b}) \\
& \text{(since } \mathsf{A} \text{ is a single threshold algorithm)} \\
= \; & \sum_{\mathbf{b} \in \{0,1\}^n} \\
& \frac{\mathbf{P}(\sum_{i:b_i=0} x_i \leq t \text{ and } \sum_{i:b_i=1} x_i \leq t \text{ and } \bigwedge_{i:b_i=0} x_i \in [0, \alpha_i] \text{ and } \bigwedge_{i:b_i=1} x_i \in [\alpha_i, 1])}{\mathbf{P}(\bigwedge_{i:b_i=0} x_i \in [0, \alpha_i] \text{ and } \bigwedge_{i:b_i=1} x_i \in [\alpha_i, 1])} \cdot \\
& \mathbf{P}_{\mathsf{A}}(\mathbf{y}_{\mathsf{A}}(\mathbf{x}) = \mathbf{b}) \\
& \text{(by definition of conditional probability)} \\
= \; & \sum_{\mathbf{b} \in \{0,1\}^n} \\
& \frac{\mathbf{P}(\sum_{i:b_i=0} x_i \leq t \text{ and } \bigwedge_{i:b_i=0} x_i \in [0, \alpha_i]) \cdot \mathbf{P}(\sum_{i:b_i=1} x_i \leq t \text{ and } \bigwedge_{i:b_i=1} x_i \in [\alpha_i, 1])}{\mathbf{P}(\bigwedge_{i:b_i=0} x_i \in [0, \alpha_i]) \cdot \mathbf{P}(\bigwedge_{i:b_i=1} x_i \in [\alpha_i, 1])} \cdot \\
& \mathbf{P}_{\mathsf{A}}(\mathbf{y}_{\mathsf{A}}(\mathbf{x}) = \mathbf{b}) \\
& \text{(since all variables } x_i \text{ are independent)}
\end{aligned}$$

$$= \sum_{\mathbf{b}\in\{0,1\}^n} \frac{\mathbf{P}(\sum_{i:b_i=0} x_i \le t \text{ and } \bigwedge_{i:b_i=0} x_i \in [0,\alpha_i])}{\mathbf{P}(\bigwedge_{i:b_i=0} x_i \in [0,\alpha_i])} \cdot \frac{\mathbf{P}(\sum_{i:b_i=1} x_i \le t \text{ and } \bigwedge_{i:b_i=1} x_i \in [\alpha_i,1])}{\mathbf{P}(\bigwedge_{i:b_i=1} x_i \in [\alpha_i,1])} \cdot \mathbf{P}_{\mathsf{A}}(\mathbf{y}_{\mathsf{A}}(\mathbf{x}) = \mathbf{b})$$

(by definition of conditional probability)

$$= \sum_{\mathbf{b}\in\{0,1\}^n} \mathbf{P}(\sum_{i:b_i=0} x_i \le t \mid \bigwedge_{i:b_i=0} x_i \in [0,\alpha_i]) \cdot \mathbf{P}(\sum_{i:b_i=1} x_i \le t \mid \bigwedge_{i:b_i=1} x_i \in [\alpha_i,1]) \cdot \mathbf{P}_{\mathsf{A}}(\mathbf{y}_{\mathsf{A}}(\mathbf{x}) = \mathbf{b})$$

$$= \sum_{\mathbf{b}\in\{0,1\}^n} \left(\frac{1}{(n-|\mathbf{b}|)! \prod_{l:b_l=0} \alpha_l} \cdot \sum_{0\le i\le n-|\mathbf{b}|} (-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{i : b_i = 0\},\\ |\mathcal{I}| = i,\\ \sum_{l\in\mathcal{I}} \alpha_l < t}} \left(t - \sum_{l\in\mathcal{I}} \alpha_l \right)^{n-|\mathbf{b}|} \right) \cdot \left(1 - \frac{1}{|\mathbf{b}|! \prod_{l:b_l=1}(1-\alpha_l)} \cdot \sum_{1\le i\le |\mathbf{b}|} (-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{i : b_i = 1\},\\ |\mathcal{I}| = i,\\ |\mathcal{I}| < |\mathbf{b}| - t + \sum_{l\in\mathcal{I}} \alpha_l}} \left(|\mathbf{b}| - t - |\mathcal{I}| + \sum_{l\in\mathcal{I}} \alpha_l \right)^{|\mathbf{b}|} \right) \cdot \mathbf{P}_{\mathsf{A}}(\mathbf{y}_{\mathsf{A}}(\mathbf{x}) = \mathbf{b})$$

(by Lemmas 2.4 and 2.7)

$$= \sum_{\mathbf{b}\in\{0,1\}^n} \left(\frac{1}{(n-|\mathbf{b}|)! \prod_{l:b_l=0} \alpha_l} \cdot \sum_{0\le i\le n-|\mathbf{b}|} (-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{i : b_i = 0\},\\ |\mathcal{I}| = i,\\ \sum_{l\in\mathcal{I}} \alpha_l < t}} \left(t - \sum_{l\in\mathcal{I}} \alpha_l \right)^{n-|\mathbf{b}|} \right) \cdot$$

$$\frac{1}{\prod_{l:b_l=1}(1-\alpha_l)} \cdot$$

$$\left(\prod_{l:b_l=1}(1-\alpha_l) - \frac{1}{|\mathbf{b}|!} \sum_{0 \leq i \leq |\mathbf{b}|} (-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{i : b_i = 1\}, \\ |\mathcal{I}| = i, \\ |\mathcal{I}| < |\mathbf{b}| - t + \sum_{l \in \mathcal{I}} \alpha_l}} \left(|\mathbf{b}| - t - |\mathcal{I}| + \sum_{l \in \mathcal{I}} \alpha_l \right)^{|\mathbf{b}|} \right) \cdot$$

$$\mathbf{P}_{\mathsf{A}}(\mathbf{y}_{\mathsf{A}}(\mathbf{x}) = \mathbf{b})$$

$$= \sum_{\mathbf{b} \in \{0,1\}^n}$$

$$\left(\frac{1}{(n-|\mathbf{b}|)! \prod_{l:b_l=0} \alpha_l} \cdot \sum_{0 \leq i \leq n-|\mathbf{b}|} (-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{i : b_i = 0\}, \\ |\mathcal{I}| = i, \\ \sum_{l \in \mathcal{I}} \alpha_l < t}} \left(t - \sum_{l \in \mathcal{I}} \alpha_l \right)^{n-|\mathbf{b}|} \right) \cdot$$

$$\frac{1}{\prod_{l:b_l=1}(1-\alpha_l)} \cdot$$

$$\left(\prod_{l:b_l=1}(1-\alpha_l) - \frac{1}{|\mathbf{b}|!} \sum_{0 \leq i \leq |\mathbf{b}|} (-1)^i \cdot \sum_{\substack{\mathcal{I} \subseteq \{i : b_i = 1\}, \\ |\mathcal{I}| = i, \\ |\mathcal{I}| < |\mathbf{b}| - t + \sum_{l \in \mathcal{I}} \alpha_l}} \left(|\mathbf{b}| - t - |\mathcal{I}| + \sum_{l \in \mathcal{I}} \alpha_l \right)^{|\mathbf{b}|} \right) \cdot$$

$$\prod_{1 \leq l \leq n} \alpha_l^{(b_l)}$$

(since A is a single threshold algorithm),

Clearly,

$$\prod_{1 \leq l \leq n} \alpha_l^{(b_l)} = \prod_{l:b_l=0} \alpha_l \cdot \prod_{l:b_l=1} (1-\alpha_l),$$

so that

$$\mathbf{P}_{\mathsf{A}}(t)$$

$$= \sum_{\mathbf{b}\in\{0,1\}^n} \left(\frac{1}{(n-|\mathbf{b}|)!} \cdot \sum_{0\leq i\leq n-|\mathbf{b}|} (-1)^i \cdot \sum_{\substack{\mathcal{I}\subseteq\{i : b_i = 0\},\\ |\mathcal{I}| = i,\\ \sum_{l\in\mathcal{I}} \alpha_l < t}} \left(t - \sum_{l\in\mathcal{I}} \alpha_l\right)^{n-|\mathbf{b}|} \right) \cdot$$

$$\left(\prod_{l:b_l=1} (1-\alpha_l) - \frac{1}{|\mathbf{b}|!} \sum_{0\leq i\leq|\mathbf{b}|} (-1)^i \cdot \sum_{\substack{\mathcal{I}\subseteq\{i : b_i = 1\},\\ |\mathcal{I}| = i,\\ |\mathcal{I}| < |\mathbf{b}| - t + \sum_{l\in\mathcal{I}} \alpha_l}} \left(|\mathbf{b}| - t - |\mathcal{I}| + \sum_{l\in\mathcal{I}} \alpha_l\right)^{|\mathbf{b}|} \right)$$

as needed. ■

Figures 1 and 2 depict the winning probabilities for some simple cases. It is already evident that the optimal non-oblivious algorithm is not uniform.

For non-oblivious algorithms, the analysis is more involved since it must take into account the conditional probabilities "created" by the knowledge of inputs by the agents. We show:

Theorem 5.2 (Optimality conditions for non-oblivious algorithms) *Assume that $\mathcal{A}$ is an optimal, randomized non-oblivious algorithm. Then, for any index k,*

$$\sum_{|\mathbf{b}|=0}^{n} \binom{n-1}{|\mathbf{b}|} \Big(\frac{1}{(n-1-|\mathbf{b}|)!}$$

$$\sum_{0\leq l\leq n-1-|\mathbf{b}|, \delta-\beta l>0} (-1)^l \binom{n-1-|\mathbf{b}|}{l} (\delta - \beta l)^{n-1-|\mathbf{b}|} \Big) \cdot$$

$$\Big(-(|\mathbf{b}|+1)(1-\beta)^{|\mathbf{b}|} - \frac{(|\mathbf{b}|+1)}{(|\mathbf{b}|+1)!}$$

$$\sum_{1\leq l\leq|\mathbf{b}|+1, \mathbf{b}+1-\delta-l+\beta l>0} (-1)^l \binom{|\mathbf{b}|}{l-1} l\, (\mathbf{b}+1-\delta-l+\beta l)^{|\mathbf{b}|} \Big) +$$

$$\sum_{|\mathbf{b}|=0}^{n-1} \binom{n-1}{|\mathbf{b}|} \Big(\Big((1-\beta)^{|\mathbf{b}|} - \Big(\frac{1}{|\mathbf{b}|!} \sum_{0\leq l\leq|\mathbf{b}|, |\mathbf{b}|-\delta-l+\beta l>0} (-1)^l \binom{|\mathbf{b}|}{l} (|\mathbf{b}| - \delta - l + \beta l)^{|\mathbf{b}|} \Big) \Big) \cdot$$

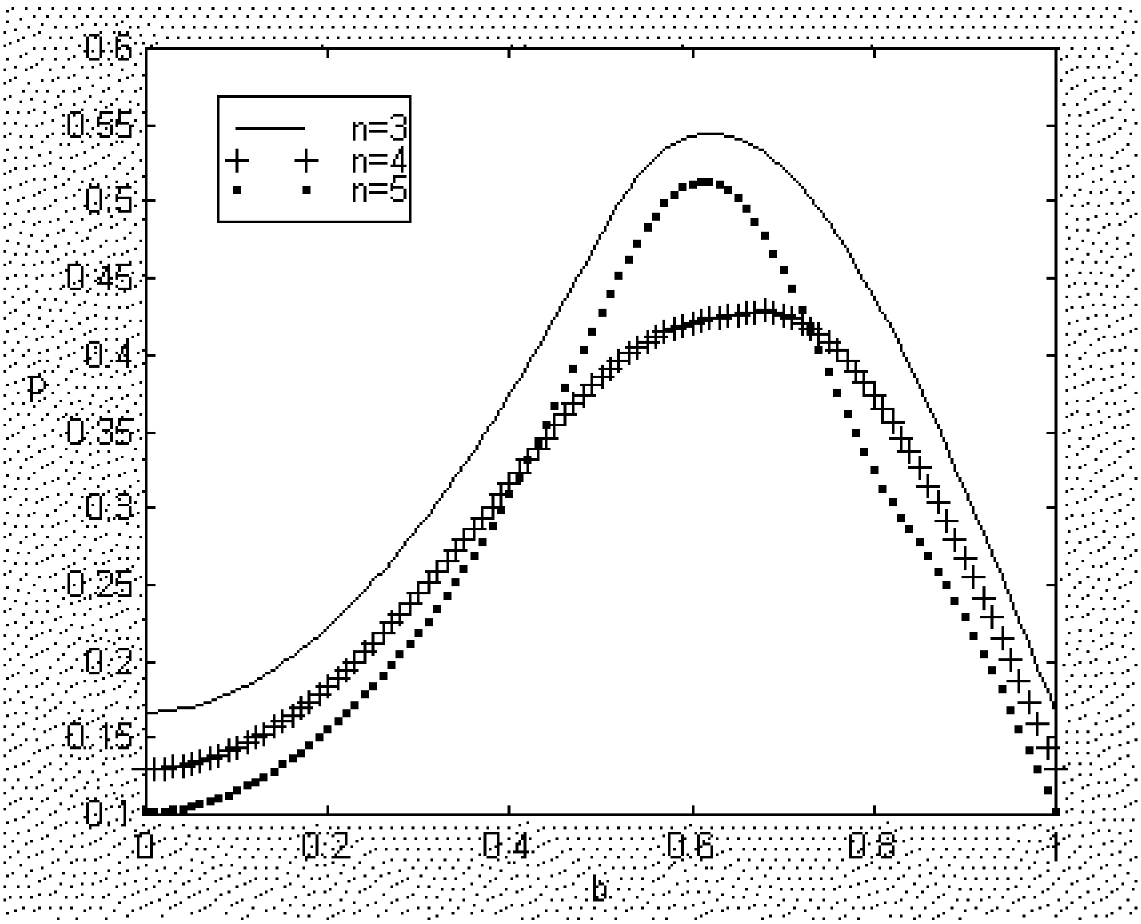

Figure 1: The winning probabilities for $n = 3$, $n = 4$ and $n = 5$

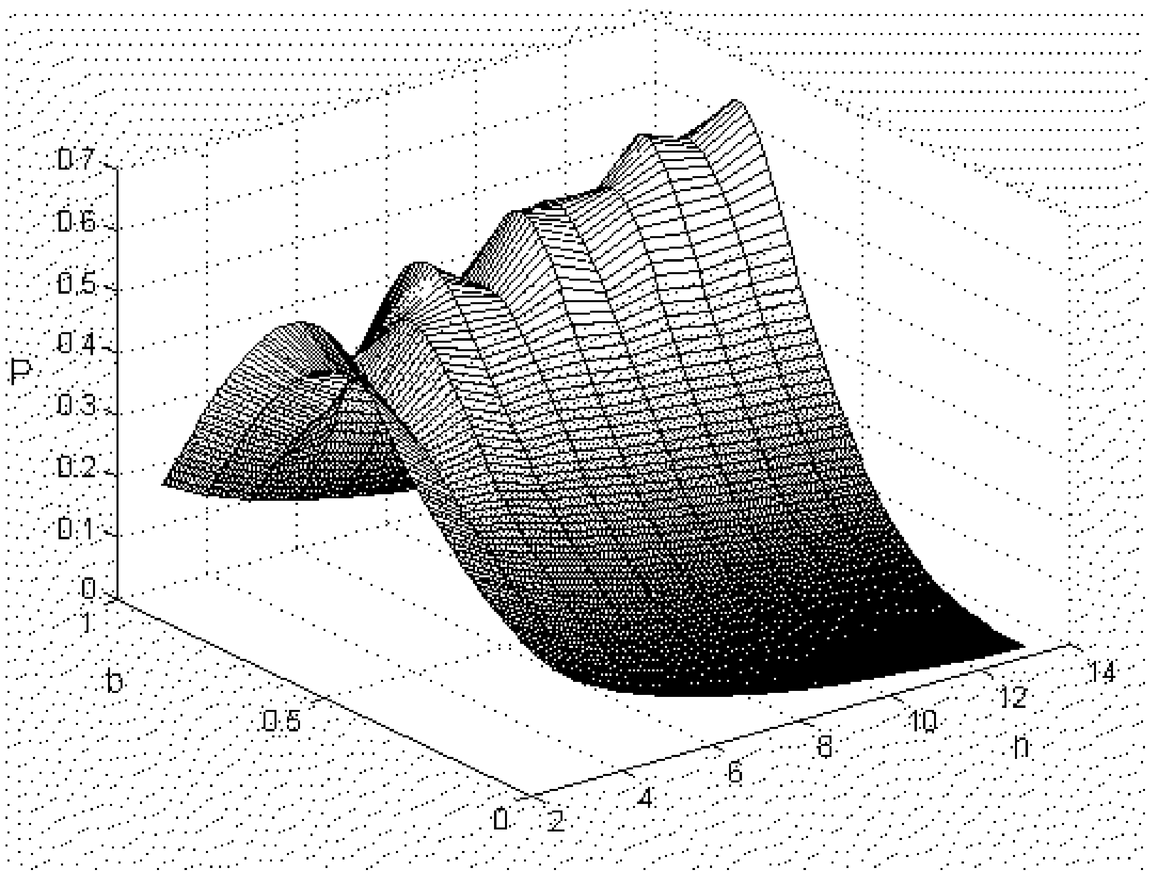

Figure 2: The winning probabilities for $n = 3$, $n = 4$ and $n = 5$

$$\frac{-(n-|\mathbf{b}|)}{(n-|\mathbf{b}|)!}\sum_{1\le l\le n-|\mathbf{b}|,\delta-\beta l>0}(-1)^l\binom{n-|\mathbf{b}|-1}{l-1}l\,(\delta-\beta l)^{n-|\mathbf{b}|-1})$$
$$= \ 0\,.$$

Unfortunately, the conditions in Theorem 5.2 do not admit a uniform solution (independent of n). We discover that the solutions for $n=3$ and $n=4$ are different. The solution for $n=3$ and $\delta=1$ satisfies the polynomial equation $\beta^2-2\beta+6/7=0$; the solution is calculated to be to $1-\sqrt{1/7}=0.622$, which is the threshold value conjectured by Papadimitriou and Yannakakis in [11] to imply optimality for the same case. (See Appendix **??** for a complete derivation.) On the other hand, the solution for $n=4$ and $\delta=4/3$ satisfies the polynomial equation $-(26/3)\beta^3+(98/3)\beta^2-(368/9)\beta-416/27=0$; the solution is calculated to be equal to approximately 0.678.

5.2 Non-Uniformity

In this section, we derive the optimal algorithms for the special cases where *(i)* $n=3$ and $\delta=1$, and *(ii)* $n=4$ and $\delta=4/3$.

5.2.1 The Case $n=3$ and $\delta=1$

We proceed by case analysis on the interval in which β lies. In each case, we first use Theorem 5.1 to derive an expression for the probability of any symmetric protocol as a function of the common threshold β. (Theorem 5.2 establishes that an optimal protocol is symmetric.) We use this expression to compute the optimal β.

$\beta\in[0,1/3]$

The optimal probability is

$$\sum_{|\mathbf{b}|=0}^{3}\binom{3}{|\mathbf{b}|}\Big(\frac{1}{(3-|\mathbf{b}|)!}\sum_{0\le l\le 3-|\mathbf{b}|,1-\beta l>0}(-1)^l\binom{3-|\mathbf{b}|}{l}(1-\beta l)^{3-|\mathbf{b}|}\Big)\cdot$$
$$[(1-\beta)^{|\mathbf{b}|}-\frac{1}{|\mathbf{b}|!}\sum_{0\le l\le|\mathbf{b}|,(|\mathbf{b}|-1-l-\beta l)>0}(-1)^l\binom{|\mathbf{b}|}{l}(|\mathbf{b}|-1-l-\beta l)^{|\mathbf{b}|}]$$
$$= \binom{3}{0}\Big(\frac{1}{3!}\Big(\binom{3}{0}(1-0)^3-\binom{3}{1}(1-\beta)^3+\binom{3}{2}(1-2\beta)^3-\binom{3}{3}(1-3\beta)^3\Big)\Big)\cdot$$
$$[(1-\beta)^0-\frac{1}{0!}\Big(\binom{0}{0}(0-1-0+0)^0]+$$

$$
\begin{aligned}
&\binom{3}{1}\Big(\frac{1}{2!}\Big(\binom{2}{0}(1-0)^2-\binom{2}{1}(1-\beta)^2+\binom{2}{2}(1-2\beta)^2\Big)\Big)\cdot \\
&[(1-\beta)^1-\frac{1}{1!}\Big(\binom{1}{0}(1-1-0+0)^1-\binom{1}{1}(1-1-1+\beta)^1\Big)]+ \\
&\binom{3}{2}\Big(\frac{1}{1!}\Big(\binom{1}{0}(1-0)^1-\binom{1}{1}(1-\beta)^1\Big)\Big)\cdot \\
&[(1-\beta)^2-\frac{1}{2!}\Big(\binom{2}{0}(2-1-0+0)^2-\binom{2}{1}(2-1-1+\beta)^2+\binom{2}{2}(2-1-2+2\beta)^2\Big)]+ \\
&\binom{3}{3}\Big(\frac{1}{0!}\Big(\binom{0}{0}(1-0)^0\Big)\Big)\cdot \\
&[(1-\beta)^3-\frac{1}{3!}\Big(\binom{3}{0}(3-1-0+0)^3-\binom{3}{1}(3-1-1+\beta)^3+\binom{3}{2}(3-1-2+2\beta)^3- \\
&\binom{3}{3}(3-1-3+3\beta)^3\Big)] \\
= \;&\frac{1}{6}(1-3(1-3\beta+3\beta^2-\beta^3)+3(1-6\beta+12\beta^2-8\beta^3)-(1-9\beta+27\beta^2-27\beta^3))+ \\
&\frac{3}{2}(1-2(1-2\beta+\beta^2)+(1-4\beta+4\beta^2))(1-\beta)+ \\
&3\beta[(1-\beta)^2-\frac{1}{2}(1-2\beta^2)]+ \\
&((1-\beta)^3-\frac{1}{6}(8-3(1+3\beta+3\beta^2+\beta^3)+\;24\beta^3)) \\
= \;&\frac{1}{6}(1-3+9\beta-9\beta^2+3\beta^3+3-18\beta+36\beta^2-24\beta^3-27\beta^2+27\beta^3+9\beta-1)+ \\
&\frac{3}{2}(1-2+4\beta-2\beta^2+1-4\beta+4\beta^2)(1-\beta)+ \\
&3\beta(1-2\beta+\beta^2-\frac{1}{2}+\beta^2)+ \\
&((1-\beta)^3-\frac{1}{6}(5-9\beta-9\beta^2+21\beta^3)) \\
= \;&(\beta^3)+(3\beta^2-3\beta^3)+(\frac{3}{2}\beta-6\beta^2+6\beta^3)+(\frac{1}{6}-\frac{3}{2}\beta+\frac{9}{2}\beta^2-\frac{9}{2}\beta^3) \\
= \;&\frac{1}{6}+\frac{3}{2}\beta^2-\frac{1}{2}\beta^3\,.
\end{aligned}
$$

The optimality condition is derived by differentiating with respect to β. We obtain that $3\beta - 3\beta^2/2 = 0$, which implies that $3\beta(1-\beta/2) = 0$. Either $\beta = 0$ or $\beta = 2$. Neither of these values both is acceptable and maximizes the probability (the first is acceptable but minimizes the probability, while the second is not acceptable, since it is greater than $1/3$, which is the upper boundary of the interval under consideration).

$\beta \in (1/3, 1/2]$

The optimal probability is

$$\begin{aligned}
&\sum_{|\mathbf{b}|=0}^{3} \binom{3}{|\mathbf{b}|}\Big(\frac{1}{(3-|\mathbf{b}|)!} \sum_{0\le l\le 3-|\mathbf{b}|, 1-\beta l>0} (-1)^l \binom{3-|\mathbf{b}|}{l} (1-\beta l)^{3-|\mathbf{b}|}\Big) \cdot \\
&[(1-\beta)^{|\mathbf{b}|} - \frac{1}{|\mathbf{b}|!} \sum_{0\le l\le |\mathbf{b}|, (|\mathbf{b}|-1-l-\beta l)>0} (-1)^l \binom{|\mathbf{b}|}{l} (|\mathbf{b}|-1-l-\beta l)^{|\mathbf{b}|}] \\
= \; & \binom{3}{0}\Big(\frac{1}{3!}\Big(\binom{3}{0}(1-0)^3 - \binom{3}{1}(1-\beta)^3 + \binom{3}{2}(1-2\beta)^3 - \binom{3}{3}(1-3\beta)^3\Big)\Big) \cdot \\
&[(1-\beta)^0 - \frac{1}{0!}\Big(\binom{0}{0}(0-1-0+0)^0] + \\
&\binom{3}{1}\Big(\frac{1}{2!}\Big(\binom{2}{0}(1-0)^2 - \binom{2}{1}(1-\beta)^2 + \binom{2}{2}(1-2\beta)^2\Big)\Big) \cdot \\
&[(1-\beta)^1 - \frac{1}{1!}\Big(\binom{1}{0}(1-1-0+0)^1 - \binom{1}{1}(1-1-1+\beta)^1\Big)] + \\
&\binom{3}{2}\Big(\frac{1}{1!}\Big(\binom{1}{0}(1-0)^1 - \binom{1}{1}(1-\beta)^1\Big)\Big) \cdot \\
&[(1-\beta)^2 - \frac{1}{2!}\Big(\binom{2}{0}(2-1-0+0)^2 - \binom{2}{1}(2-1-1+\beta)^2 + \binom{2}{2}(2-1-2+2\beta)^2\Big)] + \\
&\binom{3}{3}\Big(\frac{1}{0!}\Big(\binom{0}{0}(1-0)^0\Big)\Big) \cdot \\
&[(1-\beta)^3 - \frac{1}{3!}\Big(\binom{3}{0}(3-1-0+0)^3 - \binom{3}{1}(3-1-1+\beta)^3 + \binom{3}{2}(3-1-2+2\beta)^3 - \\
&\binom{3}{3}(3-1-3+3\beta)^3\Big)] \\
= \; & \frac{1}{6}(1 - 3(1-3\beta+3\beta^2-\beta^3) + 3(1-6\beta+12\beta^2-8\beta^3)) + \\
&\frac{3}{2}(1-2(1-2\beta+\beta^2) + (1-4\beta+4\beta^2))(1-\beta) + \\
&3\beta[(1-\beta)^2 - \frac{1}{2}(1-2\beta^2)] + \\
&((1-\beta)^3 - \frac{1}{6}(8 - 3(1+3\beta+3\beta^2+\beta^3) + 24\beta^3) - (-1+9\beta-27\beta^2+27\beta^3)) \\
= \; & \frac{1}{6}(1-3+9\beta-9\beta^2+3\beta^3+3-18\beta+36\beta^2-24\beta^3) + \\
&\frac{3}{2}(1-2+4\beta-2\beta^2+1-4\beta+4\beta^2)(1-\beta) +
\end{aligned}$$

$$
\begin{aligned}
& 3\beta(1-2\beta+\beta^2-\frac{1}{2}+\beta^2)+ \\
& ((1-\beta)^3-\frac{1}{6}(5-9\beta-9\beta^2+21\beta^3-27\beta^3+27\beta^2-9\beta+1)) \\
= \;& (\frac{1}{6}-\frac{3}{2}\beta+\frac{9}{2}\beta^2-\frac{7}{2}\beta^3)+(3\beta^2-3\beta^3)+(\frac{3}{2}\beta-6\beta^2+6\beta^3)+0 \\
= \;& \frac{1}{6}+\frac{3}{2}\beta^2-\frac{1}{2}\beta^3
\end{aligned}
$$

The optimality condition is derived by differentiating with respect to β. We obtain that $3\beta - 3\beta^2/2 = 0$, which implies that $3\beta(1-\beta/2) = 0$. Either $\beta = 0$ or $\beta = 2$. Neither of these values is acceptable (both are outside the interval $(1/3, 1/2]$ under consideration).

$\beta \in (1/2, 1]$

The optimal probability is

$$
\begin{aligned}
& \sum_{|\mathbf{b}|=0}^{3} \binom{3}{|\mathbf{b}|} \Big(\frac{1}{(3-|\mathbf{b}|)!} \sum_{0\le l\le 3-|\mathbf{b}|, 1-\beta l>0} (-1)^l \binom{3-|\mathbf{b}|}{l} (1-\beta l)^{3-|\mathbf{b}|}\Big) \cdot \\
& [(1-\beta)^{|\mathbf{b}|} - \frac{1}{|\mathbf{b}|!} \sum_{0\le l\le |\mathbf{b}|, (|\mathbf{b}|-1-l-\beta l)>0} (-1)^l \binom{|\mathbf{b}|}{l} (|\mathbf{b}|-1-l-\beta l)^{|\mathbf{b}|}] \\
= \;& \binom{3}{0}(\frac{1}{3!}(\binom{3}{0}(1-0)^3-\binom{3}{1}(1-\beta)^3+\binom{3}{2}(1-2\beta)^3-\binom{3}{3}(1-3\beta)^3))\cdot \\
& [(1-\beta)^0-\frac{1}{0!}(\binom{0}{0}(0-1-0+0)^0]+ \\
& \binom{3}{1}(\frac{1}{2!}(\binom{2}{0}(1-0)^2-\binom{2}{1}(1-\beta)^2+\binom{2}{2}(1-2\beta)^2))\cdot \\
& [(1-\beta)^1-\frac{1}{1!}(\binom{1}{0}(1-1-0+0)^1-\binom{1}{1}(1-1-1+\beta)^1)]+ \\
& \binom{3}{2}(\frac{1}{1!}(\binom{1}{0}(1-0)^1-\binom{1}{1}(1-\beta)^1))\cdot \\
& [(1-\beta)^2-\frac{1}{2!}(\binom{2}{0}(2-1-0+0)^2-\binom{2}{1}(2-1-1+\beta)^2+\binom{2}{2}(2-1-2+2\beta)^2)]+ \\
& \binom{3}{3}(\frac{1}{0!}(\binom{0}{0}(1-0)^0-))\cdot \\
& [(1-\beta)^3-\frac{1}{3!}(\binom{3}{0}(3-1-0+0)^3-\binom{3}{1}(3-1-1+\beta)^3+\binom{3}{2}(3-1-2+2\beta)^3-
\end{aligned}
$$

$$\binom{3}{3}(3-1-3+3\beta)^3)]$$

$$
\begin{aligned}
= \quad & \frac{1}{6}(1-3(1-3\beta+3\beta^2-\beta^3))+ \\
& \frac{3}{2}(1-2(1-2\beta+\beta^2))(1-\beta)+ \\
& 3\beta[(1-\beta)^2-\frac{1}{2}(1-2\beta^2+1-4\beta+4\beta^2)]+ \\
& ((1-\beta)^3-\frac{1}{6}(8-3(1+3\beta+3\beta^2+\beta^3)+24\beta^3-27\beta^3+27\beta^2-9\beta+1)) \\
= \quad & \frac{1}{6}(-2+9\beta-9\beta^2+3\beta^3)+ \\
& \frac{3}{2}(1-2+4\beta-2\beta^2)(1-\beta)+ \\
& 3\beta(1-2\beta+\beta^2-1+2\beta-\beta^2)+ \\
& (1-3\beta+3\beta^2-\beta^3-\frac{1}{6}(8-3-9\beta-9\beta^2-3\beta^3+24\beta^3-27\beta^3+27\beta^2-9\beta+1)) \\
= \quad & (-\frac{1}{3}+\frac{3}{2}\beta-\frac{3}{2}\beta^2+\frac{1}{2}\beta^3)+(-\frac{3}{2}+\frac{15}{2}\beta-9\beta^2+3\beta^3)+0+0 \\
= \quad & -\frac{11}{6}+9\beta-\frac{21}{2}\beta^2+\frac{7}{2}\beta^3.
\end{aligned}
$$

The optimality condition is derived by differentiating with respect to β. We obtain that $6/7-2\beta+\beta^2=0$, which implies that either $\beta=1+\sqrt{1/7}$ or $\beta=1-\sqrt{1/7}=0.622$. The first is not acceptable because it is greater than 1. The second maximizes indeed the optimal probability. (The second derivative at $\beta=1-\sqrt{1/7}$ becomes negative.) The corresponding optimal (maximum) probability is 0.545. This settles a conjecture of Papadimitriou and Yannakakis [11] for the case $n=3$ and $\delta=1$.

5.2.2 The Case $n=4$ and $\delta=4/3$

$$
\begin{aligned}
& \sum_{|\mathbf{b}|=0}^{4}\binom{4}{|\mathbf{b}|}(\frac{4}{(4-|\mathbf{b}|)!}\sum_{0\le l\le 4-|\mathbf{b}|,\frac{4}{3}-\beta l>0}(-1)^l\binom{4-|\mathbf{b}|}{l}\left(\frac{4}{3}-\beta l\right)^{4-|\mathbf{b}|})\cdot \\
& [(1-\beta)^{|\mathbf{b}|}-\frac{1}{|\mathbf{b}|!}\sum_{0\le l\le|\mathbf{b}|,(|\mathbf{b}|-\frac{4}{3}-l-\beta l)>0}(-1)^l\binom{|\mathbf{b}|}{l}\left(|\mathbf{b}|-\frac{4}{3}-l-\beta l\right)^{|\mathbf{b}|}] \\
= \quad & \binom{4}{0}(\frac{1}{4!}(\binom{4}{0}(\frac{4}{3}-0)^4-\binom{4}{1}(\frac{4}{3}-\beta)^4+\binom{4}{2}(\frac{4}{3}-2\beta)^4-\binom{4}{3}(\frac{4}{3}-3\beta)^4+\binom{4}{4}(\frac{4}{3}-4\beta)^4))\cdot \\
& [(1-\beta)^0-\frac{1}{0!}(\binom{0}{0}(0-\frac{4}{3}-0+0)^0]+
\end{aligned}
$$

$$\binom{4}{1}(\frac{1}{3!}(\binom{3}{0}(\frac{4}{3}-0)^3 - \binom{3}{1}(\frac{4}{3}-\beta)^3 + \binom{3}{2}(\frac{4}{3}-2\beta)^3 - \binom{3}{3}(\frac{4}{3}-3\beta)^3))\cdot$$

$$[(1-\beta)^1 - \frac{1}{1!}(\binom{1}{0}(1-\frac{4}{3}-0+0)^1 - \binom{1}{1}(1-\frac{4}{3}-1+\beta)^1)] +$$

$$\binom{4}{2}(\frac{1}{2!}(\binom{2}{0}(\frac{4}{3}-0)^2 - \binom{2}{1}(\frac{4}{3}-\beta)^2 + \binom{2}{2}(\frac{4}{3}-2\beta)^2))\cdot$$

$$[(1-\beta)^2 - \frac{1}{2!}(\binom{2}{0}(2-\frac{4}{3}-0+0)^2 - \binom{2}{1}(2-\frac{4}{3}-1+\beta)^2 + \binom{2}{2}(2-\frac{4}{3}-2+2\beta)^2)] +$$

$$\binom{4}{3}(\frac{1}{1!}(\binom{1}{0}(\frac{4}{3}-0)^1 - \binom{1}{}(\frac{4}{3}-\beta)^1))\cdot$$

$$[(1-\beta)^3 - \frac{1}{3!}(\binom{3}{0}(3-\frac{4}{3}-0+0)^3 - \binom{3}{1}(3-\frac{4}{3}-1+\beta)^3 + \binom{3}{2}(3-\frac{4}{3}-2+2\beta)^3 -$$

$$\binom{3}{3}(3-\frac{4}{3}-3+3\beta)^3)]$$

$$\binom{4}{4}(\frac{1}{0!}(\binom{0}{0}(\frac{4}{3}-0)^0\cdot$$

$$[(1-\beta)^4 - \frac{1}{4!}(\binom{4}{0}(4-\frac{4}{3}-0+0)^4 - \binom{4}{1}(4-\frac{4}{3}-1+\beta)^4 + \binom{4}{2}(4-\frac{4}{3}-2+2\beta)^4 - \binom{4}{3}(4-\frac{4}{3}-3+$$

$$\binom{4}{4}(4-\frac{4}{3}-4+4\beta)^4)]$$

$$= \frac{1}{24}(\frac{256}{81} - 4(\frac{256}{81}-\frac{256}{27}\beta+\frac{96}{9}\beta^2-\frac{16}{3}\beta^3+\beta^4)+6(\frac{256}{81}-\frac{512}{27}\beta+\frac{384}{9}\beta^2-\frac{128}{3}\beta^3+16\beta^4)-4(\frac{256}{81}-$$

$$\frac{2}{3}(\frac{64}{27}-3(\frac{64}{27}-\frac{48}{9}\beta+\frac{12}{3}\beta^2-\beta^3)+3(\frac{64}{27}-\frac{96}{9}\beta+\frac{48}{3}\beta^2-8\beta^3)-(\frac{64}{27}-\frac{144}{9}\beta+\frac{108}{3}\beta^2-27\beta^3))(1-\beta)$$

$$4\beta[(1-\beta)^3-\frac{1}{6}(\frac{125}{27}-3(\frac{8}{27}-\frac{4}{3}\beta+2\beta^2+\beta^3)+3(\frac{-1}{27}+\frac{2}{3}\beta-4\beta^2+8\beta^3))]+$$

$$[(1-\beta)^4-\frac{1}{24}(\frac{4096}{81}-4(\frac{625}{81}+\frac{500}{27}\beta+\frac{150}{9}\beta^2+\frac{20}{3}\beta^3+\beta^4)+6(\frac{16}{81}+\frac{64}{27}\beta+\frac{96}{9}\beta^2+\frac{64}{3}\beta^3+16\beta^4)-4($$

$$= [\frac{1}{24}(24\beta^4)]+[\frac{2}{3}(6\beta^3)(1-\beta)]+[6\beta^2(\frac{7}{9}-2\beta+\beta^2)]+[(1-\beta^4)-\frac{1}{24}(\frac{1688}{81}-\frac{1568}{27}\beta-\frac{80}{3}\beta^2+\frac{736}{3}\beta^3-$$

$$= \beta^4+(4\beta^3-4\beta^4)+(\frac{128}{81}\beta-\frac{32}{3}\beta^2+24\beta^3-18\beta^4)+(\frac{32}{243}-\frac{128}{81}\beta+\frac{64}{9}\beta^2-\frac{128}{9}\beta^3+\frac{32}{3}\beta^4)$$

$$= \frac{32}{243}+\frac{10}{9}\beta^2+\frac{16}{9}\beta^3-\frac{13}{3}\beta^4.$$

$$\sum_{|\mathbf{b}|=0}^{4}\binom{4}{|\mathbf{b}|}(\frac{4}{(4-|\mathbf{b}|)!}\sum_{0\le l\le 4-|\mathbf{b}|,\frac{4}{3}-\beta l>0}(-1)^l\binom{4-|\mathbf{b}|}{l}\left(\frac{4}{3}-\beta l\right)^{4-|\mathbf{b}|})\cdot$$

$$
\begin{aligned}
& [(1-\beta)^{|\mathbf{b}|} - \frac{1}{|\mathbf{b}|!} \sum_{0 \le l \le |\mathbf{b}|, (|\mathbf{b}| - \frac{4}{3} - l - \beta l) > 0} (-1)^l \binom{|\mathbf{b}|}{l} \left(|\mathbf{b}| - \frac{4}{3} - l - \beta l\right)^{|\mathbf{b}|}] \\
= \quad & \binom{4}{0} (\frac{1}{4!} (\binom{4}{0} (\frac{4}{3} - 0)^4 - \binom{4}{1} (\frac{4}{3} - \beta)^4 + \binom{4}{2} (\frac{4}{3} - 2\beta)^4 - \binom{4}{3} (\frac{4}{3} - 3\beta)^4 + \binom{4}{4} (\frac{4}{3} - 4\beta)^4)) \cdot \\
& [(1-\beta)^0 - \frac{1}{0!} (\binom{0}{0} (0 - \frac{4}{3} - 0 + 0)^0] + \\
& \binom{4}{1} (\frac{1}{3!} (\binom{3}{0} (\frac{4}{3} - 0)^3 - \binom{3}{1} (\frac{4}{3} - \beta)^3 + \binom{3}{2} (\frac{4}{3} - 2\beta)^3 - \binom{3}{3} (\frac{4}{3} - 3\beta)^3)) \cdot \\
& [(1-\beta)^1 - \frac{1}{1!} (\binom{1}{0} (1 - \frac{4}{3} - 0 + 0)^1 - \binom{1}{1} (1 - \frac{4}{3} - 1 + \beta)^1)] + \\
& \binom{4}{2} (\frac{1}{2!} (\binom{2}{0} (\frac{4}{3} - 0)^2 - \binom{2}{1} (\frac{4}{3} - \beta)^2 + \binom{2}{2} (\frac{4}{3} - 2\beta)^2)) \cdot \\
& [(1-\beta)^2 - \frac{1}{2!} (\binom{2}{0} (2 - \frac{4}{3} - 0 + 0)^2 - \binom{2}{1} (2 - \frac{4}{3} - 1 + \beta)^2 + \binom{2}{2} (2 - \frac{4}{3} - 2 + 2\beta)^2)] + \\
& \binom{4}{3} (\frac{1}{1!} (\binom{1}{0} (\frac{4}{3} - 0)^1 - \binom{1}{\ } (\frac{4}{3} - \beta)^1)) \cdot \\
& [(1-\beta)^3 - \frac{1}{3!} (\binom{3}{0} (3 - \frac{4}{3} - 0 + 0)^3 - \binom{3}{1} (3 - \frac{4}{3} - 1 + \beta)^3 + \binom{3}{2} (3 - \frac{4}{3} - 2 + 2\beta)^3 - \\
& \binom{3}{3} (3 - \frac{4}{3} - 3 + 3\beta)^3)] \\
& \binom{4}{4} (\frac{1}{0!} (\binom{0}{0} (\frac{4}{3} - 0)^0) \cdot \\
& [(1-\beta)^4 - \frac{1}{4!} (\binom{4}{0} (4 - \frac{4}{3} - 0 + 0)^4 - \binom{4}{1} (4 - \frac{4}{3} - 1 + \beta)^4 + \binom{4}{2} (4 - \frac{4}{3} - 2 + 2\beta)^4 - \binom{4}{3} (4 - \frac{4}{3} - 3 + \\
& \binom{4}{4} (4 - \frac{4}{3} - 4 + 4\beta)^4)] \\
= \quad & \frac{1}{24} (\frac{-256}{81} + \frac{1024}{27}\beta - \frac{1536}{9}\beta^2 + \frac{1024}{3}\beta^3 - 232\beta^4) + \\
& \frac{2}{3} (\frac{64}{27} - 3(\frac{64}{27} - \frac{48}{9}\beta + \frac{12}{3}\beta^2 - \beta^3) + 3(\frac{64}{27} - \frac{96}{9}\beta + \frac{48}{3}\beta^2 - 8\beta^3) - (\frac{64}{27} - \frac{144}{9}\beta + \frac{108}{3}\beta^2 - 27\beta^3))(1-\beta) \\
& 6\beta^2(1 - 2\beta + \beta^2 - \frac{1}{2}(\frac{2}{9} + \frac{4}{3}\beta - 2\beta^2) + \\
& 4\beta[(1-\beta)^3 - \frac{1}{6}(\frac{125}{27} - 3(\frac{8}{27} - \frac{4}{3}\beta + 2\beta^2 + \beta^3) + 3(\frac{-1}{27} + \frac{2}{3}\beta - 4\beta^2 + 8\beta^3))] + \\
& [(1-\beta)^4 - \frac{1}{24}(\frac{4096}{81} - 4(\frac{625}{81} + \frac{500}{27}\beta + \frac{150}{9}\beta^2 + \frac{20}{3}\beta^3 + \beta^4) + 6(\frac{16}{81} + \frac{64}{27}\beta + \frac{96}{9}\beta^2 + \frac{64}{3}\beta^3 + 16\beta^4) - 4(
\end{aligned}
$$

$$
\begin{aligned}
= \quad & (\frac{-32}{243} + \frac{128}{81}\beta - \frac{64}{9}\beta^2 + \\
& \frac{128}{9}\beta^3 - \frac{29}{3}\beta^4) + (\frac{2}{3}(6\beta^3)(1-\beta)) + (6\beta^2(\frac{8}{9} - \frac{8}{3}\beta + 2\beta^2)) + ((1-\beta^4) - \frac{1}{24}(\frac{1688}{81} - \frac{1568}{27}\beta - \frac{80}{3}\beta^2 + \frac{736}{3} \\
= \quad & (\frac{-32}{243} + \frac{128}{81}\beta - \frac{64}{9}\beta^2 + \\
& \frac{128}{9}\beta^3 - \frac{29}{3}\beta^4) + (4\beta^3 - 4\beta^4) + \\
& (\frac{128}{81}\beta - \frac{32}{3}\beta^2 + 24\beta^3 - 18\beta^4) + (\frac{32}{243} - \frac{128}{81}\beta + \frac{64}{9}\beta^2 - \frac{128}{9}\beta^3 + \frac{32}{3}\beta^4) \\
= \quad & (\frac{128}{81} - \frac{16}{3}\beta^2 + 12\beta^3 - 9\beta^4).
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{|\mathbf{b}|=0}^{4} \binom{4}{|\mathbf{b}|} (\frac{4}{(4-|\mathbf{b}|)!} \sum_{0 \le l \le 4-|\mathbf{b}|, \frac{4}{3}-\beta l > 0} (-1)^l \binom{4-|\mathbf{b}|}{l} \left(\frac{4}{3} - \beta l\right)^{4-|\mathbf{b}|}) \cdot \\
& [(1-\beta)^{|\mathbf{b}|} - \frac{1}{|\mathbf{b}|!} \sum_{0 \le l \le |\mathbf{b}|, (|\mathbf{b}| - \frac{4}{3} - l - \beta l) > 0} (-1)^l \binom{|\mathbf{b}|}{l} \left(|\mathbf{b}| - \frac{4}{3} - l - \beta l\right)^{|\mathbf{b}|}] \\
= \quad & \binom{4}{0} (\frac{1}{4!} (\binom{4}{0} (\frac{4}{3} - 0)^4 - \binom{4}{1} (\frac{4}{3} - \beta)^4 + \binom{4}{2} (\frac{4}{3} - 2\beta)^4 - \binom{4}{3} (\frac{4}{3} - 3\beta)^4 + \binom{4}{4} (\frac{4}{3} - 4\beta)^4)) \cdot \\
& [(1-\beta)^0 - \frac{1}{0!} (\binom{0}{0} (0 - \frac{4}{3} - 0 + 0)^0] + \\
& \binom{4}{1} (\frac{1}{3!} (\binom{3}{0} (\frac{4}{3} - 0)^3 - \binom{3}{1} (\frac{4}{3} - \beta)^3 + \binom{3}{2} (\frac{4}{3} - 2\beta)^3 - \binom{3}{3} (\frac{4}{3} - 3\beta)^3)) \cdot \\
& [(1-\beta)^1 - \frac{1}{1!} (\binom{1}{0} (1 - \frac{4}{3} - 0 + 0)^1 - \binom{1}{1} (1 - \frac{4}{3} - 1 + \beta)^1)] + \\
& \binom{4}{2} (\frac{1}{2!} (\binom{2}{0} (\frac{4}{3} - 0)^2 - \binom{2}{1} (\frac{4}{3} - \beta)^2 + \binom{2}{2} (\frac{4}{3} - 2\beta)^2)) \cdot \\
& [(1-\beta)^2 - \frac{1}{2!} (\binom{2}{0} (2 - \frac{4}{3} - 0 + 0)^2 - \binom{2}{1} (2 - \frac{4}{3} - 1 + \beta)^2 + \binom{2}{2} (2 - \frac{4}{3} - 2 + 2\beta)^2)] + \\
& \binom{4}{3} (\frac{1}{1!} (\binom{1}{0} (\frac{4}{3} - 0)^1 - \binom{1}{} (\frac{4}{3} - \beta)^1)) \cdot \\
& [(1-\beta)^3 - \frac{1}{3!} (\binom{3}{0} (3 - \frac{4}{3} - 0 + 0)^3 - \binom{3}{1} (3 - \frac{4}{3} - 1 + \beta)^3 + \binom{3}{2} (3 - \frac{4}{3} - 2 + 2\beta)^3 - \\
& \binom{3}{3} (3 - \frac{4}{3} - 3 + 3\beta)^3)]
\end{aligned}
$$

$$
\begin{aligned}
& \binom{4}{4}(\frac{1}{0!}(\binom{0}{0}(\frac{4}{3}-0)^0 \cdot \\
& [(1-\beta)^4 - \frac{1}{4!}(\binom{4}{0}(4-\frac{4}{3}-0+0)^4 - \binom{4}{1}(4-\frac{4}{3}-1+\beta)^4 + \binom{4}{2}(4-\frac{4}{3}-2+2\beta)^4 - \binom{4}{3}(4-\frac{4}{3}-3+ \\
& \binom{4}{4}(4-\frac{4}{3}-4+4\beta)^4)] \\
= & \frac{1}{24}(\frac{256}{81} - 4(\frac{256}{81} - \frac{256}{27}\beta + \frac{96}{9}\beta^2 - \frac{16}{3}\beta^3 + \beta^4) + 6(\frac{256}{81} - \frac{512}{27}\beta + \frac{384}{9}\beta^2 - \frac{128}{3}\beta^3 + 16\beta^4) + \\
& \frac{2}{3}(\frac{64}{27} - 3(\frac{64}{27} - \frac{48}{9}\beta + \frac{12}{3}\beta^2 - \beta^3) + 3(\frac{64}{27} - \frac{96}{9}\beta + \frac{48}{3}\beta^2 - 8\beta^3)) \cdot (1-\beta) \\
& 6\frac{1}{2}(2\beta^2)[1 - 2\beta + \beta^2 - \frac{1}{2}(\frac{4}{9} - 2(\frac{1}{9} - \frac{2}{3}\beta + \beta^2))] + \\
& 4\beta\,[(1-\beta)^3 - \frac{1}{6}(\frac{125}{27} - 3(\frac{8}{27} + \frac{4}{3}\beta + 2\beta^2 + beta^3) \\
& +3(-\frac{1}{27} + \frac{2}{3}\beta - 4\beta^2 + 8\beta^3) - (-\frac{64}{27} + 16\beta - 36\beta^2 + 27beta^3))] \\
& (1-\beta)^4 - \frac{1}{24}(\frac{1688}{81} - \frac{1568}{27}\beta - \frac{80}{3}\beta^2 + \frac{736}{3}\beta^3 - 232\beta^4 + \frac{256}{81} - \frac{1024}{27}\beta + \frac{1536}{9}\beta^2 + \\
& -\frac{1024}{3}\beta^3 + 256\beta^4 \\
= & \frac{1}{24}(\frac{768}{81} - \frac{2048}{27}\beta - \frac{1920}{9}\beta^2 - \frac{704}{3}\beta^3 + 12\beta^4) + \\
& (\frac{128}{81} - \frac{32}{3}\beta + 24\beta^2 - 14\beta^3) \cdot (1-\beta) + 6\beta^2(\frac{8}{9} - \frac{8}{3}\beta + 2\beta^2) + \\
& 4\beta(1 - 3\beta + 3\beta^2 - \beta^3 - 1 + 3\beta - 3\beta^2 + \beta^3)(1-\beta)^4 - (1 - 4\beta + 6\beta^2 - 4\beta^3 + \beta^4) \\
= & \frac{32}{81} - \frac{256}{81}\beta + \frac{80}{9}\beta^2 - \frac{88}{9}\beta^3 + \frac{92}{24}\beta^4 + \frac{128}{81} - \frac{992}{81}\beta + \frac{104}{3}\beta^2 - 38\beta^3 + 14\beta^4 + \\
& \frac{48}{9}\beta^2 - \frac{48}{3}\beta^3 + 12\beta^4 + 0 + 0 \\
= & \frac{160}{81} + \frac{416}{27}\beta + \frac{440}{9}\beta^2 + \frac{574}{9}\beta^3 - \frac{179}{6}\beta^4.
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{|\mathbf{b}|=0}^{4} \binom{4}{|\mathbf{b}|}(\frac{1}{(4-|\mathbf{b}|)!} \sum_{0 \le l \le 4-|\mathbf{b}|, \frac{4}{3}-\beta l > 0} (-1)^l \binom{4-|\mathbf{b}|}{l} \left(\frac{4}{3} - \beta l\right)^{4-|\mathbf{b}|}) \cdot \\
& [(1-\beta)^{|\mathbf{b}|} - \frac{1}{|\mathbf{b}|!} \sum_{0 \le l \le |\mathbf{b}|, (|\mathbf{b}| - \frac{4}{3} - l - \beta l) > 0} (-1)^l \binom{|\mathbf{b}|}{l} \left(|\mathbf{b}| - \frac{4}{3} - l - \beta l\right)^{|\mathbf{b}|}] \\
= & \binom{4}{0}(\frac{1}{4!}(\binom{4}{0}(\frac{4}{3}-0)^4 - \binom{4}{1}(\frac{4}{3}-\beta)^4 + \binom{4}{2}(\frac{4}{3}-2\beta)^4 - \binom{4}{3}(\frac{4}{3}-3\beta)^4 + \binom{4}{4}(\frac{4}{3}-4\beta)^4)) \cdot
\end{aligned}
$$

$$[(1-\beta)^0 - \frac{1}{0!}(\binom{0}{0}(0-\frac{4}{3}-0+0)^0] +$$

$$\binom{4}{1}(\frac{1}{3!}(\binom{3}{0}(\frac{4}{3}-0)^3 - \binom{3}{1}(\frac{4}{3}-\beta)^3 + \binom{3}{2}(\frac{4}{3}-2\beta)^3\binom{3}{3}(\frac{4}{3}-3\beta)^3))\cdot$$

$$[(1-\beta)^1 - \frac{1}{1!}(\binom{1}{0}(1-\frac{4}{3}-0+0)^1 - \binom{1}{1}(1-\frac{4}{3}-1+\beta)^1)] +$$

$$\binom{4}{2}(\frac{1}{2!}(\binom{2}{0}(\frac{4}{3}-0)^2 - \binom{2}{1}(\frac{4}{3}-\beta)^2 + \binom{2}{2}(\frac{4}{3}-2\beta)^2))\cdot$$

$$[(1-\beta)^2 - \frac{1}{2!}(\binom{2}{0}(2-\frac{4}{3}-0+0)^2 - \binom{2}{1}(2-\frac{4}{3}-1+\beta)^2 + \binom{2}{2}(2-\frac{4}{3}-2+2\beta)^2)] +$$

$$\binom{4}{3}(\frac{1}{1!}(\binom{1}{0}(\frac{4}{3}-0)^1 - \binom{1}{1}(\frac{4}{3}-\beta)^1))\cdot$$

$$[(1-\beta)^3 - \frac{1}{2!}(\binom{3}{0}(3-\frac{4}{3}-0+0)^3 - \binom{3}{1}(3-\frac{4}{3}-1+\beta)^3 + \binom{3}{2}(3-\frac{4}{3}-2+2\beta)^3) -$$

$$\binom{3}{3}(3-\frac{4}{3}-3+3\beta)^3)] +$$

$$\binom{4}{4}(\frac{1}{0!}(\binom{0}{0}(\frac{4}{3}-0)^0-))\cdot$$

$$[(1-\beta)^4 - \frac{1}{4!}(\binom{4}{0}(4-\frac{4}{3}-0+0)^4 - \binom{4}{1}(4-\frac{4}{3}-1+\beta)^4 + \binom{4}{2}(4-\frac{4}{3}-2+2\beta)^4 -$$

$$\binom{4}{3}(4-\frac{4}{3}-3+3\beta)^4)]$$

$$\binom{4}{4}(4-\frac{4}{3}-4+4\beta)^4)]$$

$$= \frac{1}{24}(\frac{256}{81} - 4(\frac{256}{81} - \frac{256}{27}\beta + \frac{96}{9}\beta^2 - \frac{16}{3}\beta^3 + \beta^4)) +$$

$$\frac{2}{3}(\frac{64}{27} - 3(\frac{64}{27} - \frac{48}{9}\beta + 4\beta^2 - \beta^3)(1-\beta)) +$$

$$3[(\frac{16}{9} - 2(\frac{16}{9} - \frac{8}{3}\beta + \beta^2)]\cdot$$

$$[(1-\beta)^2 - \frac{1}{2}(\frac{4}{9} - 2(\frac{1}{9} - \frac{2}{3}\beta + \beta^2) + \frac{16}{9} - \frac{16}{3}\beta + 4\beta^2)] +$$

$$[4\beta(1-\beta)^3 - \frac{1}{6}(8 - 3(1+3\beta+3\beta^2+\beta^3) + 24\beta^3 + 1 - 9\beta + 27\beta^2 - 27\beta^3)] +$$

$$[(1-\beta)^4 - \frac{1}{24}(\frac{4096}{81} - 4(\frac{625}{81} + \frac{500}{27}\beta + \frac{150}{9}\beta^2) + \frac{20}{3}\beta^3 + \beta^4) + 6(\frac{16}{81} + \frac{64}{27}\beta + \frac{96}{9}\beta^2) +$$

$$
\begin{aligned}
& \frac{64}{3}\beta^3 + 16\beta^4) - 4(\frac{1}{81} - \frac{4}{9}\beta + 6\beta^2) - 36\beta^3 + 81\beta^4)] \\
= \; & -\frac{478}{1215} + \frac{128}{81}\beta - \frac{16}{9}\beta^2 + \frac{8}{9}\beta^3 - \frac{1}{6}\beta^4 \\
& -(\frac{256}{81} + \frac{32}{3}\beta - 8\beta^2 + \frac{2}{3})(1-\beta) + \\
& (-\frac{16}{3} + 16\beta - 6\beta^2)(1 - 2\beta + \beta^2 - 1 + 2\beta - \beta^2) \\
& 4\beta(1 - 3\beta + 3\beta^2 - \beta^3 - \frac{1}{6}(8 - 3 - 9\beta - 9\beta^2 - 3\beta^3 + 24\beta^3 + 1 - 9\beta + 27\beta^2 - 27\beta^3)) + \\
& (1 - 4\beta + 6\beta^2 - 4\beta^3 + \beta^4 - \frac{211}{243} + \frac{196}{81}\beta + \frac{10}{9}\beta^2 - \frac{92}{9}\beta^3 + \frac{29}{3}\beta^4) \\
= \; & -\frac{478}{1215} + \frac{128}{81}\beta - \frac{16}{9}\beta^2 + \frac{8}{9}\beta^3 - \frac{1}{6}\beta^4 + \\
& (-\frac{256}{81} + \frac{1120}{81}\beta - \frac{56}{3}\beta^2 + 10\beta^3 - 2\beta^4) + 0 + 0 + 0 \\
= \; & \frac{4318}{1215} - \frac{416}{27}\beta - \frac{184}{9}\beta^2 + \frac{98}{9}\beta^3 - \frac{13}{6}\beta^4
\end{aligned}
$$

6 Discussion

We have presented a simple yet elegant combinatorial framework for the design and analysis of distributed, decision-making protocols in the presence of incomplete information. Within this framework, we have settled down completely the case where no communication is allowed among the agents. Our techniques and arguments have been purely combinatorial; as such, they are of independent interest. We feel that our work makes a significant advancement in the field of distributed optimization problems by providing a mathematical framework in which further research can be carried out.

References

[1] K. Arrow, *The Economics of Information,* Harvard University Press, 1984.

[2] G. Brightwell, T. J. Ott, and P. Winkler, "Target Shooting with Programmed Random Variables," *Proceedings of the 24th Annual ACM Symposium on Theory of Computing,* pp. 691–698, May 1992.

[3] W. J. Cook, W. H. Cunningham, W. R. Pulleyblank and A. Schrijver, *Combinatorial Optimization,* Wiley-Interscience Series in Discrete Mathematics and Optimization, 1998.

[4] P. Fizzano, D. Karger, C. Stein and J. Wein, "Job Scheduling in Rings," *Proceedings of the 6th Annual ACM Symposium on Parallel Algorithms and Architectures,* pp. 210–219, June 1994.

[5] X. Deng and C. H. Papadimitriou, "Competitive Distributed Decision-Making," *Algorithmica,* Vol. 16, No. 2, pp. 133–150, 1996.

[6] P. Fatourou, M. Mavronicolas and P. Spirakis, "Efficiency of Oblivious versus Non-Oblivious Schedulers for Optimistic, Rate-Based Flow Control," *Proceedings of the 16th Annual ACM Symposium on Principles of Distributed Computing,* pp. 139–148, August 1997.

[7] J. M. Hayman, A. A. Lazar, and G. Pacifici, "Joint Scheduling and Admission Control for ATM-Based Switching Nodes," *Proceedings of the ACM SIGCOMM,* pp. 223–234, 1992.

[8] P. C. Kanellakis and C. H. Papadimitriou, "The Complexity of Distributed Concurrency Control," *SIAM Journal on Computing,* Vol. 14, No. 1, pp. 52–75, February 1985.

[9] E. Kushilevitz and N. Nisan, *Communication Complexity,* Cambridge University Press, Cambridge, 1996.

[10] C. H. Papadimitriou, "Computational Aspects of Organization Theory," *Proceedings of the 4th Annual European Symposium on Algorithms,* September 1996.

[11] C. H. Papadimitriou and M. Yannakakis, "On the Value of Information in Distributed Decision-Making," *Proceedings of the 10th Annual ACM Symposium on Principles of Distributed Computing,* pp. 61–64, August 1991.

[12] C. H. Papadimitriou and M. Yannakakis, "Linear Programming Without the Matrix," *Proceedings of the 25th Annual ACM Symposium on Theory of Computing,* pp. 121–129, May 1993.

[13] C. Polychronopoulos and D. Kuck, "Guided Self-Scheduling: A Practical Scheduling Scheme for Parallel Computers," *IEEE Transactions on Computers,* Vol. 12, pp. 1425–1439, 1987.

[14] G. C. Rota, "On the Foundations of Combinatorial Theory I: Theory of Möbius Functions," *Z. Wahrscheinlichkeitstheorie,* Vol. 2, pp. 340–368, 1964.

[15] G. C. Rota, *Introduction to Probability Theory,* Third Preliminary Edition, Birkhäuser, 1999.

[16] G. C. Rota, *Probability,* Lecture Notes – Spring 1998, MIT Course 18.313.

[17] R. P. Stanley, *Enumerative Combinatorics: Volume 1,* The Wadsworth & Brooks/Cole Mathematics Series, 1986.